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A NATURAL DEDUCTION SYSTEM FOR MODAL LOGIC

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This paper relates a particular system of propositional calculus (hereafter referred to as **F** and described in §1 below), suggested by Dr. Milton Fisk during the seminar in symbolic logic at the University of Notre Dame, to the Lewis modal logic **S4** [4]. **F** has no axioms--its description begins by laying down certain rules as basic and it proceeds by inferring rules from its basic rules. Thus **F** may be considered as a systematic for rules which govern formulas, where Lewis's system is considered as a systematic for formulas. Indeed, if the basic rules of **F** are interpreted as claiming that certain forms of arguments are valid, for instance, if *F1* is taken to mean that any argument of the form "α, β; therefore $(\alpha \wedge \beta)$" is valid, then the basic rules can accurately be called principles of (propositional) logic. As so interpreted, *F1*-*F7* provide a basis for systematizing logical principles. **F** then becomes a systematic for evaluating individual arguments: an argument is valid if it is governed by a principle which can be derived in **F**.

In this paper a system **A** is said to *imply inferentially* a system **B** if and only if the axioms and rules of **B** stated in the primitive notation of **B** can be *inferred* in **A**. Thus §2 shows that **F** inferentially implies **S4.** But **S4** does not inferentially imply **F** (and hence **F** and **S4** are not inferentially equivalent), since the rules of **F** cannot be inferred in **S4**--they hold for wffs while those of **S4** hold only for theses. But since §3 contains a formal proof that every thesis of **F** is a thesis of **S4** and since every thesis of **S4** is a thesis of **F** (as a corollary of §2), it is shown that the two systems are formally equivalent in the sense that they have the same set of theses.

The description of system **F** that appears here differs from that description of the systematic for arguments which Dr. Fisk originally suggested, in that the metarule of replacement (**FII**) which he had taken as basic is derived from the basic rules.

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§ *1 System* **F** In this system lower case Latin letters are used as propositional variables: p, q, r, etc. There are three primitive symbols for propositional functors: '$\sim$' for 'not', '$\wedge$' for 'and', and '$\rightarrow$' for 'only if'; and parentheses are used for punctuation. Any lower case Latin letter standing alone is a well formed formula, and (i) if α is a wff then $\sim\alpha$ is well formed, (ii) if α and β are wffs then $(\alpha \wedge \beta)$ and $(\alpha \rightarrow \beta)$ are well formed, and (iii) no other formula is a wff.[1] Any other propositional functors used in the system are introduced by definitions in accordance with the method of constructing definitions for propositional calculus. Thus Definition 1 below introduces '$\leftrightarrow$'.

Definition 1 $(\alpha \leftrightarrow \beta) =_{\mathrm{df}} ((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$

Furthermore, since '$\leftrightarrow$' is not taken as a primitive term the system requires a rule of definition, namely, that a definiens may be abbreviated by its definiendum and a definiendum by its definiens. (It will be shown later that the definiens and definiendum of any definition are replaceable by each other in any wff.)

F has no axioms but is given by seven rules of inference (called basic), a definition of thesis, and a notion of proof from hypotheses. In expressing rules of inference the turnstile, '$\vdash$', is used to separate the premiss formulas from the conclusion formula and commas are used to separate individual premiss formulas. Thus, *F1* below is a rule to the effect that the conclusion $(\alpha \wedge \beta)$ follows from the premisses α and β, for any α, β.

(*BR*) The basic rules of **F** are:

F1.	$\alpha, \beta \vdash (\alpha \wedge \beta)$	Adjunction
F2.	$\alpha, (\alpha \rightarrow \beta) \vdash \beta$	Strict Detachment
F3.	$(\sim\beta \rightarrow \sim\alpha) \vdash (\alpha \rightarrow \beta)$	Transposition
F4.	$\alpha \vdash \sim\sim\alpha$	Double Negation
F5.	$\alpha, \sim(\alpha \wedge \beta) \vdash \sim\beta$	Material Detachment
F6.	$(\alpha \rightarrow \beta) \vdash (\alpha \wedge \gamma) \rightarrow (\beta \wedge \gamma)$	Factorization
F7.	$(\alpha \rightarrow \beta) \vdash (\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)$	Hypothetical Syllogism

(*TH*) A thesis is defined recursively as follows:

(i) for any natural number n which is finite, if $\alpha_1, \ldots, \alpha_{n-1} \vdash \alpha_n$ in **F**, then $(\alpha_1 \wedge, \ldots, \wedge\, \alpha_{n-1}) \rightarrow \alpha_n$ is a thesis, and
(ii) any α derived from theses alone by the rules of **F** is a thesis.[2]

(*PR*) There is a proof of α_n from a set of hypotheses, $\alpha_1, \ldots, \alpha_{n-1}$ if and only if there is a finite sequence of wffs, $\beta_1, \ldots, \beta_m$ such that $m \geq n$, α_n is β_m, and for $1 \leq i \leq m$ either

(i) β_i is an hypothesis or
(ii) β_i is a thesis or
(iii) β_i follows in the sequence according to the rules of **F**.

Since (*TH*) and (*PR*) together yield a complete definition of proof from hypotheses, with this definition, *F1* may be read as, "for any α, β there is a proof of $(\alpha \wedge \beta)$ from the set of hypotheses α, β." Similarly, giving a proof of α_n from hypotheses $\alpha_1, \ldots, \alpha_{n-1}$ is sufficient for establishing another rule of **F**, namely, $\alpha_1, \ldots, \alpha_{n-1} \vdash \alpha_n$. Thus in giving the proof immediately following the citation of *F8* below, the derived rule, "$(\alpha \rightarrow \gamma)$ follows from $(\alpha \rightarrow \beta)$, $(\beta \rightarrow \gamma)$ for any α,β,γ", is established. Finally, the turnstile shall perform yet another function in this system: "α is a thesis" shall be abbreviated as "$\vdash\alpha$". This usage of the symbol corresponds to its previous usage since a thesis can be understood as a wff which follows from the empty set of hypotheses.

§ *2* **F** *inferentially implies* **S4** In order to show that **F** contains **S4** a singulary modal functor '$\Diamond$' for 'it is possible that' is defined as follows:

Definition 2. $\Diamond\alpha =_{\text{df}} \sim(\alpha \rightarrow \sim \alpha)$

With this definition and the derivations of *F8*-*F31* below the equivalence necessary for deducing the rules and axioms of **S4** in **F** is obtained in the form of two derived rules:

F32. $(\alpha \rightarrow \beta) \vdash \sim \Diamond(\alpha \wedge \sim\beta)$
F33. $\sim \Diamond (\alpha \wedge \sim\beta) \vdash (\alpha \rightarrow \beta)$

To this end, the proofs of the required rules follow.

F8. $(\alpha \rightarrow \beta), (\beta \rightarrow \gamma) \vdash (\alpha \rightarrow \gamma)$
Proof by *F7*, hypotheses, *F2*.

F9. $\sim\sim\alpha \vdash \alpha$

1.	$\sim\sim\alpha$	[Hypothesis]
2.	$(\sim \sim\alpha \rightarrow \alpha)$	[*F4*, (*TH*), *F3*]
3.	α	[1, 2, *F2*]

F10. $(\alpha \rightarrow \beta) \vdash (\sim\beta \rightarrow \sim\alpha)$

1.	$(\alpha \rightarrow \beta)$	[Hypothesis]
2.	$(\sim\sim\alpha \rightarrow \beta)$	[1, *F9*, (*TH*), *F8*]
3.	$(\sim\beta \rightarrow \sim\alpha)$	[2, *F4*, (*TH*), *F8*, *F3*]

F11. $\alpha \vdash \alpha$
Proof by *F4*, (*TH*), hypothesis, *F2*, *F9*, (*TH*), *F2*.

F12. $\alpha, \beta \vdash \alpha$
Proof by *F11*, hypotheses.

F13. $(\alpha \wedge \beta) \vdash \alpha$
Proof by *F12*,(*TH*), hypothesis, *F2*.

F14. $(\alpha \wedge \beta) \vdash \beta$
Proof similar to *F13*.

F15. $(\alpha \wedge \beta) \vdash (\beta \wedge \alpha)$
Proof by *F14*, *F13*, hypothesis, *F1*.

F16. $(\alpha \leftrightarrow \beta) \vdash ((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$
Proof by Definition 1, hypothesis.

F17. $((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)) \vdash (\alpha \leftrightarrow \beta)$
Proof by Definition 1, hypothesis.

F18. $(\alpha \leftrightarrow \beta) \vdash (\sim\alpha \leftrightarrow \sim\beta)$

1. $(\alpha \leftrightarrow \beta)$	[Hypothesis]
2. $(\sim\beta \rightarrow \sim\alpha)$	[1, *F16*, *F13*, *F10*]
3. $(\sim\alpha \rightarrow \sim\beta)$	[1, *F16*, *F14*, *F10*]
4. $(\sim\alpha \leftrightarrow \sim\beta)$	[2, 3, *F1*, *F17*]

F19. $(\alpha \leftrightarrow \beta) \vdash ((\alpha \wedge \gamma) \leftrightarrow (\beta \wedge \gamma))$
Proof similar to *F18*, using *F6*.

F20. $(\alpha \leftrightarrow \beta) \vdash ((\gamma \wedge \alpha) \leftrightarrow (\gamma \wedge \beta))$

1. $(\alpha \leftrightarrow \beta)$	[Hypothesis]
2. $((\alpha \wedge \gamma) \rightarrow (\beta \wedge \gamma))$	[1, *F16*, *F13*, *F6*]
3. $((\beta \wedge \gamma) \rightarrow (\alpha \wedge \gamma))$	[1, *F16*, *F14*, *F6*]
4. $((\gamma \wedge \alpha) \rightarrow (\gamma \wedge \beta))$	[2, *F15*, (*TH*), *F8*]
5. $((\gamma \wedge \beta) \rightarrow (\gamma \wedge \alpha))$	[3, *F15*, (*TH*), *F8*]
6. $((\gamma \wedge \alpha) \leftrightarrow (\gamma \wedge \beta))$	[4, 5, *F1*, *F17*]

F21. $(\alpha \leftrightarrow \beta) \vdash ((\alpha \rightarrow \gamma) \leftrightarrow (\beta \rightarrow \gamma))$
Proof similar to *F18*, using *F7*

F22. $(\alpha \leftrightarrow \beta) \vdash ((\gamma \rightarrow \alpha) \leftrightarrow (\gamma \rightarrow \beta))$

1. $(\alpha \leftrightarrow \beta)$	[Hypothesis]
2. $(\sim\beta \rightarrow \sim\alpha)$	[1, *F16*, *F13*, *F10*]
3. $(\sim\alpha \rightarrow \sim\beta)$	[1, *F16*, *F14*, *F10*]
4. $((\sim\alpha \rightarrow \sim\gamma) \rightarrow (\sim\beta \rightarrow \sim\gamma))$	[2, *F7*]
5. $((\sim\beta \rightarrow \sim\gamma) \rightarrow (\sim\alpha \rightarrow \sim\gamma))$	[3, *F7*]
6. $((\gamma \rightarrow \alpha) \rightarrow (\gamma \rightarrow \beta))$	[4, *F10*, (*TH*), *F3*, (*TH*), *F8*]
7. $((\gamma \rightarrow \beta) \rightarrow (\gamma \rightarrow \alpha))$	[5, *F10*, (*TH*), *F3*, (*TH*), *F8*]
8. $((\gamma \rightarrow \alpha) \leftrightarrow (\gamma \rightarrow \beta))$	[6, 7, *F1*, *F17*]

F23. $(\alpha \leftrightarrow \beta) \vdash (\gamma \leftrightarrow \delta)$ *where* δ *results from* γ *by replacing* α *by* β (β *by* α) *in one or more places.*

Proof by induction on the length of δ, using *F11*, *F18*-*F22*, (*PR*).

With *F23* given it is possible to prove a number of metarules for the system **F**.

FI. *If* $\vdash(\alpha \rightarrow \beta)$ *then* $\alpha \vdash \beta$.

1. $(\alpha \rightarrow \beta)$	[Thesis by the assumption]
2. α	[Hypothesis]
3. β	[1, 2, *F2*]

FII. *If* $\alpha \vdash \beta$ *and* $\beta \vdash \alpha$ *then* $\gamma \vdash \delta$ *and* $\delta \vdash \gamma$ *where* δ *results from* γ *by replacing* α *by* β (β *by* α) *in one or more places.*

1. $\alpha \vdash \beta$	[The assumption]
2. $\beta \vdash \alpha$	[The assumption]
3. $\vdash(\alpha \rightarrow \beta)$	[1, (*TH*)]
4. $\vdash(\beta \rightarrow \alpha)$	[2, (*TH*)]
5. $\vdash(\alpha \leftrightarrow \beta)$	[3, 4, *F1*, *F17*, (*TH*)]
6. $\vdash(\gamma \leftrightarrow \delta)$	[5, *F23*, (*TH*)]
7. $\gamma \vdash \delta$	[6, *F16*, *F13*, (*TH*), **FI**]
8. $\delta \vdash \gamma$	[6, *F16*, *F14*, (*TH*), **FI**]

Given this metarule of replacement the definiens and definiendum of any definition are replaceable by each other in any wff: for the rule of definition which allows either definiens or definiendum to be abbreviated yields the two rules which are sufficient to carry out either replacement in accordance with **FII**.

F24. $(\alpha \wedge \alpha) \vdash \alpha$
Proof by *F13*, hypothesis.

F25. $\alpha \vdash (\alpha \wedge \alpha)$
Proof by *F11*, *F1*, hypothesis.

F26. $\alpha \wedge (\beta \wedge \gamma) \vdash (\alpha \wedge \beta) \wedge \gamma$
Proof by *F13*, *F14*, hypothesis, *F1*.

F27. $(\alpha \wedge \beta) \wedge \gamma \vdash \alpha \wedge (\beta \wedge \gamma)$
Proof similar to *F26*.

F28. $(\alpha \wedge \beta) \rightarrow \gamma \vdash (\sim\gamma \wedge \beta) \rightarrow \sim\alpha$

1. $(\alpha \wedge \beta) \rightarrow \gamma$	[Hypothesis]
2. $(\sim\gamma \wedge \beta) \rightarrow (\sim(\alpha \wedge \beta) \wedge \beta)$	[1, *F10*, *F6*]
3. $(\sim\gamma \wedge \beta) \rightarrow (\beta \wedge \sim(\beta \wedge \alpha))$	[2, *F15*, **FII**]
4. $(\sim\gamma \wedge \beta) \rightarrow \sim\alpha$	[3, *F5*, (*TH*), *F8*]

F29. $(\beta \wedge \sim\beta) \vdash \sim(\alpha \wedge \sim\beta)$

1. $(\beta \wedge \sim\beta)$	[Hypothesis]
2. $\sim\sim\beta$	[1, *F13*, *F4*]
3. $\sim(\alpha \wedge \sim\beta)$	[2, *F14*, (*TH*), *F10*, *F2*]

F30. $\Diamond\alpha \vdash \sim(\alpha \rightarrow \sim\alpha)$
Proof by Definition 2, hypothesis.

F31. $\sim(\alpha \rightarrow \sim\alpha) \vdash \Diamond\alpha$
Proof by Definition 2, hypothesis.

F32. $(\alpha \rightarrow \beta) \vdash \sim\Diamond(\alpha \wedge \sim\beta)$

1. $(\alpha \rightarrow \beta)$	[Hypothesis]
2. $(\alpha \wedge \sim\beta) \rightarrow (\beta \wedge \sim\beta)$	[1, *F6*]
3. $(\alpha \wedge \sim\beta) \rightarrow \sim(\alpha \wedge \sim\beta)$	[2, *F29*, (*TH*), *F8*]
4. $\sim\Diamond(\alpha \wedge \sim\beta)$	[3, *F4*, *F30*, *F31*, **FII**]

F33. $\sim\Diamond(\alpha \wedge \sim\beta) \vdash (\alpha \rightarrow \beta)$

1. $\sim\Diamond(\alpha \wedge \sim\beta)$	[Hypothesis]
2. $(\alpha \wedge \sim\beta) \rightarrow \sim(\alpha \wedge \sim\beta)$	[1, *F30*, *F31*, **FII**, *F9*]
3. $(\sim\sim(\alpha \wedge \sim\beta) \wedge \sim\beta) \rightarrow \sim\alpha$	[2, *F28*]
4. $(\alpha \wedge \sim\beta) \rightarrow \sim\alpha$	[3, *F4*, *F9*, *F24*-*F27*, **FII**]
5. $(\sim\sim\alpha \wedge \alpha) \rightarrow \sim\sim\beta$	[4, *F15*, **FII**, *F28*]
6. $(\alpha \rightarrow \beta)$	[5, *F4*, *F9*, *F24*, *F25*, **FII**]

It is now possible to derive the Lewis system **S4** in its primitive notation of conjunction, negation and possibility.[3] As Parry has shown [12], **S4** is inferentially equivalent to **S3** and the Aristotelian Rule, namely,

If $\vdash\alpha$ *then* $\vdash \sim\Diamond\sim\alpha$.

It suffices then, to prove analogues of Lewis's eight axioms of **S3** (cf. p. 500 of [4]), *A1-A8* below, three rules of that system (cf. pp. 125-126 of [4]), and the Aristotelian Rule, **RIV** below. Lewis's second rule of substitution need not be proved since the use of schemata in **F** is equivalent to a rule of substitution of wffs for propositional variables in **F**. Lewis's rules of adjunction and inference are special cases of adjunction and strict detachment for **F** and are given as **RII** and **RIII** respectively. Lewis's rule of substitution of equivalent expressions follows from *F23* as **RI**.

A1. $\sim\Diamond((\alpha \wedge \beta) \wedge \sim(\beta \wedge \alpha))$

1. $(\alpha \wedge \beta) \rightarrow (\beta \wedge \alpha)$ [*F15*, (*TH*)]
2. $\sim\Diamond((\alpha \wedge \beta) \wedge \sim(\beta \wedge \alpha))$ [1, *F32*, (*TH*)]

A2. $\sim\Diamond((\alpha \wedge \beta) \wedge \sim\beta)$
Proof similar to *A1*, using *F14*.

A3. $\sim\Diamond(\alpha \wedge \sim(\alpha \wedge \alpha))$
Proof similar to *A1*, using *F25*.

A4. $\sim\Diamond((\alpha \wedge (\beta \wedge \gamma)) \wedge \sim((\alpha \wedge \beta) \wedge \gamma))$
Proof similar to *A1*, using *F26*.

A5. $\sim\Diamond(\alpha \wedge \sim\sim\sim\alpha)$
Proof similar to *A1*, using *F4*.

A6. $\sim\Diamond((\sim\Diamond(\alpha \wedge \sim\beta) \wedge \sim\Diamond(\beta \wedge \sim\gamma)) \wedge \sim\sim\Diamond(\alpha \wedge \sim\gamma))$
Proof similar to *A1*, using *F8*.

F34. $\alpha \vdash \Diamond\alpha$

1. α [Hypothesis]
2. $((\alpha \rightarrow \sim\alpha) \wedge \alpha) \rightarrow \sim\alpha$ [*F2*, *F15*, **FII**, (*TH*)]
3. $(\sim\sim\alpha \wedge \alpha) \rightarrow \sim(\alpha \rightarrow \sim\alpha)$ [2, *F28*, (*TH*)]
4. $\alpha \rightarrow \sim(\alpha \rightarrow \sim\alpha)$ [3, *F4*, *F9*, *F24*, *F25*, **FII**, (*TH*)]
5. $\Diamond\alpha$ [1, 4, *F2*, *F31*]

A7. $\sim\Diamond(\sim\Diamond\alpha \wedge \sim\sim\alpha)$
Proof similar to *A1*, using *F34*, *F10*.

F35. $(\alpha \rightarrow \sim\beta) \vdash (\beta \rightarrow \sim\alpha)$
Proof by *F10*, hypothesis, *F4*, *F9*, **FII**.

F36. $(\alpha \rightarrow \beta) \vdash (\Diamond\alpha \rightarrow \Diamond\beta)$

1. $(\alpha \rightarrow \beta)$ [Hypothesis]
2. $\Diamond\alpha \rightarrow \sim(\beta \rightarrow \sim\alpha)$ [1, *F7*, *F10*, *F30*, *F31*, **FII**]
3. $\sim(\alpha \rightarrow \sim\beta) \rightarrow \Diamond\beta$ [1, *F7*, *F10*, *F30*, *F31*, **FII**]
4. $\Diamond\alpha \rightarrow \Diamond\beta$ [2, 3, *F35*, **FII**, *F8*]

A8. $\sim\Diamond(\sim\Diamond(\alpha \wedge \sim\beta) \wedge \sim\sim\Diamond(\sim\Diamond\beta \wedge \sim\sim\Diamond\alpha))$
Proof similar to *A1*, using, *F3*, *F10*, **FII**, *F36*.

RI. *If* $\vdash \sim\Diamond(\alpha \wedge \sim\beta) \wedge \sim\Diamond(\beta \wedge \sim\alpha)$ *and* $\vdash\gamma$ *then* $\vdash\delta$ *where* δ *results from* γ *by replacing* α *by* β (β *by* α) *in one or more places.*

1. $\vdash \sim\Diamond(\alpha \wedge \sim\beta) \wedge \sim\Diamond(\beta \wedge \sim\alpha)$ [Hypothesis]
2. $\vdash \gamma$ [Hypothesis]

3. $\vdash(\alpha \leftrightarrow \beta)$ [1, *F32*, *F33*, *F16*, *F17*, **FII**, (*TH*)]
4. $\vdash(\gamma \leftrightarrow \delta)$ [3, *F23*, (*TH*)]
5. $\vdash(\gamma \rightarrow \delta)$ [4, *F16*, *F17*, **FII**, *F14*, (*TH*)]
6. $\vdash \delta$ [2, 5, *F2*, (*TH*)]

RII. *If* $\vdash\alpha$ *and* $\vdash\beta$ *then* $\vdash(\alpha \wedge \beta)$
Proof by *F1*, assumptions, (*TH*).

RIII. *If* $\vdash\alpha$ *and* $\vdash\sim\Diamond(\alpha \wedge \sim\beta)$ *then* $\vdash\beta$
Proof by *F33*, assumption, *F2*, (*TH*).

FIII. *If* $\vdash\alpha$ *then* $\vdash(\sim\alpha \rightarrow \alpha)$
If $\vdash\alpha$ then there is a proof from hypotheses,
1. $\sim\alpha$ [Hypothesis]
2. α [Thesis by the assumption]

proving $\sim\alpha \vdash\alpha$.
Hence, $\vdash(\sim\alpha \rightarrow \alpha)$ by (*TH*).

RIV. *If* $\vdash\alpha$ *then* $\vdash\sim\Diamond\sim\alpha$
1. $\vdash\alpha$ [The assumption]
2. $\vdash(\sim\alpha \rightarrow \alpha)$ [1, **FIII**]
3. $\vdash\sim\sim(\sim\alpha \rightarrow \sim\sim\alpha)$ [2, *F4*, *F9*, **FII**, (*TH*)]
4. $\vdash\sim\Diamond\sim\alpha$ [3, *F30*, *F31*, **FII**, (*TH*)]

Having thus established *A1*-*A8*, and **RI**-**RIV** the proof that **F** inferentially implies **S4** is complete.

§*3* **F** *is equivalent to* **S4** It shall be shown that **F** is equivalent to **S4** in the sense that α is a thesis of **F** if and only if α is a thesis of **S4**. And since **F** inferentially implies **S4** it is clear that every thesis of **S4** is a thesis of **F**. Thus it only remains to show that any thesis of **F** is a thesis of **S4**.

To this end, the theses of **F** are ordered in the following way.

(1) Let α be a thesis of **F** by part (i) of the definition of thesis. That is, α is $(\alpha_1 \wedge, \ldots, \wedge \alpha_{n-1}) \rightarrow \alpha_n$ where n is a finite number and there is a finite sequence of wffs, $\beta_1, \ldots, \beta_m$, such that the sequence is a proof of α_n from the hypothesis $\alpha_1, \ldots, \alpha_{n-1}$. If, for $1 \leq i \leq m$, there is no β_i such that β_i is a thesis of **F** then α has order 1. If there are theses of **F** in the sequence then the order of α is one greater than the thesis of highest order among those occurring in the sequence.
(2) Let α be a thesis of **F** by part (ii) of the definition of thesis. That is, there is a finite sequence, $\beta_1, \ldots, \beta_m$, such that, for $1 \leq i \leq m$, each β_i is a thesis of **F**, α is β_m, and each β_i is inferred from one or more members of the sequence previous to β_i by one of the rules of **F**. Then the order of α is one greater than the thesis of highest order among the $\beta_i, \ldots, \beta_{m-1}$.

Since every thesis of **F** is assigned some order under the above ordering, the proof that if α is a thesis of **F** then α is a thesis of **S4** is immediate by the principle of strong induction and the following two theorems.

Theorem I *Every thesis of order 1 is a thesis of* **S4**.

Theorem II *Assuming every thesis of order less than k, for some finite number k, is a thesis of* **S4**, *any thesis of order k is a thesis of* **S4.**

Since every thesis of **F** arises by part (i) or (ii) of the definition of thesis, the proof of Theorem II is completed by establishing the two following lemmata.

Lemma I *Assuming every thesis whose order is less than that of α is a thesis of* **S4,** *if α is a thesis of* **F** *by part (i) of the definition of thesis then α is a thesis of* **S4**.

Lemma II *Assuming every thesis whose order is less than that of α is a thesis of* **S4,** *if α is a thesis of* **F** *by part (ii) of the definition of thesis then α is a thesis of* **S4.**

Moreover, the proof of Lemma I contains the proof of Theorem I. Thus, in order to prove that if α is a thesis of **F** then α is a thesis of **S4,** it only remains to prove Lemma I and Lemma II.

In proving these lemmata, Lewis's rules and the following analogues of theses of **S4** shall be employed.[4]

A1. $(\alpha \wedge \beta) \rightarrow (\beta \wedge \alpha)$
A2. $(\alpha \wedge \beta) \rightarrow \beta$
A3. $\alpha \rightarrow (\alpha \wedge \alpha)$
A4. $(\alpha \wedge (\beta \wedge \gamma)) \rightarrow ((\alpha \wedge \beta) \wedge \gamma)$
A5. $\alpha \rightarrow \sim\sim\alpha$
A6. $((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma)$
A7. $\sim \Diamond \alpha \rightarrow \sim\alpha$
A8. $(\alpha \rightarrow \beta) \rightarrow (\sim \Diamond \beta \rightarrow \sim \Diamond \alpha)$
A9. $(\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma))$

RI. *If* $\vdash(\alpha \leftrightarrow \beta)$ *and* $\vdash\gamma$ *then* $\vdash\delta$ *where* δ *results from* γ *by replacing* α *by* β (β *by* α) *in one or more places.*

RII. *If* $\vdash\alpha$ *and* $\vdash\beta$ *then* $\vdash(\alpha \wedge \beta)$

RIII. *If* $\vdash\alpha$ *and* $\vdash(\alpha \rightarrow \beta)$ *then* $\vdash\beta$

RIV. *If* $\vdash\alpha$ *then* $\vdash\sim \Diamond \sim\alpha$

11.02 $(\alpha \rightarrow \beta) \leftrightarrow \sim \Diamond (\alpha \wedge \sim\beta)$
11.7 $(\alpha \wedge (\alpha \rightarrow \beta)) \rightarrow \beta$
12.15 $(\alpha \wedge \beta) \leftrightarrow (\beta \wedge \alpha)$
12.3 $\alpha \leftrightarrow \sim\sim\alpha$
12.41 $(\sim\beta \rightarrow \sim\alpha) \rightarrow (\alpha \rightarrow \beta)$
12.5 $((\alpha \wedge \beta) \wedge \gamma) \leftrightarrow (\alpha \wedge (\beta \wedge \gamma))$
14.01 $(\alpha \supset \beta) \leftrightarrow \sim(\alpha \wedge \sim\beta)$
14.29 $(\alpha \wedge (\alpha \supset \beta)) \rightarrow \beta$
19.6 $(\alpha \rightarrow \beta) \rightarrow ((\alpha \wedge \gamma) \rightarrow (\beta \wedge \gamma))$
19.61 $((\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \gamma)) \rightarrow (\alpha \rightarrow (\beta \wedge \gamma))$
19.75 $\sim \Diamond \sim\alpha \rightarrow (\beta \rightarrow \alpha)$

RV-RVIII are also derivable in **S4**.

RV. *If* $\vdash(\alpha \rightarrow \beta)$ *and* $\vdash(\beta \rightarrow \gamma)$ *then* $\vdash(\alpha \rightarrow \gamma)$
Proof by **RII**, the assumptions, *A6*, **RIII**.

RVI. *If* $\vdash(\alpha \rightarrow \beta)$ *and* $\vdash(\alpha \rightarrow \gamma)$ *then* $\vdash(\alpha \rightarrow (\beta \wedge \gamma))$
Proof by **RII**, the assumptions, *19.61*, **RIII**.

RVII. *If* $\vdash(\alpha \rightarrow \beta)$ *and* $\vdash(\alpha \rightarrow (\beta \supset \gamma))$ *then* $\vdash(\alpha \rightarrow \gamma)$
Proof by **RVI**, the assumptions, *14.29*, **RV**.

RVIII. *If* $\vdash\alpha$ *and* $\vdash\sim(\alpha \wedge \sim\beta)$ *then* $\vdash\beta$
Proof by **RIV**, the assumptions, *12.3*, *11.02*, **RI**, **RIII**.

Lemma I. *Assuming every thesis whose order is less than that of* α *is a thesis of* **S4**, *if* α *is a thesis of* **F** *by part* (*i*) *of the definition of thesis then* α *is a thesis of* **S4**.

Let α be $(\alpha_1 \wedge, \ldots, \wedge \alpha_{n-1}) \rightarrow \alpha_n$, where $\beta_1, \ldots, \beta_m$ is the sequence of wffs which proves α_n from the hypotheses $\alpha_1, \ldots, \alpha_{n-1}$. Consider the following sequence: $(\alpha_1 \wedge, \ldots, \wedge \alpha_{n-1}) \rightarrow \beta_1, \ldots, (\alpha_1 \wedge, \ldots, \wedge \alpha_{n-1}) \rightarrow \beta_m$. In order to establish Lemma I it is sufficient to show that for any i^{th} line of this latter sequence, if $i > 1$ and Lemma I holds for the first i-1 lines, then $(\alpha_1 \wedge, \ldots, \wedge \alpha_{n-1}) \rightarrow \beta_i$ is a thesis of **S4**. Indeed, under this assumption, the lemma holds by:

A2, *12.15*, *12.5*, if β_i is an hypothesis,
RVI, if β_i follows by *F1*,
RVI, *11.7*, **RV**, if β_i follows by *F2*,
12.41, **RV**, if β_i follows by *F3*,
A5, **RV**, if β_i follows by *F4*,
14.01, **RI**, **RVII**, if β_i follows by *F5*,
19.6, **RV**, if β_i follows by *F6*,
A9, **RV**, if β_i follows by *F7*, and
RIV, *19.75*, **RIII**, if β_i is a thesis of **F**.

Lemma II. *Assuming every thesis whose order is less than that of* α *is a thesis of* **S4**, *if* α *is a thesis of* **F** *by part* (*ii*) *of the definition of thesis then* α *is a thesis of* **S4**.

Under the assumption, the proof of the lemma is completed by:

RII, if α follows by *F1*,
RIII, if α follows by *F2*,
12.41, **RIII**, if α follows by *F3*,
A5, **RIII**, if α follows by *F4*,
RVIII, if α follows by *F5*,
19.6, **RIII**, if α follows by *F6*, and
A9, **RIII**, if α follows by *F7*.

Having thus established the lemmata, the proof that systems **F** and **S4** are formally equivalent in the sense that they contain the same theses is complete.

§4 *Concluding remarks* Moh Shaw-Kwei in [9] has shown that the following deduction theorem:

If $\alpha_1, \ldots, \alpha_{n-1} \vdash \alpha_n$ *then* $\vdash (\alpha_1 \wedge, \ldots, \wedge \alpha_{n-1}) \rightarrow \alpha_n$

occurs in a system, **S4***, which has (a) the notion of proof from hypotheses given for **F**, (b) the rules of **S4** strengthened to hold for any wffs, and (c) the axioms of **S4**. **S4*** is not inferentially equivalent to **S4** (since the rules of **S4*** are stronger than the rules of **S4**), but it is formally equivalent to **S4** in the sense that the two systems have the same theses. Furthermore, he has shown that a system which he calls **V*3** and which he describes using Gentzen rules for material implication (except the paradoxical "from α infer that β materially implies α"), together with certain metarules and axioms, (cf. pp. 68-69 of [9]), is inferentially equivalent to **S4*** (and hence **F**).

It has here been shown however, that a system, namely **F**, having the same theses as **S4** can be described using no axioms but only the rules *F1-F7*, and a metarule which is the deduction theorem of **S4**. This formulation has the peculiarity of taking a symbol for 'only if' as primitive and using a definition of thesis which involves that symbol. That is, the theses of **S4** can be obtained by the natural deductive method without considering any propositional functor as primitive other than those for 'and', 'not' and 'only if'.

Clearly, a system having the same theses as **S5** can be generated by adding to **F** any rule from which $\Diamond \alpha \rightarrow \sim \Diamond \sim \Diamond \alpha$ can be inferred as a thesis. For instance, since it can be shown that

$$\sim(\alpha \rightarrow \sim\alpha) \rightarrow ((\alpha \rightarrow \sim\alpha) \rightarrow \sim(\alpha \rightarrow \sim\alpha))$$

is a thesis of **S5**, one might have **S5** by adding to **F**

$$\sim(\alpha \rightarrow \sim\alpha) \vdash ((\alpha \rightarrow \sim\alpha) \rightarrow \sim(\alpha \rightarrow \sim\alpha))$$

It is even possible to describe a system having the same theses as **T** of Feys-von Wright.[5] For this, the rule *F7* must be weakened to *F8*. It would then no longer be possible to prove *F23*, but only the weaker metarule:

If $\vdash(\alpha \leftrightarrow \beta)$ *then* $\vdash(\gamma \leftrightarrow \delta)$ *where* δ *results from* γ *by replacing* α *by* β (β *by* α) *in one or more places.*

But such a rule is sufficient for deriving **F11**, and thus all the theorems needed to show that this weaker system has all the theses of **T** can be obtained. Conversely, it can still be shown that the weakened system **F** contains no more theses than the theses of **T**, and thus, that α is a thesis of **T** if and only if α is a thesis of the weakened system **F**.

Since the theses of **T**, **S4**, and **S5** can be obtained by the natural deductive method without taking a modal functor as primitive, it has been shown that systems containing only the paradoxes of strict implication contained in **T**, **S4**, or **S5** can be obtained from the notion of proof from hypotheses. K. Matsumoto and M. Ohnishi have used this Gentzen manner of describing systems and have obtained systems containing the theses of various Lewis systems (cf. [6-8, 10, 11]). But they always take a singulary modal functor as primitive. And though their formulations have the advantage of yielding decision procedures for various Lewis systems, since they employ some singulary modal functor as primitive their formulations fall short of giving a description of a system which is non-modal in its primitive basis, that is, a system which takes as primitive only functors such as 'and', 'not', and 'only if'.

NOTES

1. Throughout the remainder of the paper lower case Greek letters shall always be used for well formed formulas.

2. Note that $(\alpha_1 \wedge, \ldots, \wedge \alpha_{n-1}) \rightarrow \alpha_n$ is a schema. Thus (i) of the definition of thesis establishes an infinite set of theses for each rule to which it is applied.

3. In this paper these functors will continue to be symbolized with '$\wedge$', '$\sim$', '$\Diamond$' respectively. Furthermore, '$\rightarrow$' shall be used for Lewis's '$\prec$', '$\leftrightarrow$' for Lewis's '$=$', '$\supset$' for Lewis's '$\supset$', and "α is a thesis" will be abbreviated by "$\vdash \alpha$".

4. The numbering of the theses is that of Lewis in [4] except that the number *A9* originates in this paper. *A9* is an unnumbered thesis of **S4** (cf. p. 500 of [4]). The remainder of the theses listed are found passim pp. 124-174. Following Lewis, the theses and rules of **S4** are abbreviated in accordance with Definitions 1 and 2.

5. For a description of system **T** cf. [2], p. 500, note 13; [15], Appendix II, pp. 85-90; and in particular [14].

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