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AXIOMATISATIONS OF THE MODAL CALCULUS $\mathbf{Q}$

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R. A. Bull has shown in [1] that the modal calculus $\mathbf{Q}$ of [2] may be axiomatised by taking as primitives a strong and a weak necessity $\mathbf{L}$ and L, and by adding to **PC** the axioms

A1. $CLpp$
A2. $C\mathbf{L}pp$
A3. $CK\mathbf{L}p\mathbf{L}q\mathbf{L}Kpq$

and the rules (beside substitution and detachment)

RQ*La*: $\vdash C\beta\gamma \rightarrow \vdash C\beta L\gamma$, for β fully modalised and with all its variables occurring in γ.
RQ*Lb*: $\vdash C\mathbf{L}\alpha C\beta\gamma \rightarrow \vdash C\mathbf{L}\alpha C\beta L\gamma$, for β fully modalised and with all its variables occurring in α or γ.
RQL: $\vdash C\mathbf{L}\alpha C\beta\gamma \rightarrow \vdash C\mathbf{L}\alpha C\beta\mathbf{L}\gamma$, for β fully modalised and with all variables of β and γ occurring in α.

From the sufficiency of these postulates it is possible to prove the sufficiency of some other postulates for $\mathbf{Q}$ which I suggest in [3]. In these, I adopt a suggestion of J. L. Mackie and use as a primitive a functor S ("always statable"), such that Sp is equivalent, in terms of Bull's primitives, to $\mathbf{L}Cpp$. The other primitive I use in [3] is a possibility-operator M (in Bull's terms NLN), but Bull's weak necessity L will do just as well, and indeed makes possible a slight simplification of the postulates. Bull's $\mathbf{L}p$ is definable in terms of my primitives as $KSpLp$. My postulates, for subjoining to **PC**, then become the one axiom *A1.* $CLpp$, and the three rules:—

RS1: $\vdash CS\alpha Sp$, where p is any variable in α.
RS2: $\vdash CSpCSq \ldots\ldots S\alpha$, where p, q, etc. are all the variables in α.
RSL: $\vdash C\alpha\beta \rightarrow \vdash CSpCSq \ldots\ldots C\alpha L\beta$, where α is fully modalised and p, q, etc. are all the variables in β that are not in α.

In view of Bull's result, the sufficiency of these for $\mathbf{Q}$ may be shown by deducing Bull's postulates from them, including a pair of implications ($CSp\mathbf{L}Cpp$ and $C\mathbf{L}CppSp$) corresponding to the definition of S in Bull's system.

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Bull's *A1* and mine are identical. His *A2* expands, with my definition of **L**, to *CKSpLpp*, which follows from *A1*, *CKpqq* and *Syll*. His **RQ***La* is simply the special case of my **RSL** in which the *p*, *q*, etc. of the consequent vanish as non-existent, and may be used to prove Bull's *A3* as follows:-

1. *CKLpLqKpq* (*A1*, *A1 p/q*, *CCpqCCrsCKprKqs*)
2. *CKLpLqLKpq* (*1*, **RQ***La*)
3. *CKSpSqSKpq* (**RS2**, *CCpCqrCKpqr*)
4. *CKKSpLpKSqLqKSKpqLKpq* (*3*, *2*, *CCKpqrCCKstuCKKpsKqtKru*)
5. *CK***L***p***L***qLKpq* (*4*, *Df.* **L**)

Bull's **RQ***Lb* expands to

$\vdash CKS\alpha L\alpha C\beta\gamma \rightarrow \vdash CKS\alpha L\alpha C\beta L\gamma$,

with provisos, which is equivalent by **PC** to

$\vdash CS\alpha CKL\alpha\beta\gamma \rightarrow \vdash CS\alpha CKL\alpha\beta L\gamma$,

with the same provisos, namely that β (and therefore $KL\alpha\beta$) is fully modalised and all its variables occur in α or γ, i.e. all the variables in β that do not occur in γ occur in α, and so all the variables in $KL\alpha\beta$ that do not occur in γ occur in α. This makes the rule, in its last formulation above, a special case of

$\vdash CS\gamma C\alpha\beta \rightarrow \vdash CS\gamma C\alpha L\beta$,

where γ is a formula containing all the variables in β that are not in α, and α is fully modalised; a rule which follows immediately from **RS1** and **RSL.**

Bull's **RQL** expands to

$\vdash CKS\alpha L\alpha C\beta\gamma \rightarrow \vdash CKS\alpha L\alpha C\beta KS\gamma L\gamma$,

with provisos, and this is equivalent by **PC** to

$\vdash CS\alpha CKL\alpha\beta\gamma \rightarrow \vdash CS\alpha CKL\alpha\beta KS\gamma L\gamma$,

with the same provisos, namely that β (and so $KL\alpha\beta$) is fully modalised and all variables of β and γ occur in α. From the antecedent here we may infer the weaker consequent $\vdash CS\alpha CKL\alpha\beta L\gamma$ as with **RQ***Lb*, and we may strengthen this to the given consequent by **PC** and $\vdash CS\alpha S\gamma$, which follows from **RS1** and **RS2** when all variables of γ are in α.

Finally, the implications corresponding to the definition of *Sp* as **L***Cpp* are provable as follows:-

1. *CCLpLpCpp* (**PC**)
2. *CCLpLpLCpp* (*1*, **RQ***La*)
3. *LCpp* (*2*, *and CLpLp from* **PC**)
4. *CSpLCpp* (*3*, *CpCqp*)
5. *CSpSCpp* (**RS2**)
6. *CSpKSCppLCpp* (*5*, *4*, *CCpqCCprCpKqr*)
7. *CSCppSp* (**RS1**)
8. *CKSCppLCppSp* (*7*, *CKpqp*)
9. *CSp***L***Cpp* (*6*, *Df.* **L**)
10. *C***L***CppSp* (*8*, *Df.* **L**).

I should add that I know no way of proving the equivalence of *Lp* to *KSpLp*, i.e. *KLCppLp*, from Lemmon's conjectured postulates for **Q** cited in [3], and was therefore guilty of an oversight in there describing as "obvious" the equivalence of Lemmon's postulates to my own in *M* and *S*.

REFERENCES

[1] R. A. Bull, An axiomatization of Prior's modal calculus **Q**, *Notre Dame Journal of Formal Logic*, v. V (1964), pp. 211-214.

[2] A. N. Prior, *Time and Modality* (Oxford 1957).

[3] A. N. Prior, Notes on a group of new modal systems. *Logique et Analyse*, v. 2, No. 6-7 (1959), pp. 122-7.

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