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# ON NOT STRENGTHENING INTUITIONISTIC LOGIC

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We wish to reexamine —in the wake of R. E. Vesley's [7] —the question of converting so-called structural and intelim rules for $PC_I$, the intuitionistic sequenzen-kalkül of Gentzen, into rules for $PC_C$, the classical sequenzen-kalkül. We shall limit ourselves here to sequenzen or turnstile statements of the form $A_1, A_2, \ldots, A_n \vdash B$, where $A_1, A_2, \ldots, A_n$, $(n \geqslant 0)$, and $B$ are wffs consisting of propositional variables, zero or more of the connectives '&', 'v', '$\sim$', '$\supset$', and '$\equiv$', and zero or more parentheses.

One can pass from $PC_I$ to $PC_C$ by amending the intelim rules for '$\sim$', a result of long standing, or by amending the intelim rules for either one of '$\supset$' and '$\equiv$', a more recent find.[1] In a talk at Yale University in 1961, however, Leblanc conjectured that amending the intelim rules for either one of '&' and 'v' will not do the trick. The point, mentioned in Leblanc [4], appears as follows in Leblanc and Belnap [5]:

> We also conjecture, by the way, that any structural rule which holds in $PC_C$ also holds in $PC_I$, that any elimination or introduction rule for '&' and 'v' which holds in $PC_C$ also holds in $PC_I$, and hence that the only way of turning standard Gentzen rules of inference for $PC_I$ into rules for $PC_C$ is to strengthen the elimination or introduction rules for '$\sim$' or those for '$\supset$', or those for '$\equiv$'.

Leblanc's conjecture, to which we devote the rest of this paper, has had a rather checkered career: proved true at one time or another by three different writers in two different ways, it has also been proved false once.[2] To resolve this seeming contradiction and sort out what has been proved true and what false, we shall have another look at some of the key terms in the above quotation. It will turn out that the readings of Leblanc's conjecture in [1], [2], and [7] are not quite apposite, and that of two fresh ones which we consider here one is unrestrictedly true, while the other holds under a slight restriction.

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# I

To simplify our analysis of the conjecture, we shall ignore the distinction between introduction and elimination rules and concentrate, for the time being, on so-called rules for 'v'. Under these provisos the conjecture comes to read:

*Any rule for 'v' which holds in $PC_C$ also holds in $PC_I$.*

What, however, are we to understand by *a rule for* 'v' and what by a rule for 'v' *holding in $PC_C$, $PC_I$, or any system S*?

Two interpretations of *holding in a system S* suggest themselves. (i) A rule **R** might be said to hold in a system *S* if the conclusion of **R** is obtainable by means of the axioms and primitive rules of *S* from the premisses of **R** or, to put it more briefly, if **R** is provable in *S*.[3] (ii) A rule **R** might also be said to hold in a system *S* if the premisses of **R** cannot be theorems of *S* without the conclusion of **R** being also a theorem of *S* or, to borrow Lorenzen's term, if **R** is admissible in *S*. We thus wind up with two possible readings of Leblanc's conjecture:

(1) Any rule for 'v' which is provable in $PC_C$ is also provable in $PC_I$ and

(2) Any rule for 'v' which is admissible in $PC_C$ is also admissible in $PC_I$.

At least three interpretations of *a rule for* 'v' come to mind. (i′) A rule **R** might be said to qualify as a rule for 'v' if **R**-thought of as a wholesale metastatement—exhibits no connective but 'v'. According to this interpretation Vesley's rule, to wit:

**R1**: *From $\vdash A^*$ and $A, P \vdash Q$ to infer $\vdash A \vee P$, where P and Q are distinct propositional variables, A contains no propositional variable but P, and $A^*$ is obtainable from A by substitution,*

passes as a rule for 'v' and is disproof of (2), as Vesley has shown. **R1**, of course, is no disproof of (1), since it is not provable in $PC_C$.[4] Other rules which also pass as rules for 'v' according to interpretation (i′) are, however, disproofs of (1), among them:

**R2**: *From $A \vdash A$ to infer $\vdash A \vee B$, where A is a propositional variable and B consists of A and one extra sign.*[5]

Since rules like **R1** and **R2** were definitely not contemplated by Leblanc, interpretation (i′) is out of the question and hence Vesley's reading of (2) is inapposite.

According to a second interpretation, (ii′) a rule **R** might be said to qualify as a rule for 'v' if every inference condoned by **R** exhibits no connective but 'v'. With *a rule for* 'v' thus understood, not only do **R1** and **R2** fail to pass as rules for 'v', but both (1) and (2) hold true. Proof of (1) is to be found in [1] and [2]. As for (2), it follows from a theorem of Leblanc's in [6] to the effect that any turnstile statement $A_1, A_2, \ldots, A_n \vdash B$ which exhibits no connective but 'v' is provable in $PC_I$ if provable in $PC_C$.

Interpretation (ii′), however, is much too limiting, and hence the reading of (1) in [1] and [2] is also inapposite. Consider indeed Gentzen's introduction rule for 'v':

**R3**: *From* $A_1, A_2, \ldots, A_n \vdash B$ *to infer* $A_1, A_2, \ldots, A_n \vdash B \vee C$ (*or* $C \vee B$).

The rule was meant to condone such inferences as $p \vdash q \vee r$ from $p \vdash q$, which exhibit no connective but 'v'. But it was also meant to condone such inferences as $p \vdash (q \,\&\, r) \vee s$ from $p \vdash q \,\&\, r$, which, though exhibiting a connective other than 'v', exhibit it—so to speak—*inessentially*.

A third and more attractive interpretation is available, however, to wit: (iii′) A rule **R** is to qualify as a rule for 'v' if every inference condoned by **R** can be gotten by substitution from some inference condoned by R that exhibits no connective but 'v'. With *a rule for* 'v' thus understood, **R2** (as well as **R1**) fails again to pass as a rule for 'v',[6] and (1) proves to be true, as we shall demonstrate below (see Conjecture Ic). (2), on the other hand, is false. Consider indeed the rule which condones only the following two inferences:

**R4**: (a) *From* $\vdash p$ *to infer* $\vdash q \vee r$;
(b) *From* $\vdash p \supset p$ *to infer* $\vdash p \vee \sim p$.

**R4** passes as a rule for 'v' according to interpretation (iii′); it is admissible in $PC_C$; and yet it is not admissible in $PC_I$.

Though false of some of the rules which according to interpretation (iii′) pass as rules for 'v', (2) is nonetheless true of all those closed under substitution, that is, of all those which condone any inference gotten by substitution from some inference already condoned by them. As a matter of fact, not only can it be shown that

(2′) Any rule for 'v' which is closed under substitution and is admissible in $PC_C$ is admissible in $PC_I$;

it can even be shown (see Conjecture IIc below) that

(2″) Any rule for 'v' which is closed under substitution and is admissible in $PC_C$ is provable in $PC_I$.

Vesley's rule, by the way, is not closed under substitution,[7] whereas all of Gentzen's rules are.

One final word before we tackle our new readings of Leblanc's conjecture. As the above attests, the inferences which a rule condones may matter more than the metaterms in which it is couched. We shall accordingly identify a rule with the set of all the inferences it condones or—to put it more briefly—with the set of all its instances. We shall also treat an instance of a rule as an ordered pair $<\Sigma, \mathbf{T}>$, $\Sigma$ consisting of all the turnstile statements which for the occasion do duty as premisses and **T** being the turnstile statement which does duty as conclusion. This departure from customary ways of thinking about a rule has further advantages into which we cannot go here.

II

A number of definitions are first in order, which we rehearse in five batches.

By a well-formed formula (wff) we shall understand any propositional variable, any expression of the form $\sim A$, where $A$ is a wff, and any expression of one of the forms $(A \& B)$, $(A \vee B)$, $(A \supset B)$, and $(A \equiv B)$, where $A$ and $B$ are wffs. By a turnstile statement (**T**-statement) we shall understand any expression of the form $A_1, A_2, \ldots, A_n \vdash B$, where $A_1, A_2, \ldots, A_n$ $(n \geq 0)$, and $B$ are wffs. And by the wff-associate of a **T**-statement $A_1, A_2, \ldots, A_n \vdash B$ we shall understand the wff B or the wff $(A_1 \& A_2 \& \ldots \& A_n) \supset B$, according as $n = 0$ or $n > 0$.

Our next batch of definitions has to do with the notion of a rule. By a premisses-conclusion pair we shall understand any ordered pair $<\Sigma, \mathbf{T}>$, where $\Sigma$ is a finite (and possibly empty) set of **T**-statements and **T** is a **T**-statement. By a rule we shall understand any set of premisses—conclusion pairs, and by an instance of a rule any member of a rule. Given two premisses—conclusion pairs $<\Sigma, \mathbf{T}>$ and $<\Sigma', \mathbf{T}'>$, we shall say that $<\Sigma, \mathbf{T}>$ yields $<\Sigma', \mathbf{T}'>$ (or, equivalently, $<\Sigma', \mathbf{T}'>$ is obtainable from $<\Sigma, \mathbf{T}>$) by substitution if, $v_1, v_2, \ldots$, and $v_p$ being all the propositional variables that occur in $<\Sigma, \mathbf{T}>$ and $A_1, A_2, \ldots$, and $A_p$ being (not necessarily distinct) wffs, $<\Sigma', \mathbf{T}'>$ is like $<\Sigma, \mathbf{T}>$ except for exhibiting, for each i from *1* to $p$, $A_i$ wherever $<\Sigma, \mathbf{T}>$ exhibits $v_i$. We shall say that **R** is a rule for $K$, where $K$ is a (possibly empty) subset of $\{\&, \vee, \sim, \supset, \equiv\}$, if every instance of **R** is obtainable by substitution from some instance of **R** which exhibits only connectives in $K$.[8] And we shall say that a rule **R** is closed under substitution if every premisses-conclusion pair obtainable by substitution from some instance of **R** is an instance of **R**.

Our third batch of definitions has to do with the semantical notion of validity. We shall say that a **T**-statement **T** is classically valid or, for short, $C$-valid (intuitionistically valid or, for short, $I$-valid) if the wff-associate of **T** is $C$-valid ($I$-valid).[9] We shall say that a premisses-conclusion pair $<\{\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_n\}, \mathbf{T}>$ $(n \geq 0)$ is $C$-valid ($I$-valid) if the **T**-statement $\mathbf{T}_1^*, \mathbf{T}_2^*, \ldots, \mathbf{T}_n^* \vdash \mathbf{T}^*$, where $\mathbf{T}_1^*, \mathbf{T}_2^*, \ldots, \mathbf{T}_n^*$, and $\mathbf{T}^*$ are the wff-associates of $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_n$, and **T**, respectively, is $C$-valid ($I$-valid). We shall say that a rule **R** is $C$-valid ($I$-valid) if every instance of **R** is $C$-valid ($I$-valid). And we shall say that a rule **R** is weakly $C$-valid (weakly $I$-valid) if for every instance $<\Sigma, \mathbf{T}>$ of **R** either some member of $\Sigma$ is not $C$-valid ($I$-valid) or else **T** is $C$-valid ($I$-valid).

Our fourth batch of definitions has to do with the syntactical notions of provability and admissibility. We shall say that a premisses-conclusion pair $<\Sigma, \mathbf{T}>$ is provable by means of a set of rules **S** if **T** is the last entry in a column of **T**-statements $\mathbf{T}_1, \mathbf{T}_2, \ldots$, and $\mathbf{T}_r$ such that, for each $i$ from *1* to $r$, $\mathbf{T}_i$ belongs to $\Sigma$ or is preceded in the column by $s$ $(s \geq 0)$ **T**-statements $\mathbf{T}_{i_1}, \mathbf{T}_{i_2}, \ldots$, and $\mathbf{T}_{i_s}$ such that $<\{\mathbf{T}_{i_1}, \mathbf{T}_{i_2}, \ldots, \mathbf{T}_{i_s}\}, \mathbf{T}_i>$ is an instance of a rule in **S**.[10] We shall say that a **T**-statement **T** is provable by means of a set of rules **S** if $<\phi, \mathbf{T}>$ is so provable. We shall say that a rule

**R** is provable by means of a set of rules **S** if every instance of **S** is so provable. We shall say that a premisses-conclusion pair, a **T**-statement, or a rule is provable in $PC_C$ ($PC_I$) if it is provable by means of the set of rules for $PC_C$ ($PC_I$) in Leblanc and Belnap.[5] And we shall say that a rule **R** is admissible in $PC_C$ ($PC_I$) if for every instance $<\Sigma, \mathbf{T}>$ of **R** either some member of $\Sigma$ is not provable in $PC_C$ ($PC_I$) or else **T** is provable in $PC_C$ ($PC_I$).

Turning, lastly, to a syntactico-semantical notion, we shall say that the rules in a set **S** are intuitionistically rule-complete if every *I*-valid rule is provable by means of **S**.

With these definitions out of the way, we next proceed to formulate our conjectures. For expository reasons we offer four equivalent versions of each.

*Conjecture Ia.* Any *C*-valid rule for {&, v} is provable in $PC_I$.
*Conjecture Ib.* Any *C*-valid rule for {&, v} is *I*-valid.
*Conjecture Ic.* Any rule for {&, v} which is provable in $PC_C$ is provable in $PC_I$.
*Conjecture Id.* Let the rules in **S** be intuitionistically rule-complete, and let **R** be a *C*-valid rule for {&, v}. Then any rule (and hence any **T**-statement) provable by means of **S** and **R** is provable by means of **S** alone.
*Conjecture IIa.* Any weakly *C*-valid rule for {&, v} which is closed under substitution is provable in $PC_I$ (and hence admissible in $PC_I$).
*Conjecture IIb.* Any weakly *C*-valid rule for {&, v} which is closed under substitution is *I*-valid (and hence weakly *I*-valid).
*Conjecture IIc.* Any rule for {&, v} which is closed under substitution and is admissible in $PC_C$ is provable in $PC_I$ (and hence admissible in $PC_I$).
*Conjecture IId.* Let the rules in **S** be intuitionistically rule-complete, and let **R** be a weakly *C*-valid rule for {&, v} which is closed under substitution. Then any rule (and hence any **T**-statement) provable by means of **S** and **R** is provable by means of **S** alone.

Of the foregoing, Ib (IIb) follows from Ia (IIa) and the fact that any rule provable in $PC_I$ is *I*-valid; Ic follows from Ia and the fact that any rule provable in $PC_C$ is *C*-valid; IIc follows from IIa and the fact that any rule admissible in $PC_C$ is weakly C-valid; and Id (IId) follows from Ia (IIa) and the fact that any rule provable in $PC_I$ is provable by means of the set of rules **S** mentioned in Id (IId). We accordingly restrict ourselves to proving Ia and IIa, the latter via a rather interesting lemma concerning classical validity.

*Conjecture Ia: Any C-valid rule for* {&, v} *is provable in* $PC_I$.

*Proof*: Let **R** be a *C*-valid rule for {&, v} and $<\Sigma, \mathbf{T}>$ be an instance of **R**. Then there is bound to be a premisses-conclusion pair $<\Sigma', \mathbf{T}'>$ which is an instance of **R**, exhibits only connectives in {&, v}, and yields $<\Sigma, \mathbf{T}>$ by substitution. But if $<\Sigma', \mathbf{T}'>$ is an instance of **R**, then $<\Sigma', \mathbf{T}'>$ is bound to be *C*-valid; if $<\Sigma', \mathbf{T}'>$ is *C*-valid and exhibits only connectives in {&, v}, then $<\Sigma', \mathbf{T}'>$ is bound by a result of Belnap and Thomason's in

[1] to be provable in $PC_I$;[1] and if $<\Sigma',\mathsf{T}'>$ is provable in $PC_I$ and yields $<\Sigma,\mathsf{T}>$ by substitution, then $<\Sigma,\mathsf{T}>$ is also bound to be provable in $PC_I$, since substitution is provability-preserving. Hence **R** is bound to be provable in $PC_I$.

*Lemma 1: Any weakly C-valid rule which is closed under substitution is C-valid.*

*Proof*: Suppose **R** is closed under substitution and is not $C$-valid. Then there is bound to be an instance of **R**, say $<\Sigma,\mathsf{T}>$, which is not $C$-valid, and hence there is bound to be an assignment of truth-values to the propositional variables in $<\Sigma,\mathsf{T}>$ which satisfies every member of $\Sigma$ but fails to satisfy **T**. Now consider the result $<\Sigma',\mathsf{T}'>$ of substituting '$p \vee \sim p$' for every propositional variable in $<\Sigma,\mathsf{T}>$ which is assigned the truth-value T in the said assignment and '$p \,\&\sim p$' for every one which is assigned the truth-value F. Every member of $\Sigma'$ is bound to be $C$-valid, while $\mathsf{T}'$ is not. Hence $<\Sigma',\mathsf{T}'>$ is not weakly $C$-valid. But $<\Sigma',\mathsf{T}'>$ is bound to be an instance of **R**, since **R** is closed under substitution. Hence **R** is not weakly $C$-valid.

*Conjecture IIa: Any weakly C-valid rule for* $\{\&,\vee\}$ *which is closed under substitution is provable in* $PC_I$.

*Proof* by Conjecture Ia and Lemma 1.

It should now be clear that any structural rule, any rule for '&', and any rule for 'v' which holds in $PC_C$ in the sense of being provable in $PC_C$ also holds in $PC_I$, and that any structural rule, any rule for '&', and any rule for 'v' which holds in $PC_C$ in the sense of being admissible in $PC_C$ also holds in $PC_I$, so long in the latter case as the rule is closed under substitution. It should likewise be clear that the only way of turning standard Gentzen rules for $PC_I$ into rules for $PC_C$ or, to be more explicit about it, of so amending the former rules as to permit proof of any $C$-valid rule and T-statement, is to strengthen the rules for '$\sim$', or those for '$\supset$', or those for '$\equiv$'.

## NOTES

1. See [3] and [5].
2. See [1] and [2] for proofs of the conjecture, [7] for a disproof of it.
3. The literature talks of primitive and derived rules; it has no label that we know of for rules which, whether or not they be derived from the primitive rules of a system, can be derived from those rules. We use the term *provable* for lack of a better one.
4. **R1** condones such an inference as $\vdash p \vee p$ from $\vdash p \supset p$ and $p, p \vdash q$, even though $\vdash p \vee p$ is not obtainable from $\vdash p \supset p$ and $p, p \vdash q$ by means of the axioms and primitive rules of $PC_C$. **R1** is therefore not provable in $PC_C$.

5. Because of the wording of **R2**, $B$ has to be $\sim A$, where $A$ is a propositional variable, say '$p$'. But $\vdash p \vee \sim p$, though obtainable from $p \vdash p$ by means of the axioms and primitive rules of $PC_C$, is not obtainable from $p \vdash p$ by means of the axioms and primitive rules of $PC_I$. Though provable in $PC_C$, **R2** is therefore not provable in $PC_I$.

6. **R2** fails to pass as a rule for 'v' according to interpretation (iii′) because such an inference as $\vdash p \vee \sim p$ from $p \vdash p$, though condoned by **R2**, cannot be gotten by substitution from any inference condoned by **R2** that exhibits no connective but 'v'. Similarly, **R1** fails to pass as a rule for 'v' according to interpretation (iii′), because such an inference as $\vdash \sim p \vee p$ from $\vdash \sim p$ and $\sim p, p \vdash q$, though condoned by **R1**, cannot be gotten by substitution from any inference condoned by **R2** that exhibits no connective but 'v'.

7. Note for proof that such an inference as $\vdash q \vee \sim q$ from $\vdash \sim q$ and $\sim q, q \vdash q$, though gotten by substitution from an inference condoned by **R1**, is not itself condoned by **R1**.

8. Note that what we called above a structural rule proves under the present terminology to be a rule for any subset of $\{\&, \vee, \sim, \supset, \equiv\}$ and hence for $\{\&, \vee\}$, and that what we called a rule for '&' or a rule for 'v' proves under the present terminology to be a rule for $\{\&, \vee\}$.

10. Axiom schemata can be viewed as rules with instances of the form $\langle \phi, \mathbf{T} \rangle$ and hence do not call here for separate mention.

11. The rules that figure in [1], though different from the Leblanc-Belnap ones in [5], are provable in $PC_I$, as the reader may verify on his own.

12. In the absence of a handy criterion of intuitionistic validity the reader may take a wff (and hence the wff-associate of a **T**-statement **T**) to be intuitionistically valid if it is provable as a theorem of Heyting's propositional calculus. We of course take a wff to be classically valid if it is a tautology.

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