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VARIATIONS ON A THEME OF BERNAYS*

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I

The system of Bernays [1] is a first-order one, having *membership* (ϵ), *abstraction* ($\{x| \ldots x \text{--}\}$) and the *Hilbert selector* (σ) as its (set-theoretic) primitives. We wish first to formulate this system without $\{x| \ldots x - \}$ or σ; this will permit a readier comparison of the reflection-principle (III below) with the weaker reflection-principles of e.g., Levi's [2].

The axioms are as follows:

I. $x = y \ \& \ x \in z \rightarrow y \in z$
II. $x \in \{y| \ldots y - \} \leftrightarrow \ \ldots x - \ \& (\exists z)(x \in z)$
III. $\phi \rightarrow (\exists x)(\mathrm{SC}\ x - \ \& \ \mathrm{Rel}(\phi, x))$
IV. $x \in y \rightarrow \sigma y \in y \ \& \ \sigma y \cap y = O.$

Before we explain the meanings of the abbreviations '=', 'SC', 'Rel', '$\cap$', 'O' occurring in I-IV, we point out two minor divergences from the original formulation in [1]. (1). The axiom $x = y \quad \sigma x = \sigma y$ has been dropped, because we showed in [3] that it is dependent. (2). Bernays' system is built on a functional calculus in the Hilbert-Ackermann style (so that in particular distinct letters are used for free and bound variables, and vacuous quantification and abstraction are prohibited). It will be more convenient for our purposes to take as a logical basis the functional calculus of Quine [4], p. 88, where however (2a) in order to make the definition of 'Rel' below unambiguous, conjunction and negation rather than joint denial are taken as primitive connectives (2b) *103 is strengthened to read $\vdash \ulcorner(\alpha)(\ldots \alpha, \text{--}) \rightarrow \ldots \zeta \text{--}\urcorner$ for any term ζ and (2c) for convenience we permit ourselves to drop initial universal quantifiers in statements of theorems. (In the axiomatic formalization B_1 below, as well as its later variants, Quine's form of the functional calculus can be taken over intact.)

$x = y$ is short for $(w)(w \in x \leftrightarrow w \in y)$

$\mathrm{SC}(x)$ is short for $(\exists z)(x \in z) \ \& \ (w)(y)(w \in x \ \& \ (y \in w \vee y \subset w) \rightarrow y \in x)$ where

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$y \subset w$ is short for $(u)(u \in y \to u \in w)$
$x \cap y$ is short for $\{z | z \in x \ \& \ z \in y\}$
O is short for $\{x | x \neq x\}$, i.e. $\{x | u(x = x)\}$.

A free occurrence of a variable α in a formula ϕ is called a *set-occurrence* if ϕ is a conjunction one of whose conjuncts has the form $\ulcorner \alpha \in \zeta \urcorner$, otherwise a *class occurrence*.

If α does not occur in ϕ (x is supposed not to occur in ϕ in III), then $\mathrm{Rel}(\phi,\alpha)$ is obtained from α by replacing every universal quantifier $\ulcorner(\beta)\urcorner$ by $\ulcorner(\beta \subset_\alpha)\urcorner$, every abstraction-prefix $\ulcorner\{\beta|\urcorner$ by $\ulcorner\{\beta_{\epsilon\alpha}|\urcorner$, and every class-occurrence of a free variable β by $\beta\cap\alpha$. Set-occurrences of free variables are left unchanged. And here

$$\ulcorner(\beta_{\subset \alpha})\psi\urcorner \text{ is short for } \ulcorner(\beta)(\beta \subset \alpha \to \psi)\urcorner$$

and

$$\ulcorner\{\beta_{\epsilon\alpha} | \psi\}\urcorner \text{ is short for } \ulcorner\{\beta | \beta \in \alpha \ \& \ \psi\}\urcorner.$$

This completes the specification of the system.

II

It is easy to eliminate σ, i.e. to replace IV by two further axioms IV_1 and IV_2 such that the consequences of I-III, IV_1, IV_2 are precisely those consequences of I-IV which do not contain σ.

Specifically we write

IV_1. $(\exists f)(x)(x \in V \to x \neq O \to f(x) \in x)$
IV_2. $x \neq O \to (\exists y)(y \in x \ \& \ y \cap x = O)$.

In IV 'V' abbreviates $\{x | x = x\}$ (so that $\vdash x \in V \leftrightarrow (\exists y)(x \in y)$ and $f(x)$ is defined as

$$\{y | (\exists z)(y \in z \ \& \ (w)((x,w) \in f \leftrightarrow w = z))\}$$

where

$$(x,w) = \{\{x\}, \{x,w\}\}$$
$$\{x,w\} = \{y | y = x \ \mathrm{V} \ y = w\}$$

and

$$\{x\} = \{x,x\}.$$

The proof that IV is equivalent to IV_1 and IV_2 is quite routine. The only point to be watched is the interpretation of σy in IV. If y is a set (i.e. $y \in V$), σy is evidently

$$f(\{z | z \in y \ \& \ z \cap y = O\})$$

with f as in IV_1; but if y is a proper class, σy has to be

$$f\{z | z \in y_O \ \& \ z \wedge y_O = O\}$$

where y_O is the intersection of y with the least rank r for which $r \cap y \neq O$.
The system of axioms I-III, IV_1, IV_2 will be denoted by B.

III

Now we seek to eliminate $\{x \mid \ldots x -\}$ from B, i.e. to replace axioms II, III and IV_1 of B by axioms II', III', and IV_1' such that the theorems of B are precisely the theorems of the system B' with axioms

$$\text{I, II', III', } IV_1', IV_2$$

together with all abbreviations of such theorems obtained by one of the standard contextual definitions of abbreviation.

We shall use the following one; $\ulcorner \ldots \{\alpha \mid \phi\} - \urcorner$ is short for

$$\ulcorner (\exists \gamma)[(\alpha)(\alpha \in \gamma \leftrightarrow (\exists \beta)(\alpha \in \beta) \ \& \ \phi) \ \& \ \ldots \gamma -]\urcorner .$$

The long formula will be called an *immediate transform* of $\ldots \{\alpha \mid \phi\} -$. If $\phi_1, \ldots, \phi_n$ are so related that, for $1 \leq i < n$, ϕ_{i+1} is an immediate transform of ϕ_1, ϕ_n is called a *transform* of ϕ_1; if further ϕ_n contains no $\{\ \}$, an *abstractionless transform.* In general a formula possesses many abstractionless transforms, but they are all demonstrably equivalent.

There is no difficulty with II or IV_1. II' will be

$$\ulcorner (\exists \alpha)(\beta)(\beta \in \alpha \leftrightarrow (\exists \gamma)(\gamma \in \beta) \ \& \ \phi)\urcorner$$

and IV_1' will be the same as IV_2 (written in primitive notation). We claim that III can be replaced by

III'. $\vdash \ulcorner \alpha_1, \ldots, \alpha_n \in V \ \& \ \phi \rightarrow (\exists \beta)(SC(\beta) \ \& \ \phi')\urcorner$

where ϕ' is obtained from ϕ by replacing every quantifier $\ulcorner(\gamma)\urcorner$ by $\ulcorner(\gamma)(\gamma \subset \beta \rightarrow \ldots)\urcorner$ and every free occurrence of a variable δ distinct from each of $\alpha_1, \ldots, \alpha_n$ by $\ulcorner \delta \cap \beta \urcorner$.

(Notice that III' contains also the result that

$$\vdash \ulcorner \alpha_1, \ldots, \alpha_n \in V \ \& \ \phi \rightarrow (\exists \beta)(SC(\beta) \ \& \ \alpha_1, \ldots, \alpha_n \in V \ \& \ \phi')\urcorner .$$

E.g., with $n = 1$ we have

$$\begin{aligned} \vdash \ulcorner \alpha \in V \ \& \ \phi &\rightarrow \alpha \in V \ \& (\exists \gamma)(\alpha \in \gamma) \ \& \ \phi \urcorner \\ &\rightarrow (\exists \beta)(SC(\beta) \ \& ((\exists \gamma)(\alpha \in \gamma) \ \& \ \phi')\urcorner \\ &\rightarrow (\exists \beta)(SC(\beta) \ \& (\exists \gamma)(\gamma \subset \beta \ \& \ \alpha \in \gamma) \ \& \ \phi')\urcorner \\ &\rightarrow (\exists \beta)(SC(\beta) \ \& \ \alpha \in \beta \ \& \ \phi')\urcorner .) \end{aligned}$$

More precisely, we prove the following

THEOREM. *Let* B *have* $\in$, σ *and* $\{x \mid \ldots x -\}$ *as its primitive ideas, and* I-II, IV_1 *and* IV_2 *as its axioms. Let* B_1 *have* $\in$ *as its sole primitive idea, and* I, II', III', IV_1 (*or* IV_1') *and* IV_2 *as its axioms* (*the abstracts occurring in these axioms to be replaced by their definitions, and* ϕ *in* III' *to* *be in primitive notation). Then* (1) *every theorem of* B *is (an abbreviation of) a theorem of* B_1 *and* (2) *every theorem of* B_1 *is a theorem of* B.

We write '$\vdash_B \phi$' ('$\vdash_1 \phi$') for 'ϕ is a theorem of B' and 'ϕ is a theorem of B_1' respectively.

Re (1). It suffices to show that every *axiom* of B is a theorem of B_1 or an abbreviation of such. This is clear except for axioms of the form III.

For every formula ϕ of B and for every sequence $\alpha_1, \ldots, \alpha_n, \beta$ of variables ($n \geq 0$) we define

$$\ulcorner \mathrm{Rel}_{\alpha_1, \ldots, \alpha_n}(\phi, \beta)\urcorner$$

to be the result of replacing in ϕ

$$\ulcorner(\gamma)\psi\urcorner \text{ by } \ulcorner(\gamma)(\gamma \subset \beta \to \psi)\urcorner$$

$$\ulcorner\{\gamma|\psi\}\urcorner \text{ by } \ulcorner\{\gamma|\gamma \in \beta \,\&\, \psi\}\urcorner$$

and every free occurrence of a variable δ distinct from each of $\alpha_1, \ldots, \alpha_n$ by $\ulcorner\delta \cap \beta\urcorner$. The result that every axiom of the form III is (an abbreviation of) a theorem of B_1 follows from the

LEMMA. *Let* ϕ *be a formula of* B *and let* ϕ' *be a transform of* ϕ. *Then*

$$\vdash_1 \ulcorner \mathrm{SC}(\beta) \,\&\, \mathrm{Rel}_{\alpha_1, \ldots, \alpha_n}(\phi', \beta) \to \mathrm{Rel}_{\alpha_1, \ldots, \alpha_n}(\phi, \beta)\urcorner$$

We first show that the lemma implies $\vdash_1$ III. Let ϕ be a formula of B, ϕ' any of its abstractionless transforms and $\alpha_1, \ldots, \alpha_n$ all the variables α such that for some term θ, $\ulcorner\alpha \in \theta\urcorner$ is a conjunctive component of ϕ. Then

$$\vdash_1 \ulcorner\phi \to \alpha_1, \ldots, \alpha_n \in V \,\&\, \phi\urcorner \text{ (since } \vdash \ulcorner\alpha \in \theta \to \alpha \in V\urcorner)$$

$$\to \ulcorner(\exists\beta)(\mathrm{SC}(\beta) \,\&\, \mathrm{Rel}_{\alpha_1 \ldots, \alpha_n}(\phi', \beta))\urcorner \text{ (by III')}$$

$$\to \ulcorner(\exists\beta)(\mathrm{SC}(\beta) \,\&\, \mathrm{Rel}_{\alpha_1 \ldots, \alpha_n}(\phi, \beta))\urcorner \text{ (by the Lemma), q.e.d.}$$

Now we prove the Lemma. It suffices to consider the case where ϕ' is an immediate transform of ϕ. Then there are expressions μ and ν, a vari- γ and a formula ψ such that ϕ is

$$\ulcorner\mu\{\gamma|\psi\}\nu\urcorner$$

and ϕ' is

$$(\exists\delta)[(\gamma)(\gamma \in \delta \leftrightarrow (\exists\gamma_1)(\gamma \in \gamma_1) \,\&\, \psi) \,\&\, \mu\delta\nu].$$

For brevity, we omit the superscripts $\alpha_1, \ldots, \alpha_n$ on 'Rel' and further we write μ^* for $\ulcorner\mathrm{Rel}(\mu, \beta)\urcorner$; likewise ν^*, ψ^*. With this notation $\ulcorner\mathrm{Rel}(\phi', \beta)\urcorner$ is

(1) $\ulcorner(\exists\delta)[\delta \subset \beta \,\&\, (\gamma)(\gamma \subset \beta \to (\gamma \in \delta \leftrightarrow (\exists\gamma_1)(\gamma_1 \in \beta \,\&\, \gamma \in \gamma_1) \,\&\, \psi^*)) \,\&\, \mu^*\delta\nu^*]\urcorner$

and we are to prove

$$\vdash_1 \ulcorner\mathrm{SC}(\beta) \,\&\, (1) \to \mu^*\{\gamma|\gamma \in \beta \,\&\, \psi^*\}\nu^*\urcorner.$$

It clearly suffices to prove (in B_1) that

(2) $\ulcorner\mathrm{SC}(\beta)\urcorner$

(3) $$\ulcorner \delta \subset \beta \urcorner$$

and

(4) $$\ulcorner (\gamma)(\gamma \subset \beta \rightarrow (\gamma \in \delta) \leftrightarrow (\exists\, \gamma_1)(\gamma_1 \subset \beta \;\&\; \gamma \in \gamma_1) \;\&\; \psi^*)) \urcorner$$

imply

$$\ulcorner \delta = \{\gamma | \gamma \in \beta \;\&\; \psi^*\} \urcorner .$$

(4) simplifies to

(5) $$\ulcorner (\gamma)(\gamma \subset \beta \rightarrow (\gamma \in \delta \leftrightarrow (\gamma \in \beta \;\&\; \psi^*)) \urcorner .$$

If $\ulcorner \gamma \in \delta \urcorner$, then

(6) $$\ulcorner \gamma \in \beta \urcorner$$ by (3)

(7) $$\ulcorner \gamma \subset \beta \urcorner$$ by (6) and (2)

and

$$\psi^*$$ by (5) and (7);

thus

(8) $$\ulcorner \delta \subset \{\gamma | \gamma \in \beta \;\&\; \psi^*\} \urcorner .$$

Conversely, if

$$\ulcorner \gamma \in \beta \;\&\; \psi^* \urcorner$$

then

$$\ulcorner \gamma \subset \beta \urcorner$$ by (2)

and

$$\ulcorner \gamma \in \delta \urcorner$$ by (5);

thus

$$\ulcorner \{\gamma | \gamma \in \beta \;\&\; \psi^*\} \subset \delta \urcorner .$$

By (8)

$$\delta = \{\gamma | \gamma \in \beta \;\&\; \psi^*\} ;$$

this completes the proof of the Lemma.

Re (2). It suffices to show that every formula of the form III' is a theorem of B. Let then ϕ be a formula without abstraction, let $\alpha_1, \ldots, \alpha_n$ be any variables and let $\gamma_1, \ldots, \gamma_n$ be all the variables γ distinct from each of $\alpha_1, \ldots, \alpha_n$ such that for some term ζ $\ulcorner \gamma \in \zeta \urcorner$ is a conjunctive component of ϕ. III yields

$$\vdash_B \ulcorner \alpha_1, \ldots, \alpha_n \in V \;\&\; \phi \rightarrow (\exists\, \beta)(\mathrm{SC}(\beta) \;\&\; \mathrm{Rel}(\alpha_1, \ldots, \alpha_n \in V \;\&\; \phi, \beta)) \urcorner$$
$$\rightarrow \ulcorner (\exists\, \beta)(\mathrm{SC}(\beta) \;\&\; \mathrm{Rel}(\phi, \beta)) \urcorner .$$

We wish to prove $\vdash_B \ulcorner \alpha_1, \ldots, \alpha_n \in V \;\&\; \phi \rightarrow (\exists\, \beta)(\mathrm{SC}(\beta) \;\&\; \phi') \urcorner$. ϕ' differs from ϕ in that it contains $\ulcorner\gamma_i \cap \beta\urcorner$ whenever ϕ contains γ_i. Hence it will suffice to prove

$$\vdash_B \ulcorner \mathrm{SC}(\beta)\mathrm{Rel}(\phi,\beta) \rightarrow \gamma_i = \gamma_i \cap \beta\urcorner ,$$

i.e.,

(9) $$\vdash_B \ulcorner \mathrm{SC}(\beta)\mathrm{Rel}(\phi,\beta) \rightarrow \gamma_i \subset \beta\urcorner .$$

γ_i appears in some conjunctive component $\ulcorner\gamma_i \in \zeta\urcorner$ of ϕ. The corresponding conjunctive component of $\ulcorner\mathrm{Rel}(\phi,\beta)\urcorner$ is $\ulcorner\gamma_i \in \{\delta | \delta \in \beta \,\&\, \mathrm{Rel}(\psi,\beta)\}\urcorner$ if ζ is an abstract $\ulcorner\{\delta|\psi\}\urcorner$; in that case surely (9) holds. Likewise if ζ is a variable which is not one of the α_i or γ_i. More generally, call a sequence

$$\zeta_1, \zeta_2, \ldots, \zeta_n$$

of terms a ϕ-*chain* if each of the formulae

$$\ulcorner\zeta_1 \in \zeta_2\urcorner, \ \ulcorner\zeta_2 \in \zeta_3\urcorner, \ldots, \ulcorner\zeta_{n-1} \in \zeta_n\urcorner$$

is a conjunctive component of ϕ. Then if there is a ϕ-chain beginning with a variable γ_i and ending either with a variable not belonging to the α_i or γ_i or with an abstract, (2) holds. For example, with n = 4 we might have

$$\gamma_1 \in \gamma_n$$

$$\gamma_2 \in \gamma_3$$

$$\gamma_3 \in \{\alpha|\psi\}$$

as conjunctive components of ϕ. Then

$$\vdash_B \ulcorner\mathrm{Rel}(\phi,\beta) \rightarrow \gamma_1 \in \gamma_2 \,\&\, \gamma_2 \in \gamma_3 \,\&\, \gamma_3 \subset \beta\urcorner$$
$$\rightarrow \ulcorner\gamma_1 \in \gamma_2 \,\&\, \gamma_2 \in \beta\urcorner$$

But

$$\vdash_B \ulcorner\mathrm{SC}(\beta) \,\&\, \gamma_1 \in \gamma_2 \,\&\, \gamma_2 \in \beta \rightarrow \gamma_1 \in \beta\urcorner$$

and

$$\vdash_B \ulcorner\mathrm{SC}(\beta) \,\&\, \gamma_1 \in \beta \rightarrow \gamma_1 \subset \beta\urcorner$$

so that

$$\vdash_B \ulcorner\mathrm{SC}(\beta) \,\&\, \mathrm{Rel}(\phi,\beta) \rightarrow \gamma_1 \subset \beta\urcorner$$

If, on the other hand, there is no ϕ-chain beginning with γ_i which satisfies the above condition, then every ϕ-chain beginning with γ_i will consist wholly of γ's and so will eventually cycle. But the existence of such a cycle contradicts the axiom of regularity (IV_2) and consequently $\vdash_B \ulcorner\sim\phi\urcorner$ and *a fortiori*

$$\vdash_B \ulcorner\alpha_1, \ldots, \alpha_n \in V \,\&\, \phi \rightarrow (\exists\beta)(\mathrm{SC}(\beta) \,\&\, \phi')\urcorner$$

in this case too. This completes the proof of the Theorem.

IV

Let us see how the system B_1 should be changed in order to accomodate the existence of *individuals*. In order to avoid new primitive ideas and permit an elegant formulation, we adopt Quine's idea ([4], p. 135) that individuals (and only they) are their own singletons. On this basis I and II' are unaltered. The notion of rank is changed by considering all individuals as of lowest rank; since the intent of III' is that β be a rank, we must modify the definition of supercompleteness so that a supercomplete set contains all individuals as members.

$$\mathrm{SC}^*(x) \leftrightarrow \mathrm{SC}(x) \;\&\; (y)(y \in I \rightarrow y \in x)$$

where

$$I = \{x \mid (y)(y \in x \leftrightarrow y = x)\}$$

and then obtain our modified axiom III'' from III' by writing SC* instead of SC.

IV_1 remains unchanged, but IV_2 (axiom of regularity) is obviously false if x is an individual. The neatest substitute seems to be a form of Tarski's *set-theoretic* induction, namely

IV_2'. $$\vdash \ulcorner\{(\alpha)(\alpha \in I \rightarrow \phi) \;\&\; (\alpha)[(\forall\beta)(\beta \in \alpha \rightarrow \phi' \rightarrow \phi]\; \rightarrow (\alpha)\phi\urcorner$$

where ϕ' is obtained by writing free β for all free α in ϕ.

Another axiom, namely the existence of the set of all individuals, is superfluous because it follows at once from III''.

V

As the last of our modifications, let us consider what changes need be made if *descriptive predicates* ([4], p. 151) are added to the system. Evidently I must be strengthened to

I'. $$\alpha = \beta \rightarrow (\phi \rightarrow \phi\ )$$

where ϕ' is like ϕ except for containing free β where ϕ contains free α. (Cf. p. 201 of [4].)

The only other axiom which needs to be changed in III'', which leads to a contradiction if ϕ is allowed to contain arbitrary descriptive predicates. (I am indebted to Dana Scott for this curious observation.)

Let Γ be a predicate such that $\Gamma(x)$ is true if and only if $x = V$. Then by III''

$$\vdash ((x)(y)(x \in y \rightarrow x \in z) \rightarrow \Gamma(z)) \rightarrow$$

$$(\exists\, u)(\mathrm{SC}^*(u) \;\&\; \{((x)(y)(x, y \subset u \;\&\; x \in y \rightarrow x \in z \cap u) \rightarrow \quad (z \cap u)\}).$$

The antecedent however is true, hence there is a set u_O satisfying

$$\mathrm{SC}^*(u_O)$$

and

$$(x)(y)(x,y \subset u_0 \ \& \ x \in y \to x \in z \cap \iota_0) \to z \cap u = V.$$

But evidently (even with the hypothesis $SC^*(u_0)$)

$$(x)(y)(x,y \subset u_0 \ \& \ x \in y \to x \in u_0 \cup u_0)$$

so that

$$u_0 \cap u_0 = V$$

$$u_0 = V$$

$$u_0 \notin V$$

and *a fortiori* $\sim SC^*(u_0)$ (and $\sim SC(u_0)$). Thus III'' must be restricted by allowing no descriptive constants in ϕ.

We conclude by stating in detail the formation and transformation rules of the final form of the system.

Symbols. Variables $x,y,z,w,\ldots$; *predicates* ϵ (of degree 2), ϵ', ϵ'', . . . (of various degrees; all predicates save ϵ are called *descriptive predicates*). $\downarrow$ ( ).

Formulas. Atomic formulas $\ulcorner(\alpha \in \beta)\urcorner$ and $\mu\alpha_1, \ldots, \alpha_n$ where μ is a descriptive predicate of degree n; $\ulcorner(\phi \downarrow \psi)\urcorner$ and $\ulcorner(\alpha)\phi\urcorner$ where ϕ, ψ are formulas and α is a variable.

DEFINITIONS. Usual definitions of $\sim$, &, v, $\to$, $\leftrightarrow$.

D1. $\ulcorner(\alpha \in \{\beta|\phi\})\urcorner$ for $\ulcorner(\exists\gamma)(\alpha \in \gamma \ \& (\beta)(\beta \in \gamma \to \phi))\urcorner$
D2. $\ulcorner(\zeta = \eta)\urcorner$ for $\ulcorner(\alpha)(\alpha \in \zeta \leftrightarrow \alpha \in \eta)\urcorner$
D3. $\ulcorner(\zeta \neq \eta)\urcorner$ for $\ulcorner\sim(\zeta = \eta)\urcorner$
D4. $\ulcorner(\{\alpha|\phi\} \in \zeta)\urcorner$ for $\ulcorner(\exists\beta)(\beta = \{\alpha|\phi\} \ \& \ \beta \in \zeta)\urcorner$
D5. 'V' for '$\{x|x = x\}$'
D6. 'I' for '$\{x|(y)(y \in x \leftrightarrow y = x)\}$'
D7. $\ulcorner(\alpha \subset \beta)\urcorner$ for $\ulcorner(\gamma)(\gamma \in \alpha \to \gamma \in \beta)\urcorner$
D8. 'SC' for '$\{x|(y)(z)(y \in z \ \& (z \in x \vee z \subset x) \to y \in x)\}$'
D9. $\ulcorner\zeta \cap \eta\urcorner$ for $\ulcorner\{\alpha|\alpha \in \zeta \ \& \ \alpha \in \eta\}\urcorner$
D10. 'SC^*' for '$SC \cap \{x|I \subset x\}$'
D11. $\ulcorner\{\zeta,\eta\}\urcorner$ for $\ulcorner\{\alpha|\alpha = \zeta \vee \alpha = y\}\urcorner$
D12. $\ulcorner\{\zeta\}\urcorner$ for $\ulcorner\{\zeta,\zeta\}\urcorner$
D13. $\ulcorner(\zeta,\eta)\urcorner$ for $\ulcorner\{\zeta,\{\zeta,\eta\}\}\urcorner$
D14. $\ulcorner(\iota\alpha)\phi\urcorner$ for $\ulcorner\{\beta|(\exists\gamma)(\beta \in \gamma \ \& (\forall\alpha)(\phi \leftrightarrow \alpha = \gamma)\}\urcorner$
D15. $\ulcorner\zeta(\eta)\urcorner$ for $\ulcorner(\iota\alpha)((\eta,\alpha) \in \zeta)\urcorner$
D16. 'O' for '$\{x|x \neq x\}$'.

AXIOMS

*100 - *104 of [4], and

1. (= I'). $\vdash \ulcorner(\alpha = \beta) \to (\phi \to \phi')\urcorner$, where ϕ' is formed from ϕ by writing free α for one or more free β.
2. (= II'). $\vdash \ulcorner(\exists\alpha)(\beta)(\beta \in \alpha \leftrightarrow \beta \in V \ \& \ \phi)\urcorner$, where α does not occur in ϕ.

3. (= III'', restricted). $\vdash \ulcorner \alpha_1, \ldots, \alpha_n \in V \ \& \ \phi \rightarrow (\exists \beta)(\beta \in SC^* \ \& \ \phi)\urcorner$ where β does not occur in ϕ, where ϕ' is formed from ϕ by first replacing all free occurrences in ϕ of variables γ distinct from $\alpha_1, \ldots, \alpha_n$ by $\ulcorner \gamma \cap \delta \urcorner$, and then in the result replacing all subformulas of the form $\ulcorner (\gamma)\psi \urcorner$ (except those occurring in the contexts $\ulcorner \gamma \cap \delta \urcorner$ just introduced by $\ulcorner (\gamma)(\gamma \subset \beta \rightarrow \psi)\urcorner$, and where ϕ does not contain any descriptive constants.
4. (= IV_1). $\vdash \ulcorner (\exists f)(x)(x \in V \ \& \ x \neq O \rightarrow f(x) \in x)$.
5. (= IV_2'). $\vdash \ulcorner (\alpha)(\alpha \in I \rightarrow \phi) \ \& \ (\beta)[(\alpha)(\alpha \in \beta \rightarrow \phi) \rightarrow \phi'] \rightarrow (\alpha)\phi \urcorner$ where β does not occur in ϕ and ϕ' is obtained by writing β for all free α in ϕ.

The sole *rule of inference* is *modus ponens* (*105 of [4]).

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