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A REMARK CONCERNING THE THIRD THEOREM ABOUT THE EXISTENCE OF SUCCESSORS OF CARDINALS

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The following three formulas about the existence of successors of cardinals:

S_1 *For every cardinal* $\mathfrak{m}$ *there is a cardinal* $\mathfrak{n}$ *such that* (i) $\mathfrak{m} < \mathfrak{n}$, *and* (ii) *the formula* $\mathfrak{m} < \mathfrak{p} < \mathfrak{n}$ *does not hold for any cardinal* $\mathfrak{p}$.

S_2 *For every cardinal* $\mathfrak{m}$ *there is a cardinal* $\mathfrak{n}$ *such that* (i) $\mathfrak{m} < \mathfrak{n}$, *and* (ii) *for every cardinal* $\mathfrak{p}$ *the formula* $\mathfrak{m} < \mathfrak{p}$ *implies* $\mathfrak{n} \leq \mathfrak{p}$.

S_3 *For every cardinal* $\mathfrak{m}$ *there is a cardinal* $\mathfrak{n}$ *such that* (i) $\mathfrak{m} < \mathfrak{n}$, *and* (ii) *for every cardinal* $\mathfrak{p}$ *the formula* $\mathfrak{p} < \mathfrak{n}$ *implies* $\mathfrak{p} \leq \mathfrak{m}$.

are discussed by Tarski in [2] who has shown there that $\mathbf{S}_1$ can be proved without the help of the axiom of choice and that $\mathbf{S}_2$ is equivalent to this axiom. Concerning $\mathbf{S}_3$ it is remarked in [2], p. 32, that it is not yet known whether $\mathbf{S}_3$ can be proved without the help of the axiom of choice, and, therefore, *a fortiori* it is not known whether $\mathbf{S}_3$ is equivalent to the said axiom. The latter problem remains open, but according to the announcement given in [1], p. 73, note 2, the former one is solved in the negative by A. Lewi who has proved that $\mathbf{S}_3$ does not follow from the axioms of the general set theory, even if the ordering principle is added to these axioms.[1] As far as I know this result of Mr. Lewi is not yet published.

In this note I show that each of the given below formulas, $\mathbf{T}_1$ and $\mathbf{T}_2$, is such that the axiom of choice follows from it and $\mathbf{S}_3$. The formulas $\mathbf{T}_1$ and $\mathbf{T}_2$ are, as I conjecture, probably neither provable without the aid of the axiom of choice nor equivalent to this axiom.

In order to present the formulas $\mathbf{T}_1$ and $\mathbf{T}_2$ and the subsequent deductions in a more compact way I introduce here the following abbreviative definition:

D1 *For any* $\mathfrak{m}$ *and* $\mathfrak{n}$, $\mathfrak{m} \prec \mathfrak{n}$ *if and only if* $\mathfrak{m}$ *and* $\mathfrak{n}$ *are cardinals*, $\mathfrak{m} < \mathfrak{n}$, *and for every cardinal* $\mathfrak{p}$ *the formula* $\mathfrak{p} < \mathfrak{n}$ *implies* $\mathfrak{p} \leq \mathfrak{m}$.

Using this definition we can present $\mathbf{T}_1$ and $\mathbf{T}_2$ as follows:

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T$_1$ *For any cardinal numbers* $\mathfrak{m}$, $\mathfrak{n}$ *and* $\mathfrak{p}$, *if* $\mathfrak{m} < \mathfrak{n}$ *and* $\mathfrak{m} \prec \mathfrak{p}$, *then* $\mathfrak{p} \leq \mathfrak{n}$.

and

T$_2$ *For any cardinal numbers* $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{p}$ *and* $\mathfrak{r}$, *if* $\mathfrak{m} < \mathfrak{n}$, $\mathfrak{m} \prec \mathfrak{p}$ *and* $\mathfrak{n} \prec \mathfrak{r}$, *then* $\mathfrak{p} < \mathfrak{r}$.

Proof:

(i) *The axiom of choice implies* **T**$_1$. Since, obviously, for $\mathfrak{m}$ being a finite cardinal **T**$_1$ holds banally, it is sufficient to prove this theorem for cardinal number $\mathfrak{m}$ which is not finite. Hence, let us assume that

(1) $\mathfrak{m}$ *is a cardinal number which is not finite*

and the remaining conditions of **T**$_1$, viz. that

(2) $\mathfrak{n}$ *and* $\mathfrak{p}$ *are cardinal numbers*

(3) $\mathfrak{m} < \mathfrak{n}$

and

(4) $\mathfrak{m} \prec \mathfrak{p}$

Then, by **D1** and *(4)*, we have

(5) *for every cardinal* $\mathfrak{r}$, *if* $\mathfrak{r} < \mathfrak{p}$, *then* $\mathfrak{r} \leq \mathfrak{m}$

and, in virtue of the axiom of choice and *(2)*, we can establish that

(6) *either* $\mathfrak{n} < \mathfrak{p}$ *or* $\mathfrak{p} \leq \mathfrak{n}$

But, the first case of *(6)*, viz. $\mathfrak{n} < \mathfrak{p}$, together with *(2)* and *(5)* implies

(7) $\mathfrak{n} \leq \mathfrak{m}$

which contradicts *(3)*. Hence, the second case of *(6)*, viz.

(8) $\mathfrak{p} \leq \mathfrak{n}$

holds. Thus, **T**$_1$ is proved with the help of the axiom of choice.

(ii) *Formula* **T**$_2$ *follows from* **T**$_1$. Let us assume the conditions of **T**$_2$, viz. that

(9) $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{p}$ *and* $\mathfrak{r}$ *are the cardinal numbers*

(10) $\mathfrak{m} < \mathfrak{n}$

(11) $\mathfrak{m} \prec \mathfrak{p}$

and that

(12) $\mathfrak{n} \prec \mathfrak{r}$

Then, it follows from **T**$_1$, *(9)*, *(10)* and *(11)* that

(13) $\mathfrak{p} \leq \mathfrak{n}$

and from **D1** and *(12)* that

(14) $\mathfrak{n} < \mathfrak{x}$

Hence, *(13)* and *(14)* imply at once

(15) $\mathfrak{p} < \mathfrak{x}$

Thus our proof is completed. It has to be noted that, as the inspection of the proof given above shows without any difficulty, $\mathbf{T}_1$ implies actually a stronger formula than $\mathbf{T}_2$, viz.

$\mathbf{T}_2^*$ *For any cardinal numbers* $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{p}$ *and* $\mathfrak{x}$, *if* $\mathfrak{m} < \mathfrak{n}$, $\mathfrak{m} < \mathfrak{p}$ *and* $\mathfrak{n} < \mathfrak{x}$, *then* $\mathfrak{p} < \mathfrak{x}$.

I do not know whether $\mathbf{T}_2$ (or $\mathbf{T}_2^*$) implies $\mathbf{T}_1$ in its turn.

(iii) *The formulas* $\mathbf{S}_3$ *and* $\mathbf{T}_2$ *imply the axiom of choice.* Let us assume that

(16) $\mathfrak{m}$ *is an arbitrary cardinal number which is not finite*

Hence, by *(16)*,

(17) *there is the least Hartogs' aleph* $\aleph(\mathfrak{m})$ *in respect to* $\mathfrak{m}$

and due to the well-known properties of Hartogs' alephs we know that the following formula

(18) $\mathfrak{m} < \mathfrak{m} + \aleph(\mathfrak{m})$

holds. Besides, obviously,

(19) $\mathfrak{m} + \aleph(\mathfrak{m})$ *is a cardinal number*

and, since we have *(16)* and *(17)*, the following statement

(20) *the formula* $\mathfrak{m} < \mathfrak{z} < \mathfrak{m} + \aleph(\mathfrak{m})$ *does not hold for any cardinal* $\mathfrak{z}$

can be established without the aid of the axiom of choice.[2] Now, in virtue of $\mathbf{S}_3$, *(16)* and *(19)*, we know that

(21) *there are cardinal numbers* $\mathfrak{p}$ *and* $\mathfrak{x}$

such that

(22) $\mathfrak{m} < \mathfrak{p}$

and

(23) $\mathfrak{m} + \aleph(\mathfrak{m}) < \mathfrak{x}$

Hence, by $\mathbf{T}_2$, *(16)*, *(19)*, *(21)*, *(18)*, *(22)* and *(23)*,

(24) $\mathfrak{p} < \mathfrak{x}$

On the other hand, due to **D1** and *(23)* we have

(25) *for every cardinal* $\mathfrak{t}$, *if* $\mathfrak{t} < \mathfrak{x}$, *then* $\mathfrak{t} \leq \mathfrak{m} + \aleph(\mathfrak{m})$

which together with *(21)* and *(24)* implies at once:

(26) $\mathfrak{p} \leq \mathfrak{m} + \aleph(\mathfrak{m})$

Since the first case of *(26)*, viz. $\mathfrak{p} < \mathfrak{m} + \aleph(\mathfrak{m})$, together with *(22)* and **D1** gives $\mathfrak{m} < \mathfrak{p} < \mathfrak{m} + \aleph(\mathfrak{m})$ which is excluded by *(20)*, the second case of *(26)*, namely

(27) $\mathfrak{p} = \mathfrak{m} + \aleph(\mathfrak{m})$

holds. Hence, by *(22)* and *(27)*,

(28) $\mathfrak{m} < \mathfrak{m} + \aleph(\mathfrak{m})$

which, in virtue of **D1**, gives

(29) *for every cardinal* $\mathfrak{f}$, *if* $\mathfrak{f} < \mathfrak{m} + \aleph(\mathfrak{m})$, *then* $\mathfrak{f} \leq \mathfrak{m}$

Now, obviously, we have

(30) $\aleph(\mathfrak{m}) \leq \mathfrak{m} + \aleph(\mathfrak{m})$

without the aid of the axiom of choice. Since the first case of *(30)*, viz. $\aleph(\mathfrak{m}) < \mathfrak{m} + \aleph(\mathfrak{m})$, together with *(17)* and *(29)* implies at once: $\aleph(\mathfrak{m}) \leq \mathfrak{m}$ which due to the properties of Hartogs' alephs is impossible, the second case of *(30)*, namely

(31) $\aleph(\mathfrak{m}) = \mathfrak{m} + \aleph(\mathfrak{m})$

holds which gives at once:

(32) $\aleph(\mathfrak{m}) \geq \mathfrak{m}$

i.e. that our arbitrary cardinal number $\mathfrak{m}$ which is not finite is an aleph. Thus, we completed the proof that the axiom of choice follows from $\mathbf{S}_3$ and $\mathbf{T}_2$.

(iv) Since, as it was shown in (ii), $\mathbf{T}_1$ implies $\mathbf{T}_2$, the former formula and $\mathbf{S}_3$ give also the axiom of choice.

NOTES

1. In this note the general set theory is understood as the set theory from which the axiom of choice and all its consequences otherwise unprovable have been removed. It is well-known that if we base a so defined set theory on an axiomatic system in which the notions of cardinal and ordinal numbers cannot be defined, we have to introduce these concepts into the system by means of special axioms.

   It is known that the addition of the ordering principle, as a new axiom, to the general set theory does not give the axiom of choice. *Cf.*, e.g., [1], p. 53.

2. Concerning Hartogs' alephs and their properties discussed in this note, *cf.*, e.g., [2], pp. 28-30, lemmas 3, 4 and 5.

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