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MODAL SYSTEMS IN THE NEIGHBOURHOOD OF T

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Unpublished is the result of B. Sobociński that if we form systems T_n by adjoining to Feys's modal system T the axiom P_n: $CL^n pL^{n+1}p$ where L^n denotes a string of n L-s ($n \geq 0$), then while obviously T_n contains T_{n+1}, the converse is not the case. Hence there are infinitely many systems between S4$=T_1$ and T. We now ask whether the addition of B_1: $LCpLNLNp$ (Lewis's C12) to T_n and T, producing T_n^+ and T^+, similarly yields infinitely many systems between S5$=T_1^+$ and T^+, and show that this is so. Further, let $S1_n^+$ be the $S1_1^+$ of [1] augmented by P_n. Clearly $S1_1^+= T_1^+ =$ S5, while the matrix used in [2] *ad* 2 shows that if $n > 1$, $S1_n^+$ is a proper subsystem of T_n. Evidently, $S1_n^+$ contains $S1_{n+1}^+$. If $S1_n^+$ and $S1_{n+1}^+$ were equivalent, the addition to each of $LCpMp$ would produce equivalent systems; but these would be T_n^+ and T_{n+1}^+ which are not equivalent. Hence $S1_{n+1}^+$ is a proper subsystem of $S1_n^+$. Since infinitely many reductions of modality thus fail in T^+, T and $S1^+$, these all have infinitely many non-equivalent modalities, as has long been known for T.

To prove that for all n, T_n^+ is independent of T_{n+1}^+, we interpret T_n in the domain of $n+1$-sequences each place of which is filled by *1* or *2*. We base the systems on N (negation), L(necessity) and C (implication). If F is N or L, $F(x_1, \ldots, x_{n+1}) = Fx_1, \ldots, Fx_{n+1}$. $Nx_i = 1$ if $x_i = 2$, $Nx_i = 2$ *if* $x_i = 1$. $Lx_i = 2$ if $x_{i-1} = 2$ or $x_i = 2$ or $x_{i+1} = 2$; *otherwise* $Lx_i = 1$. $C(x_1, x_2, \ldots, x_{n+1})(y_1, y_2, \ldots, y_{n+1}) = Cx_1y_1, Cx_2y_2, \ldots, Cx_{n+1}y_{n+1}$; $Cx_iy_i = 2$ *if* $x_i = 1$, $y_i = 2$, otherwise $Cx_iy_i = 1$. A sequence consisting only of *1*-*s* is designated. The reader may like to compare our version of L with those discussed in [3], pp. 23-4 and [4].

Convenient axioms and rules of T are I. Propositional calculus with rules of substitution, detachment and definition applied to II. M = def. NLN; III. From α infer $L\alpha$; *A1* $CLpp$; *A2* $CLCpqCLpLq$. For T^+ we add *A3* $CpLNLNp$, and for T_n^+, *A4* $CL^n pL^{n+1}p$. The method of valuation obviously satisfies I.

Ad III: If no valuation of α contains 2, the same is true of $L\alpha$.

Ad *A1*: For any valuation of p, $Lp = 1$ at any position only if $p = 1$ at that position.

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Ad *A2*: To falsify we should have to obtain at some position x in some valuation, $LCpq = Lp = 1$, $Lq = 2$. But if $LCpq = Lp = 1$ at x, then $Cpq = p = 1$ at x and in both sequences x is flanked only by 1 - s. Hence $q = 1$ at x and in the q-sequence x is flanked only by 1 - s; hence $Lq = 1$ at x.

Ad *A3*: If $p = 1$ at some position in some value-sequence, this position and its flankers (one or both of which may be absent) reads:

p:	*111* or *112* or *211* or *212*, so we proceed,
Np:	*222* or *221* or *122* or *121*
LNp:	*222*
$NLNp$:	*111*

and so $LNLNp = 1$ in the indicated central position.

Ad *A4*: If there is any mixture of *1* and *2* in a value-sequence, each application of L reduces the number of 1 - s. The slowest reduction is when n consecutive 1 - s are preceded or followed by *2*. But the application of L^n even to such a sequence reduces it to a sequence of 2 - s and so verifies P_n for that valuation of p.

From these remarks it is clear that T_n is satisfied for all n. To show that $CL^n pL^{n+1}p$ fails in T_{n+1} we take the valuation $p = 11\ldots12$ (with $n + 2$ places); then $L^n p = 12\ldots22$ and $L^{n+1}p = 22\ldots22$ so that the proposition obtains the value $21\ldots11$.

By ordering lexicographically, with *1* preceding *2*, each set of 2^{n+1} $n + 1$-sequences, numbering them from *1* through k, and considering the displacements effected by the functors within each set and by the passage from each set to the next largest, we obtain the following matrices which could also be used for the proof of our result.

$$\mathfrak{M}_0 = \begin{array}{c|cc} C_0 & 1 & 2 \\ \hline {*}1 & 1 & 2 \\ 2 & 1 & 1 \end{array} \qquad \mathfrak{M}_{n+1} = \begin{array}{c|c|c} C_{n+1} & 1 \;\ldots\ldots\ldots & k_{n+1} \\ \hline {*}1 & M_n & M_n + \tfrac{1}{2}k_{n+1} \\ \vdots & & \\ \hline \vdots & M_n & M_n \\ k_{n+1} = 2^{n+2} & & \end{array}$$

For all n, $N_n i = k_n + 1 - i$, $(1 \leq i \leq k_{n+1})$. $L_{-1} = 1$; $L_0 1 = 1$, $L_0 2 = 2$; the first quarter of L_{n+1} = the first half of L_n; the second and fourth quarters of L_{n+1} = the second half of $L_n + \frac{1}{2}k_{n+1}$; the third quarter of $L_{n+1} = L_{n-1} + \frac{3}{4}k_{n+1}$. Thus $L_1(1234) = 1444$, $L_2(12345678) = 14887888$, $L_3(1\ldots.16) = 1\ 4\ 8\ 8\ 15\ 16\ 16\ 16\ 13\ 16\ 16\ 16\ 15\ 16\ 16\ 16$, *etc.*

Either of the values *2* or $\frac{1}{2}k_{n+1}$ will reject P_n in $\mathfrak{M}_{n+1}$.

Sobociński's result relating T to S4 can be directly obtained with the same ease as ours above by either of two decompositions of our original L-valuation. We put:

$Ux_i = 2$ if $x_{i-1} = 2$ or $x_i = 2$, and otherwise $Ux_i = 1$.
$Vx_i = 2$ if $x_i = 2$ or $x_{i+1} = 2$, and otherwise $Vx_i = 1$.

Then U, V both satisfy the L of T_n, T, but $A3$ is rejected and P_n in T_{n+1}. On translation into square matrices C_n, N_n remain as before, $U_0 = V_0 = L_0$, and thereafter:

the first half of $U_{n+1} = U_n$,
the third quarter of U_{n+1} = the first quarter of $U_{n+1} + \frac{3}{4}k_{n+1}$,
the fourth quarter of U_{n+1} = twice the second quarter of U_{n+1};
the first quarter of V_{n+1} = the first half of V_n,
the second and fourth quarters of V_{n+1} = the second half of $V_n + \frac{1}{2}k_{n+1}$,
the third quarter of V_{n+1} = the first half of $V_{n+1} + \frac{1}{2}k_{n+1}$.

Thus we have:

$U_1(1234) = 1244$,
$U_2(12345678) = 12447888$
$U_3(1....16) = 1\ 2\ 4\ 4\ 7\ 8\ 8\ 8\ 13\ 14\ 16\ 16\ 14\ 16\ 16\ 16$,
$V_1(1234) = 1434$,
$V_2(12345678) = 14785878$,
$V_3(1....16) = 1\ 4\ 7\ 8\ 13\ 16\ 15\ 16\ 9\ 12\ 15\ 16\ 13\ 16\ 15\ 16$.

Using U_{n+1}, the value 2 rejects P_n in $\mathfrak{M}_{n+1}$; using V_{n+1}, the value $\frac{1}{2}k_{n+1} + 1$ effects this. Neither of these series of matrices is the one originally used.

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