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MODAL SYSTEM S4.4

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It is known that Group II of Lewis-Langford, *cf.* [3], p. 493, i.e. the matrices $\mathfrak{M}1$ and $\mathfrak{M}2$[1]

$\mathfrak{M}1$

	C	1	2	3	4	N
*	1	1	2	3	4	4
	2	1	1	3	3	3
	3	1	2	1	2	2
	4	1	1	1	1	1

$\mathfrak{M}2$

	p	M	L
*	1	1	1
	2	2	4
	3	1	3
	4	4	4

$\mathfrak{M}3$

	p	M	L
*	1	1	1
	2	1	4
	3	1	4
	4	4	4

which falsify the proper axiom of S5:

C11   *ℭMpLMp*   (i.e. C11*   *ℭMLpLp*)

are such that besides system S4, they verify several consequences of S5 which are unprovable in the former system, as, e.g., the formulas:

G1   *ℭMLpLMp*
D2   *ALCLpqLCLqp*
M1   *ℭℭℭpLpLpCMLpLp*
N1   *ℭℭℭpLppCMLpp*

The theses *G1* and *D2* are the proper axioms of the well-known systems S4.2 and S4.3 respectively, *cf.* [2], [1], [6], and [11]. In [2], p. 263, Dummett and Lemmon have proved that *M1*, i.e. their formula (8), does not hold in S4.3. Prior, [6], p. 139, pointed out that Geach showed that in the field of S4.2 theses *M1* and *N1* are equivalent.

As one can easily notice $\mathfrak{M}1$ and $\mathfrak{M}2$ verify also the following two formulas

R1   *ℭpCMLpLp*   (i.e. *R1**   *ℭNpCMpLMp*)

and

V1   *ALpALCpqLCpNq*

It is clear that *R1* is a weaker form of C11* (i.e. of C11), but, as $\mathfrak{M}1$ and $\mathfrak{M}2$ show, in the field of S4 it does not imply S5. On the other hand

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formula *V1* is such that neither addition of it to S5 reduces the latter system to the classical logic nor is it a consequence of S5. Namely,

$\mathfrak{M}4$

	C	1	2	3	4	5	6	7	8	N
*	1	1	2	3	4	5	6	7	8	8
	2	1	1	3	3	5	5	7	7	7
	3	1	2	1	2	5	6	5	6	6
	4	1	1	1	1	5	5	5	5	5
	5	1	2	3	4	1	2	3	4	4
	6	1	1	3	3	1	1	3	3	3
	7	1	2	1	2	1	2	1	2	2
	8	1	1	1	1	1	1	1	1	1

$\mathfrak{M}5$

	p	M	L
*	1	1	1
	2	1	8
	3	1	8
	4	1	8
	5	1	8
	6	1	8
	7	1	8
	8	8	8

$\mathfrak{M}6$

	p	M	L
*	1	1	1
	2	1	6
	3	3	8
	4	4	8
	5	1	5
	6	1	6
	7	3	8
	8	8	8

$\mathfrak{M}7$

	p	M	L
*	1	1	1
	2	2	6
	3	3	7
	4	4	8
	5	1	5
	6	2	6
	7	3	7
	8	8	8

matrices $\mathfrak{M}4$ and $\mathfrak{M}5$ verify S5, but they falsify *V1* for *p/2* and *q/3*: *AL2ALC23LC2N3* = *CN8CNL3LC26* = *C1CN8L5* = *C1C18* = *C18* = *8*, and matrices $\mathfrak{M}1$ and $\mathfrak{M}3$, given above, verify S5 and *V1*, but they falsify, e.g., $\mathfrak{C}pLp$ for *p/2*: $\mathfrak{C}2L2$ = *LC24* = *L3* = *4*.

In this paper I shall show that the addition of *R1*, as a new axiom to S4, gives a modal system which I call S4.4 and which, being, obviously, a proper subsystem of S5, contains theses *G1*, *D2*, *M1* and *N1*. It allows us to construct the following family of the proper extensions of S4:

1) S4.4 = {S4;*R1*}
2) S4.3.1 = {S4.3;*N1*}
3) S4.3 = {S4;*D2*}
4) S4.2.1 = {S4.2;*N1*}
5) S4.2 = {S4;*G1*}
6) S4.1.1 = {S4;*M1*}
7) S4.1 = {S4;*N1*}

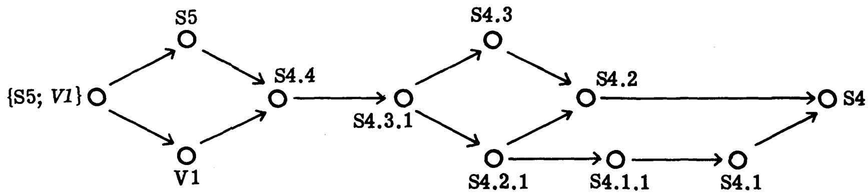
and to prove that

a) S4.4 is a proper extension of S4.3.1
b) S4.3.1 is a proper extension of S4.3.
c) S4.2.1 is a proper subsystem of S4.3.1, is a proper extension of S4.2, but it does not contain S4.3
d) S4.1.1 is a proper subsystem of S4.2.1, it contains S4.1, but it does not contain S4.2
e) S4.1 is a proper extension of S4.

I must note that I have no proof that S4.1 does not contain S4.1.1, i.e. that the latter system is a proper extension of S4.1.

Moreover, since *V1* is verified by $\mathfrak{M}1$ and $\mathfrak{M}2$, we can construct a proper extension of S4, say, system V1 obtained by the addition of *V1*, as a new axiom, to S4. Due to results of Scroogs, *cf.* [8], we know that the proper extensions of S5 are not especially interesting. On the other hand, as far as I know, the effects of the additions of the proper axioms of the possible extensions of S5, *cf.* [8], pp. 119-120, to S4 is not yet investigated. Concerning V1 I shall show here that this system contains S4.4. And, the investigations given in [9] will prove that system V1 possesses the different properties than the analogous system {S5; *V1*}.

The connections existing among the systems discussed in this paper can be presented clearly by the following diagram



1. System S4.4. Let us assume S4 and the thesis

R1 ℭ*pCMLpLp*

Then, we have:

Z1 ℭ*MpCMLpLMp* [*R1*,*p*/*Mp*;S1]
L1 ℭ*MLpLMLp* [*Z1*,*p*/*Lp*;S4]
G1 ℭ*MLpLMp* [*L1*;S2]

Thus, in virtue of *L1* or *G1* S4.4 contains S4.2.[2]

Z2 ℭ*NpCpq* [S1°]
Z3 ℭ*MLqMLCpq* [S2°]
Z4 ℭ*NpNLp* [S1°]
Z5 ℭ*MKpqMp* [S2°]
Z6 ℭ*CpCqrCCsvCCvpCCtzCCzqCsCtr* [S1°]
Z7 ℭ*NpCMKLqrCLpq* [*Z6*,*p*/*CLpq*,*q*/*MLCLpq*,*r*/*LCLpq*,*s*/*Np*,*v*/*NLp*, *t*/*MKLqr*,*z*/*MLq*;*R1*,*p*/*CLpq*;*Z4*;*Z2*,*p*/*Lp*;*Z5*,*p*/*Lq*,*q*/*r*;*Z3*,*p*/*Lp*;S1°]
Z8 *ApALCLqrLCLpq* [*Z7*,*r*/*Nr*;S1°]
Z9 ℭ*LpLCqp* [S2°]
Z10 ℭ*CpCqrCCsqCCrtCpCst* [S1°]
Z11 *CpCMKLpsLCqp* [*Z10*,*q*/*MLp*,*r*/*Lp*,*s*/*MKLps*,*t*/*LCqp*;*R1*; *Z5*,*p*/*Lp*,*q*/*s*;*Z9*;S1°]
Z12 *CpALCLpsLCqp* [*Z11*,*s*/*Ns*;S1°]
Z13 *CApqCCprArq* [S1°]
Z14 *AALCLpsLCqpALCLqrLCLpq* [*Z13*,*q*/*ALCLqrLCLpq*,*r*/*ALCLpsLCqp*; *Z8*;*Z13*;S1°]
Z15 *CAApqAqpApq* [S1°]
D2 *ALCLpqLCLqp* [*Z15*,*p*/*LCLpq*,*q*/*LCLqp*;*Z14*,*q*/*Lq*,*r*/*p*,*s*/*q*;S4]

Hence, S4.4 contains S4.3.[3]

Z16 ℭ*MLpCpLp* [*R1*;S1°]
Z17 ℭ*MLpLCpLp* [*Z16*;S2°;*L1*;S1°]
W1 ℭ*CCpLpqCMLpq* [*Z16*;S1°]
W2 ℭ*CLCpLpqCMLpq* [*Z17*;S1°]
W3 ℭ*LCCpLpqCMLpq* [*W1*;S2]
W4 ℭ*LCLCpLpqCMLpq* [*W2*;S2]
T1 ℭ*CCpLpLpCMLpLp* [*W1*,*q*/*Lp*]
T2 ℭ*CLCpLpLpCMLpLp* [*W2*,*q*/*Lp*]

T3 $\mathfrak{C}LCCpLpLpCMLpLp$ [*W3,q/Lp*]
M1 $\mathfrak{C}LCLCpLpLpCMLpLp$ [*W4,p/Lp*]

Thus, S4.4 implies *M1*.

2. Theses *W1-W4*, *T1-T3*, *M1* and *N1*. It is clear that in the field of S1 each of the theses *W1-W4* is equivalent to *R1*. It will be shown here that a) *T1* or *T2* can also serve as the proper axiom of S4.4, and that b) *T3* is provable in *S2*. Moreover, it will be proved that *M1* together with S4 implies *N1*, and it will be reconstracted Geach's proof that in the field of S4.2 *M1* is a consequence of *N1*.

2.1 Assume S1 and *T1* or *T2*. Then in the case of *T1* we have immediately

Z1 $\mathfrak{C}NCpLpCMLpLp$ [*T1*;S1°]

and in the case of T2:

Z2 $\mathfrak{C}NLCpLpCMLpLp$ [*T2*;S1°]

which in virtue of S1 implies, obviously, *Z1*. Hence

Z3 $\mathfrak{C}KpNLpCMLpLp$ [*Z1*;S1°]
R1 $\mathfrak{C}pCMLpLp$ [*Z3*;S1°]

2.2 Now, let us assume S2. Then:

Z1 $\mathfrak{C}Lpp$ [S1]
Z2 $\mathfrak{C}NCpqp$ [S1°]
Z3 $\mathfrak{C}CNprCCqrCCpqr$ [S1°]
Z4 $\mathfrak{C}CCpqLpp$ [*Z3,p/Cpq,q/Lp,r/p;Z2;Z1;S1°*]
Z5 $\mathfrak{C}LCCpqLpLp$ [*Z4;S2°*]
Z6 $\mathfrak{C}LCCpqLpCrLp$ [*Z5;S1°*]
T3 $\mathfrak{C}LCCpLpLpCMLpLp$ [*Z6,q/Lp,r/MLp*]

Hence, *T3* is a consequence of S2.

2.3 Let us assume S4 and *M1*. Then:

Z1 $\mathfrak{C}LCLLCpLpLpCMLpLp$ [*M1*;S4°]
Z2 $\mathfrak{C}LCpqLCLpLq$ [S3°]
Z3 $\mathfrak{C}LCLCpLppCMLpLp$ [*Z2,p/CLCpLp,q/p;Z1*;S4°]
N1 $\mathfrak{C}LCLCpLppCMLpp$ [*Z3*;S1]

Hence, in the field of S4 *M1* implies *N1*.

2.4 Let us assume S4.2 and *N1*. Then:

Z1 $\mathfrak{C}\mathfrak{C}qr\mathfrak{C}\mathfrak{C}prs\mathfrak{C}\mathfrak{C}pqs$ [S3°]
Z2 $\mathfrak{C}Lpp$ [S1]
L1 $\mathfrak{C}MLpLMLp$ [S4.2]
Z3 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLpCMLpp$ [*Z1,p/CpLp,q/Lp,r/p,s/CMLpp;Z2;N1*]
Z4 $\mathfrak{C}L\mathfrak{C}\mathfrak{C}pLpLpCLMLpLp$ [*Z3*;S2°]
Z5 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLpCLMLpLp$ [*Z4*;S4°]
M1 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLpCMLpLp$ [*L1;Z5*;S1°]

Thus, in the field of S4.2 *N1* implies *M1*.[4]

3 System V1. Let us assume S2 and

V1 $ALpALCpqLCpNq$

Then, we have:

V2 $LALpALCpqLCpNq$ [*V1*;S1°]
V3 $\mathfrak{C}NLpCNLCpqLCpNq$ [*V2*;S1°]
V4 $\mathfrak{C}MKpqCMKpNqLp$ [*V3*;S1°]

In [6], p. 16, Prior mentions *V4* as an odd formula verified by $\mathfrak{M}$1 and $\mathfrak{M}$2. It is clear that in the field of S1° the theses *V1* and *V4* are equivalent.

V5 $\mathfrak{C}MLpMKpLp$ [S2]
V6 $\mathfrak{C}MLpCMKpNLpLp$ [*V5*;*V4*,*q*/*Lp*;S1°]
V7 $\mathfrak{C}MLpCKpNLpLp$ [*V6*;S1]
R1 $\mathfrak{C}pCMLpLp$ [*V7*;S1°]

Thus, in the field of S2 *V1* implies *R1*. Therefore, V1 contains system S4.4.

4 Axiomatizations of S4.4 and V1. It is known, *cf.* [10], pp. 155-156, that the formula

Z1 $\mathfrak{C}CMpLq\mathfrak{C}pq$

is provable in S2. Hence, we have also in S2

Z2 $\mathfrak{C}LCMpLqLLCpq$ [*Z1*;S2°]

Therefore, the addition of *R1* to S2 yields:

Z3 $\mathfrak{C}pLCLpp$ [*Z1*,*p*/*Lp*,*q*/*p*;*R1*;S1°]
Z4 $LLCpCLMpMp$ [*Z2*,*q*/*CLMpMp*;*Z3*,*p*/*Mp*;S1°]

And, due to the fact that it was proved in **3** that *V1* together with S1 implies *V4*, the addition of *V1* to S2 allows us to make the following deductions:

V4 $\mathfrak{C}MKpqCMKpNqLp$ [*V1*;S1°]
Z5 $\mathfrak{C}MKKpqKpNqLp$ [*V4*;S2°]
Z6 $LLCKKpqKpNqp$ [*Z2*,*p*/*KKpqKpNq*,*q*/*p*;*Z5*;S1°]

Since the addition of *Z4* or *Z6* to S3 gives S4, *cf.* [4], p. 148, we have a proof that {S4.4} ⇄ {S4;*R1*} ⇄ {S3;*R1*} and that {V1} ⇄ {S4; *V1*} ⇄ {S3; *V1*}. It also shows that each of the systems {S2;*R1*} and {S2; *V1*} contains system T, *cf.* [12]. Moreover, it can be proved at once that the addition of the Brouwerian axiom, i.e.

C12 $\mathfrak{C}pLMp$

to {C2;*R1*} or to {C2; *V1*} reduces these systems to S5.

5 Connections among the discussed systems. In this section the matrices $\mathfrak{M}$4-$\mathfrak{M}$7 given above and $\mathfrak{M}$8 presented on the following page

$\mathfrak{M}8$

	C	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	N	M	L
*	1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	16	1	1
	2	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	15	1	10
	3	1	2	1	2	5	6	5	6	9	10	9	10	13	14	13	14	14	1	11
	4	1	1	1	1	5	5	5	5	9	9	9	9	13	13	13	13	13	1	12
	5	1	2	3	4	1	2	3	4	9	10	11	12	9	10	11	12	12	5	16
	6	1	1	3	3	1	1	3	3	9	9	11	11	9	9	11	11	11	6	16
	7	1	2	1	2	1	2	1	2	9	10	9	10	9	10	9	10	10	7	16
	8	1	1	1	1	1	1	1	1	9	9	9	9	9	9	9	9	9	8	16
	9	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8	8	1	9
	10	1	1	3	3	5	5	7	7	1	1	3	3	5	5	7	7	7	1	10
	11	1	2	1	2	5	6	5	6	1	2	1	2	5	6	5	6	6	1	11
	12	1	1	1	1	5	5	5	5	1	1	1	1	5	5	5	5	5	1	12
	13	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	4	5	16
	14	1	1	3	3	1	1	3	3	1	1	3	3	1	1	3	3	3	6	16
	15	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	2	7	16
	16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	16	16

will be used. It should be mentioned that a) matrix $\mathfrak{M}6$ is mine, b) $\mathfrak{M}7$ is given by Parry, *cf*. [4], p. 149, example 0.8, and, recently, in virtue of certain reasonings based on the tense logic obtained also by Prior, cf. [5], §§1-3, and that c) matrix $\mathfrak{M}8$ is constructed according to the definition of the 16-valued matrix given by Prior in [5], §6, but which is not explicitly presented in that paper.

a) Matrices $\mathfrak{M}1$ and $\mathfrak{M}3$ verify S5 and V1, but, as we know, falsify $\mathfrak{C}pLp$. On the other hand $\mathfrak{M}4$ and $\mathfrak{M}5$ verify S5, but falsify *V1*. Hence system {S5; *V1*} is a proper extension of S5 and, therefore {S4;*V1*} also is a proper extension of S4. Matrices $\mathfrak{M}1$ and $\mathfrak{M}2$ show that *V1* does not hold S5, and the deductions presented in **3** proves that S4.4 is a subsystem of V1. Since, clearly, S4.4 is a subsystem of S5, we can establish at once that

1) {S5; *V1*} → {S5} → {S4.4}

and that

2) {S5;*V1*} →{V1} → {S4.4}.

b) $\mathfrak{M}4$ and $\mathfrak{M}6$ verify S4.3 and *N1* (i.e. also *M1*), but they falsify *R1* for *p/2*: $\mathfrak{C}2CML2L2 = LC2CM66 = LC2C16 = LC26 = L5 = 5$. Hence S4.3.1 is a proper subsystem of S4.4. On the other hand, in [2], p. 263, Dummett and Lemmon have proved that *M1* (i.e. also *N1*) does not hold in S4.3. Hence S4.3.1 is a proper extension of S4.3. Thus, we have:

3) {S4.4} → {S4.3.1} → {S4.3} → {S4.2} → {S4}.

c) Since $\mathfrak{M}8$ verified S4.2.1 and falsified *D2* for *p/2* and *q/3*: $ALCL23LCL32 = CNLC103LC112 = CNL3L2 = CN1110 = C610 = 9$, system S4.2.1 is a proper subsystem of S4.3.1 and it does not contain S4.3. The proof of Dummett and Lemmon mentioned in the point b) shows that S4.2.1 is a proper extension of S4.2. On the other hand, $\mathfrak{M}4$ and $\mathfrak{M}7$ verify S4.1.1, but they falsify *G1* for *p/2*: $\mathfrak{C}ML2LM2 = LCM6L2 = LC26 = L5 = 5$. Hence S4.2.1 is a proper extension of S4.1.1 and S4.2 is not contained in S4.1.1. Thus, we know that

4) {S4.2.1} → {S4.2} → {S4}

and that

5) {S4.2.1} → {S4.1.1}.

𝔟) In virtue of the result of Dummett and Lemmon mentioned above it is clear that S4.1.1 and S4.1 are the proper extensions of S4. As I noticed previously I have no proof that S4.1.1 is a proper extension of S4.1

Thus, except of the case of the systems S4.1 and S4.1.1 the connections indicated in the diagram given above are justified by the discussions presented in the points 𝔞)-𝔟).

6 Open problems. There are several unsolved problems connected with the results given in this paper. I would like to mention the following ones:

α) to prove that S4.1 and S4.1.1 are distinct systems, which is very probable,
β) to investigate relations existing between S4.4 and the so-called Diodorian system of Prior,
γ) to constrast the normal characteristic matrices of the systems established here using, e.g., the methods given in [2] and [1],
δ) to prove that there exists or does not exist a system being a proper extension of S4.4.and at the same time being a proper subsystem of S5.

I tried, unsuccessfully, to solve only the first problem.

NOTES

1. In this paper I am using the same symbolism and in some cases the same numeration of formulas as in [11]. An acquaintance with the Lewis' modal systems and with the papers [11] and [5] is presupposed. It is assumed that all systems discussed in this paper have Lewis' primitive terms and rules of procedure. The expressions "A is a proper subsystem of B" and "A is a proper extension of B" mean respectively that system A is contained in system B, but does not contain B, and that system A contains system B, but is not contained in B. In all matrices used in this paper *1* is always the single designated value. If in the system under consideration a formula can be obtained from the formulas already given and a subsystem of the investigated theory, I mentioned always the weaker system in the proper proof line.

2. *Cf*., e.g., [11], p. 73 and p. 75, point **3.6**.

3. *Cf*. [11], p. 75, point **3.4**. I am sorry that preparing [11] I overlooked a remark of Prior given in [6], p. 139, that P. T. Geach already established the sufficiency of *D2* to be the proper axiom of S4.3.

4. Since in [6], p. 139, Prior mentioned only that Geach showed that in the field of S4.2 *M1* and *N1* are mutually interducible, the proofs given in 2.3 and 2.4 can differ from the original deductions of Geach unpresented by Prior.

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