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AN AXIOMATIZATION OF PRIOR'S MODAL CALCULUS Q

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Prior defines a model for a modal calculus **Q** (cf [1], pp. 43f):

The truth values are infinite sequences of *1*'s, *2*'s, and *3*'s, with the proviso that the first term of each sequence is not *2*. The designated values are those with no *3*'s.

The values of propositional operators are found by applying the tables

Ka_ib_i		1	2	3	b_i
	1	1	2	3	
	2	2	2	2	
a_i	3	3	2	3	

Na_i		
	1	3
	2	2
a_i	3	1

to the terms of the sequences $<a_1, a_2, a_3, \ldots>$ and $<b_1, b_2, b_3, \ldots>$. The other propositional operators can be defined from these in the usual way.

For formal convenience I shall use L for what is NMN in Prior's system, and $\mathbf{L}$ for his L. These operators are given by

$\mathbf{L}\alpha$ is
- $<1,1,1,\ldots>$ when α is $<1,1,1,\ldots>$;
- *2* where α is *2* and *3* elsewhere, when α consists of *1*'s and *2*'s;
- *2* where α is *2* and *3* elsewhere, when α has a *3*.

$L\alpha$ is
- $<1,1,1,\ldots>$ when α is $<1,1,1,\ldots>$;
- *2* where α is *2* and *1* elsewhere, when α consists of *1*'s and *2*'s;
- *2* where α is *2* and *3* elsewhere, when α has a *3*.

This paper is devoted to showing that **Q** can be axiomatized by adding to **PC** the following axioms and rules:

1 $CLpp$
2 $C\mathbf{L}pp$
3 $CK\mathbf{L}p\mathbf{L}q\mathbf{L}Kpq$
RQ$L\alpha$ $C\beta\gamma \implies C\beta L\gamma$,

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where (1) β is fully modalized,
and (2) the variables of β each occur in γ.

$$\mathbf{RQ}Lb\ C\mathbf{L}\alpha C\beta\gamma \Longrightarrow C\mathbf{L}\alpha C\beta L\gamma,$$

where (1) β is fully modalized,
and (2) the variables of β each occur in α or γ.

$$\mathbf{RQL}\ C\mathbf{L}\alpha C\beta\gamma \Longrightarrow C\mathbf{L}\alpha C\beta\mathbf{L}\gamma,$$

where (1) β is fully modalized,
and (2) the variables of β and γ each occur in α.

I do this by giving a reduction of words to a normal form, in Lemma 2, and then showing that I can either construct a derivation for such a normal form in my axiom system (Lemma 3), or construct allocations rejecting it from **Q** (Lemma 4), Lemma 1 gives some rules used in the other sections. The proofs of these Lemmas involve some lengthy but straightforward derivations, which I shall omit.

I use α, β, γ, etc, and these letters with subscripts metatheoretically for words. I sometimes use $(C\alpha_i)\beta$ as an abbreviation for $C\alpha_1 C\alpha_2 \ldots C\alpha_n\beta$ when there is no danger of confusion arising from not stating the subscripts more explicitly. I use $\sim$ as an equivalence relation between sets of words which can be derived from each other. A word α with a part β can be regarded as the value of a function with β as its argument, and with this in mind I sometimes write $\alpha(\beta)$ for such an α. I then use $\alpha(\gamma)$ for the word obtained by replacing that occurrence of β by γ; in using this device the particular part being replaced must, of course, be stated in the context.

Lemma 1. The following rules can be derived in our axiom system:

I $\alpha \sim CL\beta\alpha,\ CNL\beta\alpha$

II $\alpha \sim C\mathbf{L}\beta\alpha,\ CN\mathbf{L}\beta\alpha$

III $CLC\beta\gamma CLC\gamma\beta\alpha(\beta) \sim CLC\beta\gamma CLC\gamma\beta\alpha(\gamma)$, *where β and γ have the same variables.*

IV $CL\beta\alpha \sim CLCL\beta C\beta\beta CLCC\beta\beta L\beta\alpha$

V $CNL\beta\alpha \sim CLCL\beta NC\beta\beta CLCNC\beta\beta L\beta\alpha$

VI $C\mathbf{L}\beta\alpha \sim CLC\mathbf{L}\beta C\beta\beta CLCC\beta\beta\mathbf{L}\beta\alpha$

VII $CN\mathbf{L}\beta\alpha \sim CLC\mathbf{L}\beta NC\beta\beta CLCNC\beta\beta\mathbf{L}\beta\alpha$

Lemma 2. In our axiom system each word is equivalent to words of the form

$(C\mathbf{L}\alpha_i)(CL\beta_j)(CNL\gamma_k)(CN\mathbf{L}\delta_\ell)\epsilon$, where the α_i's, the β_j's, the γ_k's, the δ_ℓ's, and ϵ have no modal operators.

Proof. Given a word α with m L's and $\mathbf{L}$'s, define words $\alpha_1, \alpha_2, \ldots, \alpha_{2^{m+1}-1}$ as follows:

(1) $\alpha_1 = \alpha$.

(2) β_n is the first part of α_n to the right of the added antecedents which contains no modal operators but lies immediately under one. I shall use $\alpha_n(L\beta_n)$ (or $\alpha_n(\mathbf{L}\beta_n)$) for α_n with this occurrence of $L\beta_n$ (or $\mathbf{L}\beta_n$) as argument.

(3) $\alpha_{2n} = CL\beta_n \alpha_n(C\beta_n\beta_n)$ if β_n lies under an L;
$\alpha_{2n} = C\mathbf{L}\beta_n \alpha_n(C\beta_n\beta_n)$ if β_n lies under an $\mathbf{L}$.
(4) $\alpha_{2n+1} = CNL\beta_n \alpha_n(NC\beta_n\beta_n)$ if β_n lies under an L;
$\alpha_{2n+1} = CN\mathbf{L}\beta_n \alpha_n(NC\beta_n\beta_n)$ if β_n lies under an $\mathbf{L}$.

(Note that this does give a unique definition of α_n.) With each step of the defining process a modal operator is removed from the part to the right of the added antecedents, which originally contained m modal operators, so each branch of the defining process terminates after m steps. The terminal words, $\alpha_{2m}, \alpha_{2m+1}, \ldots, \alpha_{2^{m+1}-1}$, will be commutants of the required normal form.

Using the rules of Lemma 1 it can be shown that $\alpha_n \sim \alpha_{2n}, \alpha_{2n+1}$.

(When β_n lies under an L use I, III, IV, and V; when β_n lies under an $\mathbf{L}$ use II, III, VI, and VII.) Thus we have that

$$\alpha \sim \alpha_{2m}, \alpha_{2m+1}, \ldots, \alpha_{2^{m+1}-1},$$

which gives the required result.

In what follows I shall use $\beta_{j(\alpha)}$ and $\delta_{\ell(\alpha)}$ for β_j's and δ_ℓ's with all their variables in the α_i's; and $\beta_{j(k)}$'s for β_j's with all their variables in the α_i's and γ_k.

Lemma 3. The normal form

$(C\mathbf{L}\alpha_i)(CL\beta_j)(CNL\gamma_k)(CN\mathbf{L}\delta_\ell)\epsilon$

can be derived from any of the (propositional) words

$(C\alpha_i)(C\beta_j)\epsilon$
$(C\alpha_i)(C\beta_{j(k)})\gamma_k$
$(C\alpha_i)(C\beta_{j(\alpha)})\delta_{\ell(\alpha)}$

in our axiom system.

Proof. Derive 4 $C\mathbf{L}KpqK\mathbf{L}p\mathbf{L}q$; the derivations are then straightforward applications of 1, 2, 3, 4, **RQ**Lb, and **RQ$\mathbf{L}$**.

Lemma 4. The normal form

$$(C\mathbf{L}\alpha_i)(CL\beta_j)(CNL\gamma_k)(CN\mathbf{L}\delta_\ell)\epsilon$$

can be rejected from **Q** *if all the (propositional) words*

$(C\alpha_i)(C\beta_j)\epsilon$
$(C\alpha_i)(C\beta_{j(k)})\gamma_k$
$(C\alpha_i)(C\beta_{j(\alpha)})\delta_{\ell(\alpha)}$

are rejected from **PC**.

Proof. We know that there must be allocations of *1*'s and *3*'s which reject each of these propositional words in turn. Let us suppose that k ranges from *1* to r, and that ℓ ranges from *1* to s. Assign values to the terms of the sequences for the variables as follows:

(1) To the first terms: give the variables in the α_i's, the β_j's, and ϵ

values which reject $(C\alpha_i)(C\beta_j)\epsilon$; and give the other variables value *1*. Thus the α_i's and the β_j's will have value *1*; and ϵ will have value *3*.

(2) To the $(k+1)$th terms: give the variables of the α_i's and γ_k values which reject $(C\alpha_i)(C\beta_{j(k)})\gamma_k$; and give the other variables value *2*. Thus the α_i's and the $\beta_{j(k)}$'s will have value *1*; the other β_j's will have value *2*; and γ_k will have value *3*.

(3) To the $(r + \ell(\alpha)+1)$th terms: give the variables of the α_i's values which reject $(C\alpha_i)(C\beta_{j(\alpha)})\delta_{\ell(\alpha)}$; and give the other variables value *2*. Thus the α_i's and the $\beta_{j(\alpha)}$'s will have value *1*; the other β_j's will have value *2*; and $\delta_{\ell(\alpha)}$ will have value *3*.

(4) To the other $(r+\ell+1)$th terms: give the variables of the α_i's and β_j's values which would, with appropriate values for ϵ, reject $(C\alpha_i)(C\beta_j)\epsilon$; and give the other variables value *2*. Thus the α_i's and the β_j's will have value *2*; and δ_ℓ will have value *2*.

(5) This defines the values for the first $(r+s+1)$ terms of the sequences; repeat this block of allocations for the other terms.

The sequences will now have the following properties: the α_i's will have all *1*'s; the β_j's will have *1*'s and *2*'s; the γ_k's will have some *3*'s; the $\delta_{\ell(\alpha)}$'s will have some *3*'s; and the other δ_ℓ's will have some *2*'s. Thus each antecedent $\mathbf{L}\alpha_i$, $L\beta_j$, $NL\gamma_k$, $N\mathbf{L}\delta_\ell$ will have a sequence of *1*'s and *2*'s; in particular each antecedent will have *1* for its first term. Further, ϵ will have a *3* for its first term, so the normal form will have *3* for its first term and be rejected.

Theorem. The system **Q** *is axiomatized by adding to* **PC** *the axioms 1, 2, 3 and the rules* **RQ***La*, **RQ***Lb*, **RQL**.

Proof. We see from Lemmas 3 and 4 that a normal form can either be derived from our basis or rejected from **Q**. If all the normal forms for a word can be derived from our basis then that word can be derived from our basis by way of them, by Lemma 2. If one of the normal forms for a word is rejected from **Q** then the word itself must be rejected from **Q**, since the property of being verified in **Q** is preserved by derivations in our axiom system. (For the verification of the rule of detachment see [1], p. 46.)

Corollary. The system **Q** *is decidable.*

Proof. Examination of the lemmas will show that a word with m modal operators is rejected from **Q** if and only if it is rejected with sequences where the terms are repetitions of the first $(m+1)$ terms. (In Lemma 4 the blocks are of $(r+s+1)$ terms; these can be expanded to blocks of $(m+1)$ terms by repetition within the block, if necessary.) Given m, the number of such sequences is finite, so the word is decided by a finite model.

REFERENCES

[1] A. N. Prior, *Time and Modality*, Oxford, 1957.

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