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A STRONGER FORM OF A THEOREM OF FRIEDBERG

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In Friedberg [1], Theorem I, it is shown that every nonrecursive recursively enumerable (n.r.e.) set is the union of two disjoint n.r.e. sets. The proof is based on the simultaneous enumeration of recursively enumerable (r.e.) sets. On the other hand, Suzuki, [2], presents the simultaneous enumeration of recursive sets.

In this note, applying the method of Friedberg to the simultaneous enumeration of recursive sets, we will prove the following theorems:

Theorem 1. *For any given n.r.e. set* S, *there is a finite sequence of r.e. sets* $S_1, S_2, \ldots S_n$ *such that*

1 $\displaystyle\bigcup_{j=1}^{n} S_j = S,$

2 $S_i \cap S_j = \phi$ *(empty set)* for $i \neq j$,

3 *for any* j $(j = 1, 2, \ldots, n)$, *there is no recursive set* R *such that* $S_j = S \cap R$.

Theorem 2. *For any given n.r.e. set* S, *there is an infinite sequence of r.e. sets* $S_1, S_2, \ldots$ *such that*

1 $\displaystyle\bigcup_{j=1}^{\infty} S_j = S$

2 $S_i \cap S_j = \phi \quad (i \neq j)$

3 *for any integer* $j > 0$, *there is no recursive set* R *such that* $S_j = S \cap R$.

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Proof of Theorem 1. Let $E = \{S, R_1, R_2, \dots\}$ be a simultaneous enumeration of S and all recursive sets, and S^a or R_i^a denote the set of integers which, at or before step a in E, have been listed as members of S or R_i.

S_j^a is the set of integers which, at or before step a in E, have been listed as members of S_j according to the rule below. R_i is called satisfied at step a if $R_i^a \cap R_j^a \neq \phi$ for any $j = 1, 2, \dots, n$.

S_j is the r.e. set which is constructed as follows; $S_j^0 = \phi$ $(j = 1, 2, \dots, n)$. Suppose $n \in S^a - S^{a-1}$.

(1) Let i_a be the lowest integer such that $n \in R_{i_a}^a$ and $R_{i_a}^a$ is unsatisfied at step a in E. If j_a is the lowest integer such that $R_{i_a}^a \cap S_{j_a}^{a-1} = \phi$, then let n be listed as members of S_{j_a}. (i_a is attacked at step a).

(2) If any R_i that contains n is satisfied or if there is no R that contains n, then let n be listed as a member of S_i. It is evident that S_j $(j = 1, 2, \dots, n)$ is r.e., and $S = \bigcup_{j=1}^{n} S_j$. Suppose that there are integers i, j such that $S_j = S \cap R_i$. Let R_k be the complement of R_i. Since both R_i and R_k are unsatisfied at any step a and there is at most a finite number of S_j, there is a step a_0 such that neither i nor k is attacked after step a_0. Hence, after step a_0, if $n \bar{\in} S^a$ and $n \in R_i^a \cup R_k^a$, then $n \in \bar{S}$, where $\bar{S}$ is the complement of S. Moreover, for any integer n, there is an integer a_n such that $n \in R_i^a \cup R_k^a$ for $a > a_n$. Thus

$$S = (R_i^{a_0} \cup R_k^{a_0} - S^{a_0}) \cup \{\bigcup_{a > a_0} [x \mid (x \bar{\in} S^a - S^{a-1}) \;\&\; (x \in R_i^a \cup R_k^a)]\},$$

and then $\bar{S}$ is r.e. that is contrary to the hypothesis of S being n.r.e. set. Therefore, there is no j such that $S_j = S \cap R_i$ for some i. Thus the proof is complete.

Proof of Theorem 2. R_i is called k-satisfied if $R_i^a \cap S_j^a \neq \phi$ and $i + j = k$. Let S_j be the r.e. set that is constructed as follows;

$$S_j^0 = \phi \quad (j = 1, 2, \dots)$$

Suppose $n \in S^a - S^{a-1}$.

(1) Let k_a be the lowest integer such that there are S_j^a, R_i^a, where R_i^a is k_a-unsatisfied and $n \in R_i^a$. Moreover, let i_a be the lowest integer such that $R_{i_a}^a$ is k_a-unsatisfied and $n \in R_{i_a}^a$. (i is attacked.) Then let n be listed as member of $S_{k_a - i_a}$.

(2) If there is no R_i^a that contains n, let n be listed as member of S_1.

It is evident that S_j $(j = 1, 2, \dots)$ is r.e. and that $S = \bigcup_{j=1}^{\infty} S_j$.

Suppose there are integers i, j such that $S_j = S \cap R_i$, and that R_k is the complement of R_i. R_i is $i+j+1$ - unsatisfied and R_k is $k+j$ - unsatisfied. Moreover, for R_t s - unsatisfied, t can not be attacked more than $s - t$ times. Hence, there is a step a_0 such that neither i nor k is attacked after step a_0. Therefore,

$$\bar{S} = (R_i^{a_0} \cup R_k^{a_0} - S^{a_0}) \cup \{\bigcup_{a > a_0} [x \mid (x \in S^a - S^{a-1}) \;\&\; (x \in R_i^a \cup R_k^a)]\}$$

and then $\bar{S}$ (the complement of S) is r.e. that is contrary to the hypothesis of S to be n.r.e. Thus, Theorem 2 is proved.

Theorem 3 (Friedberg). *For given n.r.e. set S, there is a finite sequence of n.r.e. sets $S_1, S_2 \ldots, S_n$ such that*

(1) $$S = \bigcup_{j=1}^{n} S_j ,$$

(2) $$S_i \cap S_j = \phi \qquad (i \neq j)$$

Proof. Let S_j be the r.e. set given in Theorem 1. If S_j is a recursive set, then $S_j = R_i$ for some i. Hence,

$$S_j = S \cap S_j = S \cap R_i .$$

That is contrary to Theorem 1.

Similarly, we have

Theorem 4. *For any given n.r.e. set S, there is an infinite sequence of n.r.e. sets $S_1, S_2, \ldots$ such that*

(1) $$S = \bigcup_{j=1}^{\infty} S_j ,$$

(2) $$S_i \cap S_j = \phi \qquad (i \neq j) .$$

REFERENCES

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