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A DIAGRAM OF THE FUNCTORS OF THE TWO-VALUED PROPOSITIONAL CALCULUS

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By means of arranging the functors of the two-valued propositional calculus in a certain array (to be described below), we find that several properties of the functors are related. Such properties are connected to the possibilities of defining some functors by others, and thus in the diagram we have displayed definitional connections between certain sets of functors. In this paper we first present the method of diagramming, and certain helpful connections within the diagram, then several theorems on definitions within the propositional calculus. We are then able to show that there are three exhaustive classes for single functors in terms of definitions, of such a nature that we are able to give axioms for a large number of functors. The paper is concluded with some further consideration on definability in special cases.

Let us arrange the unary functors of the two-valued propositional calculus[1] in the array $\mathfrak{A}_1$.

$$\mathfrak{A}_1 \qquad \begin{array}{ccc} & I & \\ O & & U \\ & N & \end{array}$$

We may then extend this array to include also the binary functors by placing a given binary functor, X, having the properties, that, for certain unary functors Y and Z,

$$Xpp = Yp$$
$$XpNp = Zp$$

in an array similar to $\mathfrak{A}_1$, and half its size, centered at the functor Y, with X at the position corresponding to that occupied by Z in the array $\mathfrak{A}_1$. This

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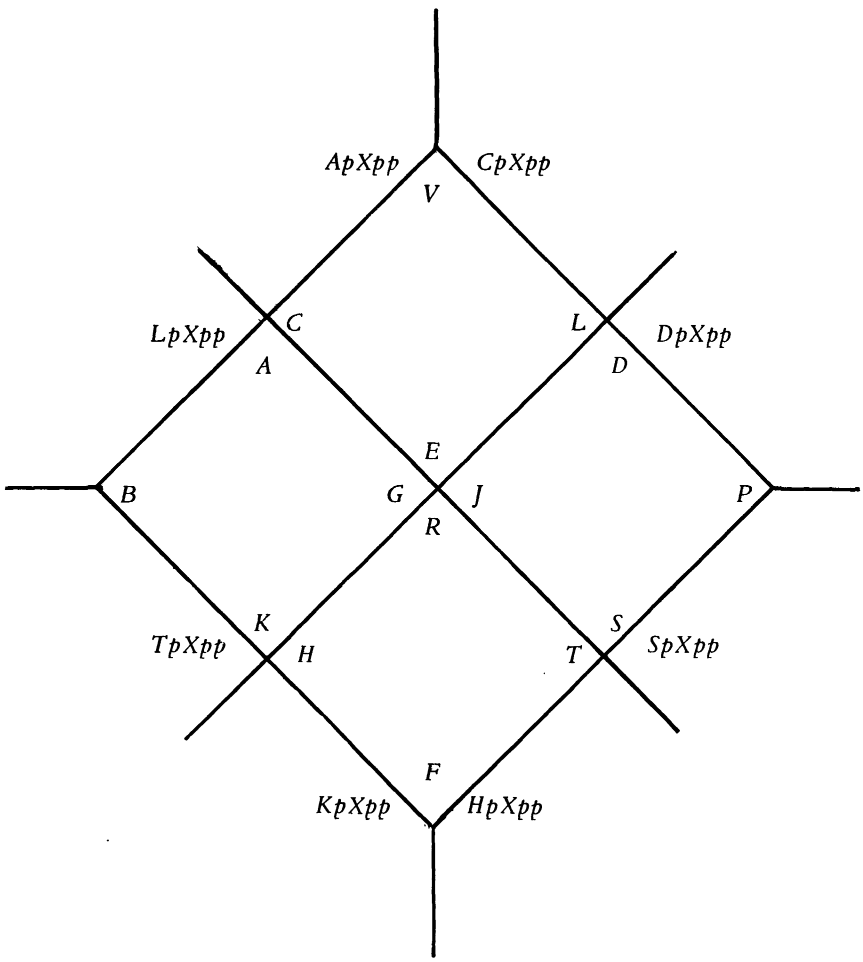
procedure gives us the array $\mathfrak{A}_2$. The array $\mathfrak{A}_2$ is quite similar to a diagram of C. S. Peirce, which we will call $\mathfrak{B}$.[2]

				G			
			K		A		
		H		B		L	
$\mathfrak{A}_2$	F		R		E		V
		T		P		C	
			S		D		
				J			

$\mathfrak{B}$



Using certain features of this diagram, Peirce was able to construct true propositions of certain forms. For example, Xpp is true if X lies in the upper quadrant. We are, in addition, able to find interesting properties concerning definability in the propositional calculus by the diagram.

Let us, then, extend the diagram so that all n-ary functors are included. If X is an $(n + 1)$-ary functor and Y and Z are n-ary functors such that

$$\text{1)} \qquad Xp_1p_2 \ldots p_np_n = Yp_1p_2 \ldots p_n$$

$$\text{2)} \qquad Xp_1p_2 \ldots p_nNp_n = Zp_1p_2 \ldots p_n$$

then X is in an array similar to $\mathfrak{A}_n$, which is centered at Y, with X at the position occupied by Z in $\mathfrak{A}_n$. The resulting array of $(n + 1)$-ary functors[3] we call $\mathfrak{A}_{n+1}$.

To formulate this in a more exact way:

Let $\mathfrak{C}$ be a one-to-one function from a subset of points in the complex plane onto the set of functors of the two-valued propositional calculus such that

$$\mathfrak{C}(\tfrac{1}{2}) = U \qquad \mathfrak{C}(\tfrac{1}{2}i) = I$$
$$\mathfrak{C}(-\tfrac{1}{2}) = O \qquad \mathfrak{C}(-\tfrac{1}{2}i) = N$$

and if X is an $(n + 1)$-ary functor, Y and Z n-ary functors with

$$\mathfrak{C}(x) = X, \qquad \mathfrak{C}(y) = Y, \qquad \mathfrak{C}(z) = Z,$$

and (1) and (2) holding, then

$$x = y + z(2^{-2^n}).$$

Now, consider the reflections and rotations of the complete array. We obtain these theorems:

Theorem 1. Let δ be the reflection in the vertical axis. Then δX is the dual of X.

Proof. It is clear that the unary functors have this property, for $\delta I = I$ and $\delta N = N$, while $NINp = Ip$ and $NNNp = Np$; and $\delta O = U$, while $NONp = Up$. If we have for each n-ary functor X

$$\delta Xp_1p_2 \ldots p_n = NXNp_1Np_2 \ldots Np_n \, ,$$

then for appropriate X, Y, Z such that (1) and (2) hold,

$$\delta Xp_1p_2 \ldots p_np_n = \delta Yp_1p_2 \ldots p_n$$

$$\delta Xp_1p_2 \ldots p_nNp_n = \delta Zp_1p_2 \ldots p_n$$

so that all $(n + 1)$-ary functors are also transformed by δ into their duals.

Theorem 2. Let β be the reflection in the diagonal: real part of x = imaginary part of x. Then

$$\beta Xp_1 \ldots p_n = Ep_1Xp_1 \ldots p_n$$

Proof. Similar to the proof of theorem 1.

It is well known that the two mentioned reflections are sufficient to determine all reflections and rotations of the diagram. But one further example which will be of use is the rotation of 180°.

Theorem 3. Let η be the rotation of 180°. Then

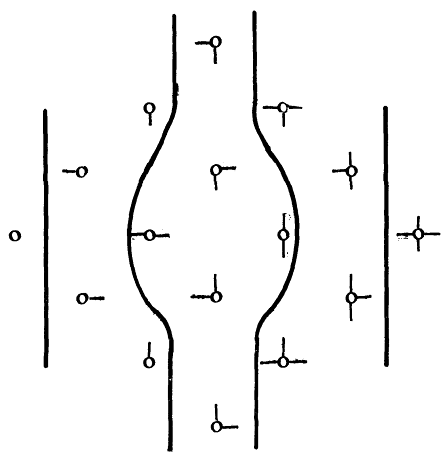
$$\eta X p_1 \ldots p_n = N X p_1 \ldots p_n$$

Proof. Follows from theorems 1 and 2, since $\delta\beta\delta\beta = \eta$:

$$\begin{aligned}
\delta\beta\delta\beta X p_1 \ldots p_n &= \delta\beta\delta E p_1 X p_1 \ldots p_n \\
&= \delta\beta N E N p_1 X N p_1 \ldots N p_n \\
&= \delta E p_1 N E N p_1 X N p_1 \ldots N p_n \\
&= \delta E p_1 E p_1 X N p_1 \ldots N p_n \\
&= \delta X N p_1 \ldots N p_n \\
&= N X p_1 \ldots p_n
\end{aligned}$$

We obtain certain regularities in our array $\mathfrak{A}_2$ if we use the symbolism of Lesniewski for the binary functors (see diagram $\mathfrak{D}$).

$\mathfrak{D}$



In each of the quadrants, there is a uniform positioning of the vertical strokes, and the horizontal stroke placement follows the same pattern within each quadrant. Also, the lines as drawn in diagram $\mathfrak{D}$ serve as "contours", dividing the functors according to the number of strokes on the functor symbols.

Let us now consider the definition characteristics according to the diagram.[4] We will say that functor X lies in the Y quadrant, where Y is a unary functor, if $Xpp \ldots p = Yp$, so that, for example, V, C, L, and E are the binary functors in the U quadrant.

Theorem 4. (i) If X is defined by functors lying in the I quadrant, then X lies in the I quadrant. (ii) If X is defined by functors lying in the U and

I quadrants, then X lies in the U or I quadrant. (iii) If X is defined by functors lying in the 0 and I quadrants, then X lies in either the 0 or I quadrant.

Proof. (i) Let X be defined by functors lying in the I quadrant. Then, clearly, $Xpp\ldots p = p$, i.e., X is also in the I quadrant. (ii) Let X be defined by functors lying in the U and I quadrants, then $X11\ldots 1 = 1$, or X lies in the U or I quadrants. (iii) Similarly, we find that on the hypothesis $X00\ldots 0 = 0$, i.e., X lies in the 0 or I quadrant.

This gives us immediately the

Corollary. If we use only functors from the 0 and I quadrants, we are unable to form a true proposition.

Theorem 5. If X is defined by functors lying on the vertical axis, i.e., self-dual functors, then X lies on the vertical axis.

Proof. Say we have

$$Xp_1p_2\ldots p_n = Y\ldots Zp_i\ldots$$

then, on negating each of the p_i we find

$$\begin{aligned} XNp_1Np_2\ldots Np_n &= Y\ldots ZNp_i\ldots \\ &= Y\ldots NZp_i\ldots \\ &= NY\ldots Zp_i\ldots \\ &= NXp_1p_2\ldots p_n \end{aligned}$$

Thus X is self-dual.

Theorem 6. X is a Sheffer functor[5] *if and only if it lies in the N quadrant, not on the vertical axis.*

Proof. (i) If a functor does not lie in the N quadrant, then, from theorem 4, it cannot define any functor lying in the N quadrant, and hence is not a Sheffer functor. If a functor lies on the vertical axis, by theorem 5 it cannot define any functor not on the vertical axis.

(ii) If X is a functor in the N quadrant, and not on the vertical axis, then $Xpq\ldots q = Spq$, Dpq, Jpq or Ppq. If $Xpq\ldots q = Spq$ or Dpq, then, since S and D are Sheffer functors, X is also a Sheffer functor. So assume $Xpq\ldots q = Ppq$ or Jpq. In such a case there is a combination of I and N, say Q_i, such that

$$3) \qquad XpqQ_1qQ_2q\ldots = Ypq$$

where Y is a binary functor not on the vertical axis. Since we are able to define (with X alone) N and I, we are able to define one of the well-known pairs of complete functors, unless Y is on the horizontal axis (V, E, R, F). So now consider the cases in which Y is on the horizontal axis. In the frame (3) for the definition of Y with X, we place 'p' for every occurrence of a non-negated 'q', and 'q' for every occurrence of 'Nq'. (For example, if $XpqNqNqq = Vpq$, we would consider $Xppqqp$). The functor so defined, say Z, has these properties: $Z11 = 0$, $Z00 = 1$, $Z01 = Z10$. And thus Z is either S or D, one of the Sheffer functors, so that X is a Sheffer functor.

As mentioned above (corollary to theorem 4), only those functors occurring in the N and U quadrants (and not on the vertical axis) are able to serve as a single functor for the propositional calculus. Thus, for example, the only such unary functor is U, and the only binary functors are V, C, L, E, D and S. We are now able to show that if we have a fragment of the propositional calculus based on the n-ary functor X alone, then either

1) X defines C;
2) X defines E and E defines X; or
3) X is verum for n arguments.

From this result we will be able to give an axiom set for any such single functor fragment of the propositional calculus. But, to prove these statements, some lemmas are required.

Lemma 1. If X is a functor in the U quadrant, not lying on the horizontal axis, then X defines C.

Proof. If $Xpq\ldots q = Cpq$ or Lpq, we immediately have this proven. Suppose that $Xpq\ldots q = Epq$ or Vpq. In such a case, there are Q_i which are either I or N, such that (3) holds, where Y is not a binary functor on the horizontal axis. We let Zpq be the result of replacing, in the left side of (3), each occurrence of 'Nq' by 'p'. It is then easy to show that Zpq is either Cpq or Lpq, except in case Y is A, R, D, K, F, E, or S.

Now, consider the left side of (3) with each occurrence of 'Nq' replaced by 'q' and each occurrence of 'q' or 'Iq' replaced by 'p', and let the resulting expression be $Z'pq$. Now we are able to show that Z' is either C or L except when Y is E, L, C, V, H, F, T, or R. Comparing the requirements for Z and Z' to be neither C nor L, we see that Y can only be a functor lying on the axis, contrary to our assumption. Thus the lemma is proved.

We will need this definition to simplify the subsequent proofs: (YZ) is that unique functor X such that the equations (1) and (2) hold. We may then use the symbolism for transformations of functors—e.g., $(X\eta X)$ is that functor such that

$$(X\eta X)p_1p_2\ldots p_np_n = Xp_1p_2\ldots p_n$$

and

$$(X\eta X)p_1p_2\ldots p_nNp_n = NXp_1p_2\ldots p_n$$

We have now

Lemma 2. If X is a functor on the axis in the U quadrant, and is definable by E, then E defines both (XX) and $(X\eta X)$.

Proof. We have immediately

$$(XX)p_1p_2\ldots p_{n+1} = Xp_1p_2\ldots p_n$$

$$(X\eta X)p_1p_2\ldots p_{n+1} = EEp_np_{n+1}\, Xp_1p_2\ldots p_n$$

Lemma 3. If X lies on the axis in the U quadrant, either X defines E or X is verum.

Proof. If $Xpqq...q = Epq$, then X defines E. Assume, then, that $Xpq...q = Vpq$. Then there are such Q_i, either I or N, such that (3) holds, and say that Y is not V (for if all such equations hold only for $Y = V$, then X is verum). Following the same sort of procedure again, replace 'Nq' by 'p', yielding Zpq such that $Z = E$ unless $Y = R$. But in such a case, in (3) we replace 'Nq' by 'q', and 'q' and 'Iq' by 'p', yielding $Z'pq$, which is Epq if $Y = R$. Thus the lemma is proved.

Lemma 4. If X is an $(n + 1)$-ary functor in the U quadrant and on the horizontal axis, and E defines X, then there is a Y such that E defines Y and $X = (YY)$ or $X = (Y\eta Y)$.

Proof. Let $X = (YZ)$, then clearly E defines Y. Let

$$X'p_1...p_n = Xp_1...p_n 1$$

Then

$$X'p_1...p_n = \begin{cases} Yp_1...p_n, \text{ if } p_n = 1 \\ Zp_1...p_n, \text{ if } p_n = 0 \end{cases}$$

Consider the effect of the reflection in the horizontal axis (which we will call α) on the functor X':

$$\begin{aligned} \alpha X'p_1...p_n &= X'Np_1...Np_n \\ &= \begin{cases} YNp_1...Np_n, \text{ if } p_n = 0 \\ ZNp_1...Np_n, \text{ if } p_n = 1 \end{cases} \\ &= \begin{cases} Yp_1...p_n, \text{ if } p_n = 0 \\ Zp_1...p_n, \text{ if } p_n = 1 \end{cases} \end{aligned}$$

since Y and Z are on the horizontal axis. Then, if X' is on the horizontal axis, $\alpha X' = X'$, and thus $Y = Z$. If X' is on the vertical axis, $\alpha X' = \eta X'$, and thus $Y = \eta Z$. Our lemma is proven.

If X' is in quadrant U, not on the horizontal axis, by lemma 1 we have that X' defines C, and thus cannot be defined by E. If X' is in quadrant I, not on the vertical axis, X' is a Sheffer functor, which means that X' and N can define all functors, so that E cannot define X'.[6] But, since E defines X, and thus also X', either $Y = Z$ or $Y = \eta Z$.

We are now able to prove the theorem

Theorem 7. If X lies in the U quadrant, either C can be defined by X, or X defines E and E defines X, or X is verum.

Proof. We have shown in lemmas 3 and 4 a necessary and sufficient condition for a functor to define C, or to be defined by E, provided it lies on the axis. But any functor not lying on the axis (in the U quadrant) defines C, by lemma 1. And by lemma 2, any functor on the axis which is not verum defines E. Thus the theorem is proved, for we have found necessary and sufficient conditions for each of the three possibilities for functors in the U quadrant.

Now we may complete the characterization of functors which may serve as a single functor basis for propositional calculus (or, in other words, those functors which can define U). If a functor is a Sheffer functor, then it clearly defines C, and this takes care of possible functors from the N quadrant, by theorems 5 and 6.

There have been given the following sorts of axiom sets for two-valued propositional calculus:[7]

1) with E as single functor
2) with C and any other functor

We are able to give a constructive method for axiomatizing the propositional calculus with any single functor with which a true proposition can be formulated: consider the three possibilities

1) If X is a functor which defines E and which E defines, we may write as an axiom set for X: a complete axiom set for E, with every occurrence of E replaced by a definition of E using X; and add to this an axiom written in the form of an equivalential definition of X in terms of E, but with E replaced throughout by its definition in terms of X. Then, we must adjust the rule of detachment for E to the corresponding one for X. Therefore, we will have not only a complete system for E expressed in terms of X (i.e., if we have a true formula containing E, we can prove the formula with every occurrence of E replaced by its definition in X), but also, since the substitution theorem is provable, the final axiom will permit us to derive any true formula in X—if we have a true formula with X, there is a corresponding true formula with E, which is provable in the system, and the equivalence of the two formulas is provable.

2) If X is a functor which can define C, then we use the axiom set of Henkin for C and X, in which C is replaced throughout by its definition in terms of X. With the rule for detachment for C adjusted to X, we clearly have a complete axiom set for X.

3) If X is verum for n arguments, we require only $Xp_1p_2\ldots p_n$ for an axiom, and no rule of detachment.

There are a few more theorems suggested by the diagram, among which, e.g.

Theorem 8. If X is a functor from the I or U quadrant, then X is definable by C and E.

Proof. We first show that the theorem holds for unary functors. $Ip = EEppp$, $Up = Epp$. Also note that $Apq = CCpqq$ and $Kpq = EpCpq$.

Assume that the theorem is true for all m-ary functors. Then let

$$Y_i p_0 \ldots p_{i-1} p_{i+1} \ldots p_m = X p_0 \ldots p_{i-1}\ 1\ p_{i+1} \ldots p_m\,,$$

for $0 \leqslant i \leqslant m$.

Note that a functor Z is in the I or U quadrant if and only if $Z11 \ldots 1 = 1$. If X is a functor in the U or I quadrant, each Y_i is in the U or I quadrant, and, since they are m-ary functors, each Y_i is definable by E and C. Note that, if X is in the I quadrant, we may write

$$X p_0 \ldots p_m = AKp_0 Y_0 p_1 \ldots p_m \ldots AKp_i Y_i p_0 \ldots p_{i-1} p_{i+1} \ldots p_m \ldots$$

and if X is in the U quadrant, then

$$X p_0 \ldots p_m = KCp_0 Y_0 p_1 \ldots p_m \ldots KCp_i Y_i p_0 \ldots p_{i-1} p_{i+1} \ldots p_m \ldots$$

Since the Y_i are definable by C and E, and A and K are definable by C and E, X is definable by C and E. And the theorem is proved.

Let us call a set of functors self-dual and complete if 1) the set of the duals of the functors is identical with the original set, 2) every functor of the propositional calculus can be defined by the members of the set, and 3) no functor in the set can be defined by the others. It is well known that there are only two such sets of binary functors, viz. $\{C, T\}$ and $\{L, H\}$. Alan Rose[8] has found two self-dual ternary functors which are able to define all functors with the constants 1 and 0. Using methods suggested by the diagram of the functors, one is able to find large groups of self-dual complete sets of functors.

First, note that the vertical axis (on which all self-dual functors lie) is generated by the reflection β on the horizontal axis. So, if a functor X on the horizontal axis is not definable by E, then the functor βX is not definable by E and R, and, in addition, βX defines E. Therefore, by lemmas 2-4, we have

Theorem 9. A set of the form $\{X, E, R\}$ *is a complete self-dual set if and only if we can write* X *in the forms* $(X_1' \; X_2')$, $((X_1'' \; X_2'')\,(X_3'' \; X_4''))$, . . ., *where for some* m, k, *neither*

$$X^{(k)}_m = X^{(k)}_{m+1} \quad \text{nor} \quad X^{(k)}_m = \eta\, X^{(k)}_{m+1}\,.$$

For example, the ternary functors satisfying the conditions of theorem 9 are (GB), (GP), (BJ), and (BG). And among these ternary functors, since

$$(BG)\ pqr = AKrApqKpq$$

$$(GB)\ 1pq = Cpq$$

$$(GP)\ pq1 = Cqp$$

$$(BJ)\ pq1 = Cpq$$

only the three (GB), (GP), and (BJ) are also complete with 0 and 1.

To complete these considerations on functors, we may note first

Theorem 10. The set $\{X, \delta X\}$ is self-dual and complete if and only if either X or δX is from the U quadrant, and is not definable by E.

Proof. If X is from the I quadrant, δX is also, and, by theorem 4, the pair is unable to define any functor outside of the I quadrant. If X and δX are from the N quadrant, either $X = \delta X$ or X is a Sheffer functor, which means that X defines δX. We have shown that if X lies in the U quadrant, and is not definable by E, then it defines C; and if X lies in the U quadrant, δX lies in the 0 quadrant and thus defines 0. And, as is well known, C and 0 are able to define all functors, proving the theorem.

Among the ternary functors in the U quadrant there are precisely four which do not satisfy the conditions of theorem 10, viz. (VV), (VF), (EE), (ER).

And, finally, considering the triples of ternary functors, we see that the possibilities reduce (because of the self-dual complete sets of two ternary functors given) to consideration of the above mentioned four with the various self-dual functors. By similar methods to those above we are able to find that there are exactly 15 self-dual complete triples of ternary functors:

$\{(GB), (VV), (FF)\}$ $\{(GB), (VF), (FV)\}$ $\{(GB), (EE), (RR)\}$

$\{(GP), (VV), (FF)\}$ $\{(GP), (VF), (FV)\}$ $\{(GP), (EE), (RR)\}$

$\{(BJ), (VV), (FF)\}$ $\{(BJ), (VF), (FV)\}$ $\{(BJ), (EE), (RR)\}$

$\{(BG), (VF), (FV)\}$ $\{(BG), (EE), (RR)\}$

$\{(GB), (ER), (RE)\}$ $\{(GP), (ER), (RE)\}$

$\{(BJ), (ER), (RE)\}$ $\{(BG), (ER), (RE)\}$

Having proven these various definitional properties in the two-valued propositional calculus, the problem presents itself whether a similar arrangement of the functors in an n-valued propositional calculus would be of any help—for example, in finding the Sheffer functors of an n-valued propositional calculus. In the first place, it would be quite difficult to picture even the binary functors in three-valued logic merely because of the great number of them, there being $3^{3^2} = 19{,}683$ of such functors.

In addition, instead of the number of $(k + 1)$-ary functors being the square of the number of k-ary functors, in n-valued propositional calculus we proceed by the nth power at each step. In analogy to the bracketing of two functors, we must bracket n functors. For example, we might write in the three-valued logic (XYZ) for the functor such that

4) $$(XYZ)p_1 \ldots p_n p_n = Xp_1 \ldots p_n$$

5) $$(XYZ)p_1 \ldots p_n Mp_n = Yp_1 \ldots p_n$$

6) $$(XYZ)p_1 \ldots p_n Np_n = Zp_1 \ldots p_n$$

where M and N have the matrices:

p	Mp	Np
1	2	3
2	3	1
3	1	2

But beyond simple considerations such as these, it is not clear how to proceed, nor of what value further investigations might be.

NOTES

1. The symbolism to be used in this paper is after that of Łukasiewicz as presented in [11], with certain modifications for the unary functors: Ip is "assertium" for p, i.e. $Ip = p$; Np is the negation of p; Op is "falsum" for p—always false; Up is "verum" for p—always true.

   For the binary functors, we have the explanatory table:

S	**P**	**L**	**C**	explanations
F	O	o	(none)	$Fpq = NCpCqq$
K	K	ϙ		$Kpq = NCpNq$
T	M	o–	¢	$Tpq = NCqp$
S	X	ȯ	⊽	$Spq = NCNpq$
H	L	–o	⊅	$Hpq = NCpq$
B	H	ϙ–	(none)	$Bpq = q = CCppq$
E	E	ϕ	≡	$Epq = KCpqCqp$
G	I	–ϙ	(none)	$Gpq = p = CCqqp$
J	F	ȯ–	(none)	$Jpq = Np = CCqqNp$
R	J	–o–	≢	$Rpq = NEpq$
P	G	–ȯ	(none)	$Ppq = Nq = CCppNq$
C	C	ϕ–	⊃	$Cpq = NKpNq$
A	A	–ϙ–	∨	$Apq = CNpq$
L	B	–ϕ	⊂	$Lpq = Cqp$
D	D	–ȯ–	|	$Dpq = NKpq$
V	V	–ϕ–	(none)	$Vpq = CpCqq$

   In this table, the column headed "**S**" is the symbolism in [11]; that headed "**P**", in [8] p. 12; "**L**", in [3] (see also [4], pp. 21-22, and [6]); "**C**" in [1], p. 37.

2. [7], 4.268, from his "Minute Logic" of 1902. Since the symbolism used is little known, the diagram is presented with the same symbolism as

the rest of the paper. I had investigated this method of diagramming the functors independently, before finding it quite accidentally in the writings of Peirce. It is interesting to note that, although Peirce was acquainted with the property of duality (see [7], 4.295), he did not remark on the obvious relation that reflection in the horizontal axis takes each functor into its dual.

3. That, in fact, each propositional functor is included by such a process can be seen by a simple combinatorial argument. Obviously, no functor is represented more than once, and since the number of n-ary functors is 2^{2^n}, we see that the number of $(n + 1)$-ary functors is the square of the number of n-ary functors.

4. Theorems 4 and 5 occur in [12], and theorem 6 occurs in [8], although clearly in different forms.

5. By "Sheffer functor" is meant any functor X, such that, by the use of X alone every functor can be defined. The name is a concession to universal use, although Sheffer's discovery was anticipated by about 30 years by Peirce—see [7], 4.12 and also 4.264.

6. For a proof of this, see [12].

7. For propositional calculus based on E alone, see Leśniewski [3], and also Łukasiewicz [5]. For that based on C and any arbitrary functor X, see Henkin [2].

8. In [10]. The two such functors which he investigated are (GB) and (GP)—Rose gives the definitions $(GB)\ pqr = AAKpqKpNrKqNr$ (which is his $Gpqr$) and $(GP)\ pqr = AAKpNqKprKNqr$ (which is his $Hpqr$).

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