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TYPES IN COMBINATORY LOGIC[1]

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The introduction of types in a formal system amounts to giving a classification of the entities of the system in categories, in such a way that the category of complex entities is determined by the categories of others that are simpler. In applications this procedure is used to define the class of formal entities that are propositions, i.e., those for which the rules of the propositional and predicate calculus are valid. In most systems the rules governing types are trivial, and it seems there is little interest in the study of those rules in a more general setting. In combinatory logic the situation has evolved in a different way. Curry has studied in several papers a theory of functionality.[2] In this system the combinators are allowed to belong to many distinct types; this is an important difference from ordinary type theories in which an entity belongs at most to one type. As a consequence the theory is not trivial, and very elaborate arguments are used in [7] to prove some fundamental properties.

In this paper the results of [7] are extended in several directions. For this purpose some properties of pure combinatory logic are necessary. Some of them are available in [7] or [9]; others are new, and the proofs are given in detail.

1. *The system* Γ. We shall consider a system of combinatory logic in which there are three primitive combinators: **S**, **K** and **I**, and possibly other atoms.[3] The atoms that are not combinators are called indeterminates. Combinations are the following formal entities: the primitive combinators and indeterminates are combinations; if X and Y are combinations, then the ordered pair consisting of X and Y in that order is also a combination which is denoted (XY).[4] When writing expressions denoting combinations we shall omit certain parentheses with the understanding that the association is to the left; parentheses at the end and at the beginning of an expression are also omitted. In this way any combination can be expressed in a unique way in the form $X_0 \ldots X_t$ $(t \geq 0)$, where X_0 is a primitive atom, called the *leading atom* of the combination. This can be shown easily by induction on the structure of the combination.

Sometimes we shall say that a combination U is a part of a combination X and that Y is obtained by replacing the part U by the combination V. We

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assume that the meaning of this process is clear. For a more technical definition the reader is referred to [7].

Letters $X, Y, Z, U, V, \ldots$, with subscripts if necessary, will be used to denote arbitrary combinations; letters x, y, z, u, v to denote indeterminates. By $X \equiv Y$ we mean that X and Y are the same combination.

Given a combination X and an indeterminate x we define a combination $[x]X$ by the following inductive rules:

a) *If* $X \equiv x$ *then* $[x]X \equiv \mathsf{I}$
b) *If* X *is an atom distinct from* x *then* $[x]X \equiv \mathsf{K}X$
c) *If* $X \equiv YZ$, $U \equiv [x]Y$, $V \equiv [x]Z$ *then* $[x]X \equiv \mathsf{S}UV$.

Now if X is a combination, and for $n > 1$, $x_1, \ldots, x_n$ are indeterminates, we define $[x_1, \ldots, x_n]X \equiv [x_1][x_2, \ldots, x_{n-1}]X$.

We need also the following definitions. If $X \equiv X_0 \ldots X_t$ where X_0 is the leading atom, then any combination $X_0 \ldots X_i$ for $i \leq t$ is called a *head* of X. If X_0 is an indeterminate we say that X is a *closed combination.* If X is of one of the forms: S, K, I, $\mathsf{S}U$, $\mathsf{S}UV$, $\mathsf{K}U$ we say that X is an *open combination.* A combination that is neither closed nor open is said to be *reducible.* If Y is a combination obtained from X replacing a part U by V, where $U R V$ is a case of one of rules (S), (K) or (I) introduced later, we say that Y is obtained by a *contraction.* If U is a head of X it is said to be a *head contraction,* otherwise an *internal contraction.*

To avoid writing long expressions we use abbreviations as follows:

$\mathsf{B} \equiv \mathsf{S}(\mathsf{KS})\mathsf{K}$
$\Phi \equiv \mathsf{B}(\mathsf{BS})\mathsf{B}$
$\mathsf{J} \equiv \mathsf{SB}(\mathsf{KI})$

In the system Γ we shall define several binary relations by giving rules, with the understanding that the relation holds in a particular case only if that follows by a finite number of applications of the rules. To simplify the definitions we state several rules for an unspecified binary relation R.

Rule (ρ) $X R X$
Rule (σ) *If* $X R Y$ *then* $Y R X$
Rule (τ) *If* $X R Y$ *and* $Y R Z$ *then* $X R Z$
Rule (μ) *If* $X R Y$ *then* $ZX R ZY$
Rule (ν) *If* $X R Y$ *then* $XZ R YZ$
Rule (S) $\mathsf{S}XYZ R XZ(YZ)$
Rule (K) $\mathsf{K}XY R X$
Rule (I) $\mathsf{I}X R X$
Rule (C1) $\mathsf{BS}(\mathsf{BK}) R \mathsf{S}(\mathsf{KK})\mathsf{J}$
Rule (C2) $\mathsf{B}(\mathsf{BS})(\mathsf{BS}(\mathsf{BS})) R \Phi\mathsf{B}(\mathsf{B}(\Phi\mathsf{S})\mathsf{S})(\mathsf{KS})$
Rule (C3) $\mathsf{B}(\mathsf{S}(\mathsf{BBS})(\mathsf{KK}))\mathsf{K} R \mathsf{BK}$
Rule (C4) $\mathsf{B}(\mathsf{BJ})\mathsf{S} R \mathsf{S}$
Rule (C5) $\mathsf{BJK} R \mathsf{K}$
Rule (C6) $\mathsf{SK} R \mathsf{KI}$
Rule (C7) $\mathsf{S}(\mathsf{KI}) R \mathsf{J}$

Rule (C8) $\quad\quad$ $\mathbf{S}(\mathbf{KI})\, R\, \mathbf{I}$

Rules ($\mathbf{S}$), ($\mathbf{K}$) and ($\mathbf{I}$) are called *reduction rules* and rules (C1)-(C8) are called *combinatory rules.*

Theorem 1. *If R_0 is a binary relation defined by a subset of the rules for R that contains rules (ρ), (τ), (μ), (ν), some reduction and combinatory rules, but not rule (σ), then for arbitrary combinations X and Y, $X\, R_0\, Y$ if and only if there is a sequence of combinations $X_1, \ldots, X_k$ with $k \geq 1$ such that $X_1 \equiv X$, $X_k \equiv Y$ and for $i \geq 1$, X_i is obtained from X_{i-1} by replacing a part U by V where $U\, R_0\, V$ is a case of one of the reduction or combinatory rules in the definition of R_0.*

Necessity follows by induction on the derivation of $X\, R_0\, Y$. To prove sufficiency it is enough to consider the case $k = 2$ and in this case the proof is by induction on the structure of X.

When rule (σ) enters in the definition of R_0 we obtain a similar property requiring that either $U\, R_0\, V$ or $V\, R_0\, U$ is a case of one of the reduction or combinatory rules in the definition of R_0.

The binary relation defined by rules (ρ), (τ), (μ), (ν), ($\mathbf{S}$), ($\mathbf{K}$), and ($\mathbf{I}$) is called reduction and is denoted by $X \geq Y$.

The binary relation defined by rules (ρ), (σ), (τ), (μ), (ν), ($\mathbf{S}$), ($\mathbf{K}$), ($\mathbf{I}$) is called α-equality and is denoted $X =_\alpha Y$.

The binary relation defined by rules (ρ), (σ), (τ), (μ), (ν), ($\mathbf{S}$), ($\mathbf{K}$), (C1), (C2), (C3), (C4), (C5) and (C6) is called β-equality and is denoted $X =_\beta Y$.

The binary relation defined by rules (ρ), (σ), (τ), (μ), (ν), ($\mathbf{S}$), ($\mathbf{K}$), (C1), (C2), (C3), (C6), (C7) and (C8) is called equality and is denoted $X = Y$.

Let X and Y be two combinations and x an indeterminate. The combination $[Y/x]X$, called *the result of substituting Y for x in X,* is defined inductively as follows:

a) *If* $X \equiv x$ *then* $[Y/x]X \equiv Y$
b) *If* X *is an atom distinct from* x *then* $[Y/x]X \equiv X$
c) *If* $X \equiv YZ$, $U \equiv [Y/x]Y$, $V \equiv [Y/x]Z$, *then* $[Y/x]X \equiv UV$.

Lemma 1. $([x]X)Y \geq [Y/x]X$.

The proof, by induction on the structure of X, is easy.

The following two lemmas were proved by Rosser in his dissertation.[6]

Lemma 2. *If $X \geq Y$, then there is a combination Z such that Z is obtained from X by a series of head contractions and Y is obtained from Z by a series of internal contractions.*

Lemma 3. *If $X =_\alpha Y$, then there is a combination Z such that $X \geq Z$ and $Y \geq Z$.*

The rules for the equality relations are formulated in such a way that some extensionality properties are valid. These properties are expressed in the following two theorems. Since they are not important for the problem considered in this paper the proofs are only outlined.[7]

Theorem 2. *The following properties hold for β-equality:*

(i) *If $X =_\alpha Y$ then $X =_\beta Y$*
(ii) *If X does not contain x then $[x]X =_\beta \mathbf{K}X$*
(iii) *If X is an open combination then there are combinations Y and Z such that $X =_\beta \mathbf{S}YZ$*
(iv) *If X is an open combination not containing x then*
$$X =_\beta \mathbf{J}X =_\beta \mathbf{B}X\mathbf{I} =_\beta \mathbf{S}(\mathbf{K}X)\mathbf{I} =_\beta [x](Xx)$$
(v) *If $X =_\beta Y$ then $[x]X =_\beta [x]Y$*
(vi) *If X and Y are open combinations not containing x and $Xx =_\beta Yx$ then $X =_\beta Y$*

To prove (i) we need only to show that rule ($\mathbf{I}$) holds for β-equality and this follows from rule (C6). Part (ii) follows by induction on the structure of X using rule (C3), and part (iii) using rules (C4), (C5) and (C6). For part (iv) use part (iii), rule (C4) and the definitions. Part (v) follows by induction on the derivation of $X =_\beta Y$, and part (vi) using parts (iv) and (v).

Theorem 3. *The following properties hold for equality:*

(i) *If X does not contain x then $[x]X = \mathbf{K}X$*
(ii) *If X does not contain x then $[x](Xx) = X$*
(iii) *If $X = Y$ then $[x]X = [x]Y$*
(iv) *If x does not occur in X or Y and $Xx = Yx$ then $X = Y$.*
(v) *If $X =_\beta Y$ then $X = Y$.*

Parts (i) and (iii) are proved as in Theorem 2. Part (ii) follows from rules (C7) and (C8). Part (iv) is proved using parts (ii) and (iii). To prove part (v) we need only to show that rules (C4) and (C5) are valid for equality. For this use part (iv).

2. *Some properties of reduction.* In this section we shall consider some relations between reduction and equality. These results will be used later in the study of normal combinations.

Lemma 4. *If X reduces to $xX_1 \ldots X_k$ by head contractions, and $X =_\alpha Y$, then Y reduces by head contractions to a combination $xY_1 \ldots Y_k$ such that $X_i =_\alpha Y_i$.*

We have $xX_1 \ldots X_k =_\alpha Y$, hence by Lemma 3 there is a combination Z such that $xX_1 \ldots X_k \geq Z$ and $Y \geq Z$. Clearly Z has to be of the form $xZ_1 \ldots Z_k$, where $X_i \geq Z_i$. Further by Lemma 2 there is a combination Z' such that Y reduces to Z' using only head contractions, and Z' reduces to Z using only internal contractions. Hence Z' has to be of the form $xY_1 \ldots Y_k$ where $Y_i \geq Z_i$. From this it follows that $X_i =_\alpha Y_i$.

Lemma 5. *If X reduces by head contractions to an open combination, and $X =_\alpha Y$, then Y also reduces by head contractions to an open combination.*

The proof is similar to that of Lemma 4.

Lemma 6. *If X reduces by head contractions to an open combination, and Y is obtained from X by replacing occurrences of U by V, where either $U\ R\ V$ or $V\ R\ U$ is a case of one of rules* (C1)-(C7), *then Y also reduces by head contractions to an open combination.*[8]

The proof is by induction on the number of head contractions in the reduction of X. First suppose that U does not occur as a head of X replaced by V: then if X is open, Y is also open and if X is reducible, Y is also reducible, and the same head contraction can be performed in both X and Y, and after this the induction hypothesis can be applied. Hence we may suppose that U is a head of X replaced by V. Then $X \equiv UU_1 \ldots U_k$, and $Y \equiv VV_1 \ldots V_k$ where V_i is obtained from U_i by replacing occurrences of U by V. Now the following property can easily be checked for each of the rules (C1)-(C7): depending on k, either both X and Y reduce to an open combination, or they reduce by head contractions to combinations X' and Y', such that there exists a combination Y'', obtained from X' by replacing occurrences of U by V, and $Y'' =_\alpha Y'$. By the induction hypothesis Y'' reduces by head contractions to an open combination, hence by Lemma 5, Y' (and Y) reduces to an open combination.

Theorem 4. *If X reduces by head contractions to an open combination, and $X =_\beta Y$, then Y also reduces by head contractions to an open combination.*

We know there is a sequence $X_1, \ldots, X_t$, where $X_1 \equiv X$, $X_t \equiv Y$, and for $i > 1$, X_i is obtained from X_{i-1} by replacing one occurrence of U by V where either $U =_\beta V$ or $V =_\beta U$ is a case of one of rules (**S**), (**K**), (**I**), (C1)-(C6). The proof is by induction on t. The case $t = 1$ is trivial. For $t > 1$ we have either $X_{t-1} =_\alpha X_t$, in which case we use Lemma 5, or X_t is obtained from X_{t-1} by a replacement corresponding to one of rules (C1)-(C6) in which case we use Lemma 6.

Lemma 7. *If X reduces by head contractions to a closed combination $xX_1 \ldots X_k$, and Y is obtained from X by replacing occurrences of U by V, where $U\ R\ V$ or $V R\ U$ is a case of one of rules* (C1)-(C7), *then there are combinations $Y_1, \ldots, Y_k$ where Y_i is obtained from X_i by replacing occurrences of U by V, and Y reduces by head contractions to a closed combination $xZ_1 \ldots Z_k$, where $Y_i =_\alpha Z_i$.*

The proof is by induction on the number of head contractions. First suppose that U does not occur as a head of X replaced by V; then, if X is closed the lemma is trivial, and if X is reducible the same head contraction can be applied to both X and Y, and after this the induction hypothesis can be applied. Hence we assume that U is a head of X replaced by V. By the argument used in the proof of Lemma 6 it follows that X reduces by head contractions to a combination X', and Y reduces to a combination Y', such that there exists a combination Y'', obtained from X' by replacing occurrences of U by V, and $Y'' =_\alpha Y'$. By the induction hypothesis and Lemma 4 the lemma follows.

Theorem 5. *If X reduces by head contractions to a closed combination $xX_1 \ldots X_k$, and $X =_\beta Y$, then Y reduces by head contractions to a combination $xY_1 \ldots Y_k$, where $X_i =_\beta Y_i$.*

The proof is similar to that of Theorem 4 using Lemmas 4 and 7.

Corollary. *If $X =_\beta x$ then $X \geq x$.*

Lemma 8. *If X reduces by head contractions to a closed combination $xX_1 \ldots X_k$, and Y is obtained from X by replacing occurrences of $\mathbf{I}$ by $\mathbf{S}(\mathbf{KI})$, then there are combinations $Y_1, \ldots, Y_k$, where Y_i is obtained from X_i by replacing occurrences of $\mathbf{I}$ by $\mathbf{S}(\mathbf{KI})$, and for any indeterminate y, Yy reduces by head contractions to $xY_1 \ldots Y_k y$.*

The proof is by induction on the number of head contractions, and we need to consider only the case in which $\mathbf{I}$ is a head of X replaced by $\mathbf{S}(\mathbf{KI})$. In this case it is easy to check that X and Yy reduce by head contractions to combinations U and Vy where V is obtained from U by replacing occurrences of $\mathbf{I}$ by $\mathbf{S}(\mathbf{KI})$. Using the induction hypothesis the lemma follows.

Lemma 9. *If X reduces by head contractions to a combination $xX_1 \ldots X_k$, and Y is obtained from X by replacing occurrences of $\mathbf{S}(\mathbf{KI})$ by $\mathbf{I}$, then Y reduces by head contractions to a combination $xY_1 \ldots Y_k$, where Y_i is obtained from X_i by replacing occurrences of $\mathbf{S}(\mathbf{KI})$ by $\mathbf{I}$.*

The proof is similar to that of Lemma 8.

Theorem 6. *If X reduces by head contractions to a combination $xX_1 \ldots X_k$, and $X = Y$, then for some $n \geq 0$, given indeterminates $y_1, \ldots, y_n$, the combination $Yy_1 \ldots y_n$ reduces by head contractions to a combination $xY_1 \ldots Y_k U_1 \ldots U_n$ where $X_i = Y_i$ and $U_j = y_j$, $i = 1, \ldots, k$, $j = 1, \ldots, n$.*

The proof is similar to that of Theorem 4, using Theorem 5 and Lemmas 8 and 9.

Corollary. *If $X = Y$, $X \equiv xX_1 \ldots X_k$, $Y \equiv yY_1 \ldots Y_t$, then $k = t$, $x \equiv y$ and $X_i = Y_i$.*

3. *Normal combinations*. In the system of lambda conversion combinations in normal form play a special role. A similar notion has been introduced in combinatory logic using strong reduction. In this section the class of normal combinations is introduced by an intrinsic definition that is independent of the algorithm used for the abstraction operator. Only the definition of the normal form of a normal combination depends on that algorithm.

First we introduce two new algorithms for abstraction that are more convenient for the purposes of this section. The first, called the strong algorithm, is defined by the following rules:

a) *If $X \equiv x$ then $[x]X \equiv \mathbf{I}$*
b) *If X does not contain x then $[x]X \equiv \mathbf{K}X$*
c) *If $X \equiv Yx$, where Y does not contain x, then $[x]X \equiv Y$*
d) *If $X \equiv YZ$, $U \equiv [x]Y$, $V \equiv [x]Z$, then $[x]X \equiv \mathbf{S}UV$*

The weak algorithm is obtained dropping rule c). When two different rules can be applied to a combination, the earliest rule in the list is to be applied.

Lemma 10. *If* $Y \equiv [y/x]X$ *where* y *does not occur in* X *then* $[x]X \equiv [y]Y$, *for the weak and strong algorithms.*

The proof by induction on the structure of X is easy.

Lemma 11. *If* x *does not occur in* Y, *and is distinct from* y, *then*

$$[x][Y/y]X \equiv [Y/y][x]X$$

The proof is by induction on the structure of X, and covers both the strong and the weak algorithms.

(i) X is x. Then

$$[x][Y/y]X \equiv [x]X \equiv \mathbf{I} \equiv [Y/y]\mathbf{I} \equiv [Y/y][x]X$$

(ii) X does not contain x. Then

$$[x][Y/y]X \equiv \mathbf{K}([Y/y]X) \equiv [Y/y](\mathbf{K}X) \equiv [Y/y][x]X$$

(iii) $X \equiv Zx$ where Z does not contain x. Set $U \equiv [Y/y]Z$. Then

$$[x][Y/y]X \equiv [x]Ux \equiv U \equiv [Y/x][x]X$$

(iv) $X \equiv UV$. In this case use the induction hypothesis for U and V, and the definitions.

We define now a special class of combinations, called *normal combinations*. The definition is by induction according to the following rules that associate with each normal combination a *length* and a *degree*.

(i) *Every indeterminate is a normal combination of length 1 and degree* 0.

(ii) *If* $X \equiv xX_1 \ldots X_k$, $k \geq 1$, *and* $X_1, \ldots, X_k$ *are normal combinations of lengths* $m_1, \ldots, m_k$, *respectively, then* X *is a normal combination of length* $1 + max(m_1, \ldots, m_k)$ *and degree 0.*

(iii) *If* X *is an open combination and* Xx *is a normal combination of length* m *and degree* n, *then* X *is a normal combination of length* $m + 1$ *and degree* $n + 1$.

(iv) *If* Y *is obtained from* X *by a head contraction and* Y *is a normal combination of length* m *and degree* n, *then* X *is a normal combination of length* $m + 1$ *and degree* n.

Lemma 12. *If* $Y \equiv [y/x]X$ *and* Y *is a normal combination of length* m *and degree* n, *then* X *is a normal combination of length* m *and degree* n.

The proof is by induction on m. Note that Y is open, closed, or reducible if and only if X is open, closed, or reducible respectively. Cases (i) and (ii) of the definition are trivial. Suppose X and Y are open. Then for some indeterminate z, Yz is a normal combination of length $m - 1$ and degree $n - 1$. Take an indeterminate w not occurring in X, and distinct from x and y. Then

$$Yz \equiv [z/w][y/x](Xw)$$

hence, by the induction hypothesis, Xw is a normal combination of length $m - 1$ and degree $n - 1$. It follows that X is a normal combination of length m and degree n. If both X and Y are reducible, and reduce by a head contraction to X_1 and Y_1 respectively, then $Y_1 \equiv [y/x]X_1$. By the induction hypothesis the lemma follows.

Lemma 13. *If X is a normal combination of length m and degree n, then m and n are uniquely determined.*

The proof is by induction on m. All cases are trivial except when X is open, and here we use Lemma 12.

The next step is to associate with every normal combination a *normal form* invariant under the equality relations. This normal form depends on the algorithm for abstraction. The normal form obtained with the weak algorithm is called the *weak normal form,* and that obtained with the strong algorithm is called the *strong normal form*. The defining rules are as follows.

(i) *If X is an indeterminate the normal form of X is X itself.*

(ii) *If $X \equiv xX_1 \ldots X_k$, where X_i is a normal combination with normal form U_i, then $xU_1 \ldots U_k$ is the normal form of X.*

(iii) *If X is an open combination not containing the indeterminate x, and U is the normal form of Xx, then $[x]U$ is the normal form of X.*

(iv) *If X is reducible, Y is obtained from X by a head contraction, and U is the normal form of Y, then U is also the normal form of X.*

For a fixed abstraction algorithm this definition assigns a unique normal form to each normal combination. In fact, from Lemma 12 it follows that in part (iii) of the definition of the normal combinations the indeterminate x can be assumed not occurring in X; further from Lemma 10 it results that the normal form is independent of the indeterminate x.[9]

Theorem 7. *If X is a normal combination of length m and degree n, and $x_1, \ldots, x_k$ are distinct indeterminates not occurring in X, $(0 \leq k \leq n)$, then $Xx_1 \ldots x_k$ reduces by head contractions to a normal combination Y, of degree $n - k$, with normal form U such that Y is open if $k < n$, and is closed if $k = n$, and the normal form of X is $[x_1, \ldots, x_k]U$.*

The proof is by induction on m, and covers both the weak and the strong normal forms.

(i) and (ii) $X \equiv xX_1 \ldots X_t$, $t \geq 0$. Then $n = k = 0$, and the theorem follows taking $Y \equiv X$.

(iii) X is open, hence $n > 0$. For $k = 0$ the theorem is trivial. Assume $k > 0$. Now Xx_1 is a normal combination of length $m - 1$ and degree $n - 1$. By the induction hypothesis $Xx_1 \ldots x_k$ reduces by head contractions to a combination Y, with normal form U, such that the normal form of Xx_1 is $[x_2, \ldots, x_k]U$. Further Y is open if $k - 1 < n - 1$ and is closed if $k - 1 = n - 1$. By definition the normal form of X is $[x_1][x_2, \ldots, x_k]U$, and this is $[x_1, \ldots, x_k]U$ by definition.

(iv) When X is reducible the theorem follows immediately from the induction hypothesis.

Theorem 8. *If X is a normal combination, and $x_1, \ldots, x_k$ are distinct indeterminates not occurring in X, then $Xx_1 \ldots x_k$ is a normal combination, and the strong normal form of X is $[x_1, \ldots, x_k]U$ where U is the strong normal form of $Xx_1 \ldots x_k$, and the strong algorithm is used.*

That $Xx_1 \ldots x_k$ is a normal combination follows easily from the definition and Lemma 12. The second part is proved by induction on the length of X.

(i) and (ii) $X \equiv xX_1 \ldots X_t$, $t \geq 0$. The normal form of $xX_1 \ldots X_t x_1 \ldots x_k$ is $xU_1 \ldots U_t x_1 \ldots x_k$, where U_i is the normal form of X_i. Also the normal form of X is $xU_1 \ldots U_t$. Since every indeterminate occurring in U_i must occur also in X_i, we have by clause c) of the strong algorithm that

$$[x_1, \ldots, x_k](xU_1 \ldots U_t x_1 \ldots x_k) \equiv xU_1 \ldots U_t$$

(iii) and (iv) The cases in which X is open or reducible follow easily using the induction hypothesis and the definitions.

Lemma 14. *If X is a normal combination of degree n, and $X =_\beta Y$, then Y is a normal combination of degree n.*

The proof is by induction on the length of X.

(i) and (ii) $X \equiv xX_1 \ldots X_t$, $t \geq 0$. By Theorem 5, Y reduces by head contractions to a combination $xY_1 \ldots Y_t$ such that $X_i =_\beta Y_i$; by the induction hypothesis Y_i is a normal combination. Since Y reduces by head contractions to a normal combination of degree 0 it is a normal combination of degree 0.

(iii) X is open. By Theorem 4, Y reduces by head contractions to an open combination Z. Then $Xx =_\beta Zx$, and by the induction hypothesis Zx is a normal combination of degree $n - 1$. Hence Z and Y are normal combinations of degree n.

(iv) X is reducible. This case follows immediately using the induction hypothesis.

Theorem 9. *If X is a normal combination, with normal form U, and $X =_\beta Y$, then U is also the normal form of Y.*

The proof is by induction on the length of X and covers both the weak and the strong normal forms. Suppose the degree of X and Y is $n \geq 0$. Take indeterminates $x_1, \ldots, x_n$, all distinct and not occurring in X or Y. By Theorem 7, $Xx_1 \ldots x_n$ reduces by head contractions to a combination $xX_1 \ldots X_k$ such that

$$[x_1, \ldots, x_n](xU_1 \ldots U_k)$$

is the normal form of X and U_i is the normal form of X_i. Hence, by Theorem 5, $Yx_1 \ldots x_n$ reduces by head contractions to $xY_1 \ldots Y_k$ where $X_i =_\beta Y_i$. By the induction hypothesis U_i is the normal form of Y_i. Hence, by Theorem 7, U is the normal form of Y.

Lemma 15. *If X is a normal combination, with strong normal form U, and Y is obtained from X by replacing occurrences of* **S(KI)** *by* **I**, *then Y is also a normal combination with strong normal form U.*

The proof is by induction on the length of X. Suppose X is of degree $n \geq 0$. Take indeterminates $x_1, \ldots, x_n$, all distinct and not occurring in X or Y. Then $Xx_1 \ldots x_n$ reduces by head contractions to a combination $xX_1 \ldots X_k$, with strong normal form $xU_1 \ldots U_k$, where U_i is the strong normal form of X_i. By Lemma 9, $Yx_1 \ldots x_n$ reduces by head contractions to $xY_1 \ldots Y_k$, where Y_i is obtained from X_i by replacing occurrences of **S(KI)** by **I**. By the induction hypothesis U_i is the strong normal form of Y_i, hence $xU_1 \ldots U_k$ is the strong normal form of $Yx_1 \ldots x_n$. It follows that Y is a normal combination and by Theorem 8 the strong normal form of Y is

$$U \equiv [x_1, \ldots, x_n](xU_1 \ldots U_k)$$

Lemma 16. *If X is a normal combination, with strong normal form U, and Y is obtained from X by replacing occurrences of* **I** *by* **S(KI)**, *then Y is a normal combination with strong normal form U.*

The proof is by induction on the length of X. Suppose X is of degree $n \geq 0$. Take indeterminates $x_1, \ldots, x_n, y$, all distinct and not occurring in X. Then $Xx_1 \ldots x_n$ reduces by head contractions to a combination $xX_1 \ldots X_k$, with strong normal form $xU_1 \ldots U_k$. By Lemma 8, $Yx_1 \ldots x_n y$ reduces by head contractions to a combination $xY_1 \ldots Y_k y$, where Y_i is obtained from X_i by replacing occurrences of **I** by **S(KI)**. By the induction hypothesis the normal form of $Yx_1 \ldots x_n y$ is $xU_1 \ldots U_k y$, and by Theorem 8 the normal form of Y is

$$[x_1, \ldots, x_n, y](xU_1 \ldots U_k y) \equiv [x, \ldots, x](xU_1 \ldots U_k)$$

since y does not occur in $xU_1 \ldots U_k$.

Theorem 10. *If X is a normal combination, with strong normal form U, and $X = Y$ then Y is a normal combination with strong normal form U.*

This follows from Theorem 9, Lemma 15 and Lemma 16 noting that rule (C7) is valid for β-equality.

Theorem 11. *If X is a normal combination of degree n, with normal form U, Y is a normal combination of degree 0 with normal form V, y is some indeterminate, and $Z \equiv [Y/y]X$, then Z is a normal combination of degree n with normal form $[V/y]U$.*

The proof is by induction on the length of X, and covers both the weak and the strong normal forms.

(i) $X \equiv xX_1 \ldots X_k$, $k \geq 0$, where x is not y. Then $Z \equiv xZ_1 \ldots Z_k$ where $Z_i \equiv [Y/y]X_i$. By the induction hypothesis Z_i is a normal combination with normal form $[V/y]U_i$, hence Z is a normal combination with normal form $[V/y]U$.

(ii) $X \equiv yX_1 \ldots X_k$, $k \geq 0$. Then $Z \equiv YZ_1 \ldots Z_k$, where $Z_i \equiv [Y/y]X$. By the induction hypothesis, Z_i is a normal combination with normal form

$U_i' \equiv [V/y]U_i$, where U_i is the normal form of X_i. Since Y is of degree 0, it reduces by head contractions to a combination $xY_1 \ldots Y_t$, and $V \equiv xV_1 \ldots V_t$ where V_j is the normal form of Y_j. Hence Z reduces by head contractions to $xY_1 \ldots Y_t Z_1 \ldots Z_k$, which is a normal combination with normal form $xV_1 \ldots V_t U_1' \ldots U_k'$. Since

$$[V/y]U \equiv VU_i' \ldots U'_k \equiv xV_1 \ldots V_t U_1' \ldots U'_k$$

the theorem is proved in this case.

(iii) X is an open combination. Take an indeterminate x not occurring in X or Y, and distinct from y. Let U_1 be the normal form of Xx. By the induction hypothesis $Zx \equiv [Y/y](Xx)$ is a normal combination of degree $n - 1$ with normal form $[V/y]U_1$. Since Z is open, this means that Z is a normal combination of degree n, and the normal form of Z is

$$[x][V/y]U_1 \equiv [V/y][x]U_1 \equiv [V/y]U$$

by Lemma 11.

(iv) The case in which X is reducible follows easily by the induction hypothesis.

Theorem. 12. *If X is a normal combination, and U is the strong (weak) normal form of X, then $X = U (X =_\beta U)$.*

The proof is by induction on the length of X. All cases are trivial except when X is open. Here we apply the induction hypothesis, for equality Theorem 3 (iv), and for β-equality Theorem 2 (vi), noting that for the weak algorithm $[x]U$ is always an open combination.

We have proved the properties of the normal combinations that are needed for our subject. It is convenient here to make a remark about the role of indeterminates in the definitions and proofs given in this section. It is clear that in order to be able to show that a combination is normal we must have at least one indeterminate in the system Γ. The function of this indeterminate is quite similar to that of variables in the predicate calculus when used to prove closed formulas without free variables. In both cases we use some kind of dummy element, to prove properties involving some constants that are completely independent of that element. This also shows that the process of proving that a combination is normal involves some kind of quantification. Further, our proofs seem to require the existence of n distinct indeterminates for arbitrary n, i.e. that we must have an infinite set of indeterminates. This is not the case: for a given X, those proofs describe inductions in which only a finite number of indeterminates is necessary. We may say that for each given case, we extend our system adding enough indeterminates in order to perform the induction described in the proof. Since in all cases the properties proved are independent of the number of indeterminates in the system, we get a proof for the original system.

4. *Normal combinations and logical paradoxes.* Our system of combinatory logic allows us to construct complex functions out of the constants given as primitives, and to derive equations that correspond to some extensionality principles. The primitives are **S**,**K**,**I** and possibly other con-

stants introduced with some intended meaning. With respect to the relations studied in the preceding sections, these constants are of course indeterminates. When speaking of constants, we shall exclude the combators **S**,**K** and **I**.

To formalize the intended meaning of the constants we must introduce new relations, with adequate defining rules. Suppose there is a constant P representing implication, and that some class of combinations, called propositions, is defined in such a way that if X and Y are propositions then PXY is a proposition, and if X is a proposition and $X = Y$, then Y is a proposition. The definition of the combinations that are propositions involves several difficulties. First we show that under some very general conditions the system would become inconsistent. Suppose we have a unary relation denoted with the prefix $\vdash$, such that for arbitrary propositions X and Y the following conditions are satisfied:[10]

(i) $\vdash P(PY(PYX))(PYX)$

(ii) *If* $\vdash X$, *and* $X =_{\alpha} Y$, *then* $\vdash Y$

(iii) *If* $\vdash PXY$, *and* $\vdash X$, *then* $\vdash Y$

For a given proposition U, consider the combination

$$Z \equiv [x]\,(P(xx)(P(xx)\,U)\,]$$

The combination Z is intuitively a class; suppose we have defined the class of propositions in such a way that the combination ZZ is a proposition. Then we have

$$ZZ \geq P(ZZ)(P(ZZ)\,U) \geq P(P(ZZ)(P(ZZ)\,U))(P(ZZ)\,U)$$

$\vdash P(P(ZZ)(P(ZZ)\,U))(P(ZZ)\,U)$	by (i)
$\vdash P(ZZ)(P(ZZ)\,U)$	by (ii)
$\vdash P(ZZ)\,U$	by (iii)
$\vdash ZZ$	by (ii)
$\vdash U$	by (iii)

The preceding argument is just a generalization of the Russell paradox due to Curry. It is easy to see that the combination ZZ is not normal. Now suppose that the class of propositions has been so defined that every proposition is a normal combination. We assume also that there is a constant representing negation and the class of proposition is closed under negation. In place of condition (i) we assume now some set of axiom schemata that are tautologies in the usual sense, and take conditions (ii) and (iii) as the only rules of derivation. The system so obtained is consistent. For if $\vdash X$ holds with some derivation, and U is the normal form of X, we can obtain a derivation of $\vdash U$ if in the derivation of $\vdash X$ we replace each combination by its normal form. In this new derivation rule (iii) is not used, since combinations in normal form are equal if and only if they are identical. Hence we have a derivation in the sense of the predicate calculus; since the normal form of a tautology is again a tautology, it follows that U is a tautology.

This consistency result is not specially deep, since no definition of the class of propositions has been given that satisfies the requirements and is acceptable on any reasonable grounds. Later it will be shown that if a classification in types is used in order to define the propositions, then the requirement of normality is satisfied. Our aim here is to show that in order to avoid certain kinds of paradoxes that involve only the propositional calculus and rules for equality, we are not obliged, in principle, to introduce types.

Suppose again a class of propositions has been defined in such a way that the normality condition is satisfied. We want to introduce a universal quantifier with the usual properties. Let π be a primitive constant representing the quantifier. We assume there are in the system atomic entities with the properties of variables. Then, if X is a proposition and x is a variable, the combination $\pi([x]X)$ is a proposition. We want the following rule:

(iv) *If* $\vdash \pi Y$ *then* $\vdash YU$

where U is a combination satisfying some conditions. The problem now is to give conditions in this rule to assure that the combination YU is normal. For, even if Y and U are both normal, it may happen that YU is not normal. For instance if we take both Y and U to be the combination Z defined at the beginning of this section.

A possible solution would be to restrict U to be a combination of degree *0*. Then, by Theorem 11, YU is also a normal combination. In this way we exclude open combinations from the range of the quantifier. Since classes and functions are represented by open combinations, the system obtained in this way would be similar to first order logic.

Another possible solution is to introduce types; this will be considered in detail in the next sections.

5. *Types in combinatory logic.* The preceding discussion intended to suggest the following conclusion: types are necessary to give conditions for Y and U in order to secure that YU is a normal combination even if U is of positive degree. In this section there is considered a formalization of the theory of types of a very general character. In a forthcoming paper I hope to apply the results obtained here to a more conventional system.

As before, the system Γ consists of the combinations generated by **S**,**K**,**I** and possibly some indeterminates. We consider also a set of formal objects generated by an atom denoted by the letter **F**, and a non-empty set of elements called atomic types. These formal objects are called *types*, and are defined by the following rules:

a) *Each atomic type is a type.*

b) *If* α *and* β *are types, then the triple consisting of* **F**, α *and* β *in this order is also a type. This triple is denoted:* $\mathbf{F}\alpha\beta$.

Lower case Greek letters will denote types. We also use the following notation:[11]

$\mathbf{F}_n\alpha_1\ldots\alpha_n\beta$	for	$\mathbf{F}\alpha_1\mathbf{F}_{n-1}\alpha_2\ldots\alpha_n\beta$
where $n > 1$	and	$\mathbf{F}_1$ is $\mathbf{F}$

We assume that with each indeterminate x of the system Γ a type α has been associated, and x is said to be of type α. A relation between combinations and types is defined by the following rules, where X and Y stand for arbitrary combinations, and α,β and μ stand for arbitrary types. For this relation we use the notation: X/α which is read: *X is of type* α.

Rule (Ax) *If x is an indeterminate of type* α, *then* x/α
Rule (**S**) $\mathbf{S}/\mathbf{F}_3\mathbf{F}_2\alpha\beta\mu\mathbf{F}\alpha\beta\alpha\mu$
Rule (**K**) $\mathbf{K}/\mathbf{F}_2\alpha\beta\alpha$
Rule (**I**) $\mathbf{I}/\mathbf{F}\alpha\alpha$
Rule (**F**) *If* $X/\mathbf{F}\alpha\beta$, Y/α *then* XY/β

Several elementary properties of this relation are stated in the following lemmas. The proofs can be easily obtained using the definitions.

Lemma 17. *If* X/α *and* $X \equiv X_0X_1 \ldots X_k$, *where* X_0 *is the leading atom, then there are types* $\beta_1 \ldots, \beta_k$ *such that*

$$X_0/\mathbf{F}_k\beta_1 \ldots \beta_k\alpha$$

$$X_i/\beta_i \qquad i = 1, \ldots, k$$

Lemma 18. *If* X/α *and X is an open combination, then* α *is not atomic.*

Theorem 13. *If* X/β, x/α *and* $Y \equiv [x]X$, *then* $Y/\mathbf{F}\alpha\beta$.

The proof is by induction on the derivation of X/β, and covers both the weak and the strong algorithm. We have the following cases:

(i) $X \equiv Zx$ where Z does not contain x, and $Y \equiv Z$. The only way to derive Zx/β is from $Z/\mathbf{F}\alpha\beta$ and x/α.

(ii) X does not contain x, hence $Y \equiv \mathbf{K}x$. Here from $\mathbf{K}/\mathbf{F}_2\beta\alpha\beta$ and X/β we obtain by rule (**F**), $\mathbf{K}X/\mathbf{F}\alpha\beta$.

(iii) X is x, hence $\beta \equiv \alpha$, and $Y \equiv \mathbf{I}$. By rule (**I**) we have $\mathbf{I}/\mathbf{F}\alpha\alpha$.

(iv) $X \equiv X_1X_2$. Suppose $Y_1 \equiv [x]X_1$, $Y_2 \equiv [x]X_2$, and $Y \equiv \mathbf{S}Y_1Y_2$.

We have the following derivation:

$$\frac{X_1/\mathbf{F}\mu\beta \qquad X_2/\mu}{X_1X_2/\beta}$$

Using the induction hypothesis we have

$$\cfrac{\cfrac{\mathbf{S}/\mathbf{F}_3\mathbf{F}_2\alpha\mu\beta\mathbf{F}\alpha\mu\alpha\beta \qquad Y_1/\mathbf{F}_2\alpha\mu\beta}{\mathbf{S}Y_1/\mathbf{F}\mathbf{F}\alpha\mu\mathbf{F}\alpha\beta} \qquad Y_2/\mathbf{F}\alpha\mu}{\mathbf{S}Y_1Y_2/\mathbf{F}\alpha\beta}$$

Theorem 14. *If* X/α *and* $X \geq Y$ *then* Y/α.

The proof by induction on the derivation of $X \geq Y$ is easy.[12]

Theorem 15. *If* X/β, Y/α, x/α *and* $Z \equiv [Y/x]X$ *then* Z/β.

By Theorem 13 we have $[x]X/\mathbf{F}\alpha\beta$, hence by rule (**F**), $([x]X)Y/\beta$. Since $([x]X)Y \geq Z$, we have Z/β by Theorem 14.

A combination Y is said to be a β-*normal combination* if the following condition is satisfied: given X,α, and y such that X/α, y/β and X is a normal combination, then $[Y/y]X$ is a normal combination.

Taking X to be y, it follows that if Y is β-normal it is a normal combination.

Lemma 19. *If $X/\mathbf{F}\beta\alpha$, and X is normal and Y is β-normal, then XY is normal.*

Let y be an indeterminate of type β not occurring in X. Then Xy/α. Since Y is β-normal, $[Y/y](Xy) \equiv XY$ is normal.

Theorem 16. *If Y is normal and, Y/β, then Y is β-normal.*

The proof is by induction on the structure of β. We assume that if Z is a normal combination, Z/μ, and μ is a proper part of β, then Z is μ-normal. Now suppose μ is fixed. Let X be some normal combination of length m, such that for some α, X/α. We show that given a normal combination Y, with Y/β and y/β, then the combination $[Y/y]X$ is normal. The proof of this is by induction on m. We have the following cases:

(i) X is an indeterminate. This case is trivial.

(ii) $X \equiv xX_1 \ldots X_k$, $k \geq 1$, where x is not y. Since X/α, we have for some types $\beta_1, \ldots, \beta_k$ that

$$x/\mathbf{F}_k\beta_1 \ldots \beta_k\alpha$$

$$X_i/\beta_i \qquad i = 1, \ldots, k.$$

By the induction on the length, $Z_i \equiv [Y/y]X_i$ is normal. Then $[Y/y]X \equiv xZ_1 \ldots Z_k$ is also normal.

(iii) $X \equiv yX_1 \ldots X_k$, $k \geq 1$. Suppose $[Y/y]X \equiv YZ_1 \ldots Z_k$, where $Z_i \equiv [Y/y]X_i$. Again we have

$$y/\mathbf{F}_k\beta_1 \ldots \beta_k\alpha$$

$$X_i/\beta_i \qquad i = 1, \ldots k$$

It follows that

$$\beta \equiv \mathbf{F}_k\beta_1 \ldots \beta_k\alpha$$

As in case (ii) Z_i is normal. Further since Y/β, by Theorem 15 we have Z_i/β_i. Since β_i is a proper part of β, it follows that Z_i is β_i-normal. Now, from Y/β, using k times Lemma 22 we obtain that $YZ_1 \ldots Z_k$ is a normal combination.

(iv) X is an open combination. Then α is not atomic. Suppose $\alpha \equiv \mathbf{F}\alpha_1\alpha_2$. Take an indeterminate x, distinct from y, and not occurring in X. Then Xx/α_2 and by the induction hypothesis $[Y/y](Xx) \equiv ([Y/y]X)x$ is normal. Hence $[Y/y]X$ is normal.

(v) X is reducible. Suppose that X_1 is obtained from X by a head contraction. By the induction hypothesis $[Y/y]X_1$ is normal. Since $[Y/y]X_1$ is obtained from $[Y/y]X$ by a head contraction, it follows that $[Y/y]X$ is normal.

Theorem 17. *If X/α, then X is a normal combination.*[13]

The proof is by induction on the derivation of X/α. Rules (Ax), (**S**), (**K**) and (**I**) are trivial. Consider rule (**F**) and suppose we have the following derivation, where $X \equiv X_1X_2$.

$$\frac{X_1/\mathbf{F}\beta\alpha \qquad\qquad X_2/\beta}{X_1X_2/\alpha}$$

By the induction hypothesis X_1 and X_2 are normal. By Theorem 16, X_2 is β-normal. Hence, by Lemma 19, X_1X_2 is normal.

Theorem 18. *If $X/\mathbf{F}_k\beta_1 \dots \beta_k\alpha$, where α is atomic then the degree of the normal combination X is not greater than k.*

For indeterminates $y_1, \dots, y_k$ of type $\beta_1, \dots, \beta_k$ respectively, we have $Xy_1 \dots y_k/\alpha$. Since $Xy_1 \dots y_k$ is normal, it reduces by head contractions to a combination Z that is open or closed, according to whether its degree is greater or equal to 0. Further Z/α. Since α is atomic, Z must be closed. Hence the degree of X is not greater than k.

Theorem 19. *If X/α and U is the normal form of X, then U/α.*

The proof is by induction on the length of X, and covers the weak and strong normal forms. We have the following cases:

(i) X is an indeterminate. This case is trivial.

(ii) $X \equiv xX_1 \dots X_k$, $k \geq 1$. This case follows easily using the induction hypothesis.

(iii) X is open. Then α is not atomic, say $\alpha \equiv \mathbf{F}\mu\beta$. Take an indeterminate x not occurring in M, of type μ. Then Xx/β. If V if the normal form of Xx, by the induction hypothesis we have V/β. Since $U \equiv [x]V$, by Theorem 13 we have $U/\mathbf{F}\mu\beta$.

(iv) X is reducible. This case follows easily using the induction hypothesis.

Lemma 20. *If for a closed combination X we have both X/α and X/β then $\alpha \equiv \beta$.*

This follows from the definitions.

Lemma 21. *If X/α, Y/β, $X = Y$ and the strong normal form of X and Y is closed, then $\alpha \equiv \beta$.*

This follows from Theorem 19, and Lemma 20.

Corollary. If X/α, Y/β, $X = Y$, and α is atomic, then $\alpha \equiv \beta$.

We remark here that there are normal combinations for which no type can be assigned. The combination **SII** is an example of such a normal combination.

6. *General models for type theory.* With each type α associate a set α^* in the following way: if α is atomic, then α^* is some given non-empty set; $(\mathbf{F}\beta\alpha)^*$ is a non-empty set of functions, each one with domain β^* and range included in α^*. We shall use the following notation: if $f \in (\mathbf{F}\beta\alpha)^*$, and $b \in \beta^*$, then $[f](b)$ denotes the value obtained when the function f is applied to b. For $n > 1$, if $f \in (\mathbf{F}_n\beta_1 \dots \beta_n\alpha)^*$, and $b_i \in \beta_i^*$, then $[f](b_1, \dots, b_n) = [[f](b_1, \dots, b_{n-1})](b_n)$.

The system of sets α^* is called a *general model* if it satisfies the following conditions:

(i) *For arbitrary* μ,β *and* α, $(\mathbf{F}_3\mathbf{F}_2\mu\beta\alpha\mathbf{F}\mu\beta\mu\alpha)^*$ *contains the function* $\mathbf{S}_{\mu\beta\alpha}$ *defined as follows: for* $f \epsilon (\mathbf{F}_2\mu\beta\alpha)^*$, $g \epsilon (\mathbf{F}_\mu\beta)^*$ *and* $c \epsilon \mu^*$, $[\mathbf{S}_{\mu\beta\alpha}](f,g,c) = [[f](c)]([g](c))$.

(ii) *For arbitrary* μ *and* β, *the set* $(\mathbf{F}_2\mu\beta\mu)^*$ *contains the function* $\mathbf{K}_{\mu\beta}$, *defined as follows: for* $c \epsilon \mu^*$ *and* $b \epsilon \beta^*$ $[\mathbf{K}_{\mu\beta}](c,b) = c$.

(iii) *For arbitrary* β, *the set* $(\mathbf{F}\beta\beta)^*$ *contains the function* $\mathbf{I}_\beta$ *defined for* $b \epsilon \beta^*$, $[\mathbf{I}_\beta](b) = b$.

Note that condition (i) means also that $(\mathbf{F}\mathbf{F}\mu\beta\mathbf{F}\mu\alpha)^*$ contains the function $[\mathbf{S}_{\mu\beta\alpha}](f)$, and that $(\mathbf{F}\mu\alpha)^*$ contains the function $[[\mathbf{S}_{\mu\beta}](f)](g)$. Condition (ii) means also that $(\mathbf{F}\beta\mu)^*$ contains the function $[\mathbf{K}_{\mu\beta}](c)$.

A valuation V, is a mapping which to each indeterminate x of type α gives a value $V(x) \epsilon \alpha^*$. If V is a given valuation, then for each type α and comvination X, a set $V(X,\alpha)$ is defined by induction on the structure of X, as follows:

(i) *If* X *is an indeterminate of type* α, *the* $V(X) \epsilon V(X,\alpha)$.
(ii) $\mathbf{S}_{\mu\beta\alpha} \epsilon V(\mathbf{S}, \mathbf{F}_3\mathbf{F}_2\mu\beta\alpha\mathbf{F}\mu\beta\mu\alpha)$
(iii) $\mathbf{K}_{\mu\beta} \epsilon V(\mathbf{K},\mathbf{F}_2\mu\beta\mu)$
(iv) $\mathbf{I}_\beta \epsilon V(\mathbf{I}, \mathbf{F}\beta\beta)$
(v) *If* $X = YZ$, *and for some type* β, $f \epsilon V(Y,\mathbf{F}\beta\alpha)$, $b \epsilon V(Z,\beta)$ *then* $[f](b) \epsilon V(X,\alpha)$.

Lemma 22. *$V(X,\alpha)$ is non-empty if and only if X/α.*

The proof by induction on the structure of X is trivial.

Lemma 23. *If V_1 and V_2 are two valuations that give the same value to all the indeterminates occurring in X, then for all types α, $V_1(X,\alpha) = V_2(X,\alpha)$.*

By induction on the structure of X it can be shown that $V_1(X,\alpha) \subset V_2(X,\alpha)$ and $V_2(X,\alpha) \subset V_1(X,\alpha)$.

Lemma 24. *If $X \geq Y$, then $V(X,\alpha) \subset V(Y,\alpha)$.*

The proof by induction on the derivation of $X \geq Y$ is easy.

Theorem 20. *If $V(X,\alpha)$ is non-empty, then it contains exactly one element.*

By Lemma 22 X is a normal combination. The proof is by induction on the length of X. We have the following cases:

(i) X is an indeterminate. In this case $V(X)$ is the only element in $V(X,\alpha)$.

(ii) $X \equiv xX_1 \ldots X_k$, $k \geq 1$. Then for given $\beta_1, \ldots, \beta_k$, any element in $V(X,\alpha)$ is of the form $[f](b_1, \ldots, b_k)$ where f is the unique element in $V(x, \mathbf{F}_k\beta_1 \ldots \beta_k\alpha)$ and, by the induction hypothesis, b_i is the unique element in $V(X_i,\beta_i)$.

(iii) X is open. Then α is not atomic, say $\alpha \equiv \mathbf{F}\beta\mu$. Take an indeterminate x of type β not occurring in X. Then Xx/μ, and by the induction hypothesis $V(Xx,\mu)$ contains exactly one element for all valuations V. Now let V be some valuation, and suppose $f \epsilon V(X,\alpha)$, $g \epsilon V(X,\alpha)$. Take $b \epsilon \mu^*$. Consider the valuation V_1 which gives to x the value b, and otherwise is identical with V. Then, by Lemma 24, $V(X,\alpha) = V_1(X,\alpha)$, hence f and g are in $V_1(X,\alpha)$. Also we have $[f](b) \epsilon V_1(Xx,\mu)$ and $[g](b) \epsilon V_1(Xx,\mu)$, hence $[f](b) = [g](b)$. This is true for arbitrary b in β^*; hence $f = g$.

(iv) X is reducible. This case follows easily using the induction hypothesis and Lemma 24.

We now introduce the following notation: if $V(X,\alpha)$ is not empty, then $V^{\alpha}(X)$ denotes the unique element in $V(X,\alpha)$.

Lemma 25. *If X/α, and $X \geq Y$, then $V^{\alpha}(X) = V^{\alpha}(Y)$.*

This is a consequence of Lemma 24.

Lemma 26. *Let $X/\mathbf{F}\beta\mu$, $f = V^{\mathbf{F}\beta\mu}(X)$, and $b \epsilon \beta^*$. Then $[f](b) = V_1^{\mu}(Xx)$, where x is some indeterminate of type β not occurring in X, and V_1 is a valuation that gives to x the value b, and otherwise is identical with V.*

By Lemma 24, $V_1^{\mathbf{F}\beta\mu}(X) = V^{\mathbf{F}\beta\mu}(X) = f$, and by definition $[f](b) = V_1^{\mu}(Xx)$.

Theorem 21. *If X/α, and U is the normal form of X, then $V^{\alpha}(X) = V^{\alpha}(U)$.*

The proof is by induction on the length of X, and covers both the weak and the strong normal forms.

(i) X is an indeterminate. This case is trivial.

(ii) $X \equiv xX_1 \ldots X_k$, $k \geq 1$. This case follows easily using the induction hypothesis.

(iii) X is open. Then α is not atomic, say $\alpha \equiv \mathbf{F}\beta\mu$. Take an indeterminate x of type β, not occurring in X. Suppose $b \epsilon \beta^*$ and V_1 is a valuation that gives to x the value b, and otherwise is identical with V. By Lemma 26 we have

$$[V^{\alpha}(\mathbf{X})](b) = V_1^{\mu}(Xx)$$

$$[V^{\alpha}(U)](b) = V_1^{\mu}(Ux)$$

If Y is the normal form of Xx, by the induction hypothesis

$$V_1^{\mu}(Xx) = V_1^{\mu}(Y)$$

Also $Ux \geq Y$, hence by Lemma 25

$$V_1^{\mu}(Ux) = V_1^{\mu}(Y)$$

It follows that for arbitrary $b \epsilon \beta^*$

$$[V^{\alpha}(X)](b) = [V^{\alpha}(U)](b)$$

Hence that

$$V^{\alpha}(X) = V^{\alpha}(U)$$

(iv) X is reducible. This case follows using the induction hypothesis and Lemma 25.

Corollary. *If* X/α, Y/α *and* $X = Y$, *then* $V^{\alpha}(X) = V^{\alpha}(Y)$.

NOTES

1) This paper is based on part of a doctoral thesis submitted to the Pennsylvania State University in 1963. The author wishes to express sincere thanks to Dr. H. B. Curry for his assistance and to the Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina for a fellowship that allowed him to study in the U.S.A.

2) See [3], [4], [5] and [7].

3) For all the purposes of this paper it would be sufficient to take **S** and **K** as the only primitive combinators. Since **I** plays a fundamental role we think it is convenient to take it as a primitive.

4) In the terminology of Curry there is an operation, called the application operation, which when applied to X and Y produces (XY).

5) In connection with this definition see [7], Chapter 6A.

6) See [9], Section C, Theorems 10 and 12.

7) For a more detailed exposition see [7], Chapter 6.

8) In this paper the expression: "replacing occurrences" includes the case in which no replacement is made.

9) We avoid giving here the details of the proof of uniqueness of the normal form. They can be easily furnished by the reader.

10) See also [6] and [7], Chapter 8.

11) The notation we use for types satisfies the conditions of the Łukasiewicz notation, hence we may avoid writing parentheses. In any formula $\mathbf{F}_n$ must be followed by $n + 1$ arguments.

12) See also [7], Theorem 9C2.

13) A proof of this theorem is given in [7] by means of a formalization in which the technique of Gentzen is used, and a form of the elimination theorem is proved.

BIBLIOGRAPHY

[1] Church, A., A formulation of the simple theory of types, *The Journal of Symbolic Logic*, 5:56-68 (1940).

[2] Church, A., Review of Curry [6], *The Journal of Symbolic Logic*, 7:170-171 (1942).

[3] Curry, H. B., Functionality in combinatory logic, *Proc. Nat. Acad. Sci. U.S.A.*, 20:584-590 (1934).

[4] Curry, H. B., First properties of functionality in combinatory logic. *Tohôku Math. J.*, 41:371-401 (1936).

[5] Curry, H. B., Some advances in the combinatory theory of quantification, *Proc. Nat. Acad. Sci. U.S.A.*, 28:564-569 (1942).

[6] Curry, H. B., The inconsistency of certain formal logics, *The Journal of Symbolic Logic*, 7:115-117 (1942).

[7] Curry, H. B., and Feys, R., *Combinatory logic*, Amsterdam, 1958.

[8] Rosser, J. B., Deux esquisses de logique, Paris and Louvain, 1955.

[9] Rosser, J. B., A mathematical logic without variables, *Ann. of Math.*, 36:127-150 (1935) and *Duke Math., J*, 1:328-355 (1935).

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