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SIX NEW SETS OF INDEPENDENT AXIOMS FOR DISTRIBUTIVE LATTICES WITH O AND I

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In [4] Grau defined and discussed the following ternary Boolean functor*

A $\quad \phi(a\ b\ c) = (a \cap b) \cup (b \cap c) \cup (c \cap a)$

which, since the formula

B $\quad (a \cap b) \cup (b \cap c) \cup (c \cap a) = (a \cup b) \cap (b \cup c) \cap (c \cup a)$

holds in Boolean algebra, is, obviously, the self-dual operation.

In [1] Birkhoff and Kiss have shown that, if this connective of Grau is considered as lattice operation (called the *median* of a, b, c), then a distributive lattice with O and I can be defined in terms of this single functor. This result is formulated in [2], pp. 137-138, theorem 4, as follows

Let **A** *be any algebric system with a ternary operation* $\phi(a\ b\ c)$ *and elements O and I such that it satisfies*

(i) $\quad \phi(O\ a\ I) = a$

(ii) $\quad \phi(a\ b\ a) = a$

(iii) $\quad \phi(a\ b\ c) = \phi(b\ a\ c) = \phi(b\ c\ a)$

(iv) $\quad \phi(\phi(a\ b\ c)\ d\ e) = \phi(\phi(a\ d\ e)\ b\ \phi(c\ d\ e))$

identically. Then if we define

(v) $\quad a \cup b = \phi(a\ I\ b)$ $\quad$ and $\quad a \cap b = \phi(a\ O\ b)$

A *is a distributive lattice in which A holds.*

As problem 64, in [2], p. 138, Birkhoff put the question whether at least part of (iii) can be dispensed with, if a suitable permutation of (iv) is

*Instead of Birkhoff's notation for this ternary operation: $(a\ b\ c)$, cf. e.g. [2], p. 137, I use the symbol: $\phi\ (a\ b\ c)$ throughout this paper.

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used. The various solutions to this problem are already published by several authors. Namely:

a) In [7] Vassiliou has proved that conditions (i) - (iv̈) of Birkhoff follow from (i), (ii) and the following formula

(v̈i) $\phi(d\ \phi(a\ b\ c)\ e) = \phi(\phi(e\ d\ c)\ b\ \phi(e\ d\ a))$

b) In [3], pp. 24-25, Croisot has proved that (i), (ii) and

(v̈ii) $\phi(d\ \phi(a\ b\ c)\ e) = \phi(b\ \phi(c\ d\ e)\ \phi(a\ d\ e))$

imply (iii) and (iv̈).

c) In [5], p. 49, Hashimito has shown that we can deduce (iii) and (iv̈) from (i), (ii) and

(v̈iii) $\phi(d\ \phi(a\ b\ c)\ e) = \phi(\phi(e\ b\ d)\ a\ \phi(e\ c\ d))$

d) In [6], p. 30, Sholander announced without proof that conditions (i) - (iv̈) follow from the following two formulas

(iẍ) $\phi(O\ a\ \phi(I\ b\ I)) = a$

and

(ẍ) $\phi(d\ \phi(a\ b\ c)\ e) = \phi(\phi(d\ b\ e)\ c\ \phi(a\ d\ e))$

Many other axiom-systems satisfying Birkhoff's problem for distributive lattices with O and I can be established and added in this list. I present here six such sets of postulates. These axiom-systems possess a certain common feature, since the same permutation of (iv̈) is involved in their construction. Namely, I shall show that

α) Conditions (iii) and (iv̈) follow from (i), (ii) and

(ẍi) $\phi(d\ \phi(a\ b\ c)\ e) = \phi(\phi(d\ c\ e)\ \phi(d\ a\ e)\ b)$

β) Each of the following formulas

(ẍii) $\phi(O\ \phi(b\ a\ a)\ I) = a$

(ẍiii) $\phi(O\ \phi(a\ a\ b)\ I) = a$

and

(ẍiv̈) $\phi(O\ \phi(a\ b\ a)\ I) = a$

together with (ẍi) implies (i) and (ii).

c) Conditions (i) and (ẍi) follow from (ii) and either

(ẍv̈) $\phi(O\ \phi(d\ \phi(a\ b\ c)\ e)\ I) = \phi(\phi(d\ c\ e)\ \phi(d\ a\ e)\ b)$

or

(ẍv̈i) $\phi(d\ \phi(a\ b\ c)\ e) = \phi(O\ \phi(\phi(d\ c\ e)\ \phi(d\ a\ e)\ b)\ I)$

Proof:

Since, obviously, conditions (v̈i) - (ẍv̈i) follow from (i) - (iv̈) at once, it is sufficient to prove that the latter formulas follow from the respective sets of postulates mentioned in a) - c). Hence:

§1. Assume conditions (i), (ii) and (ẍi), i.e. the formulas

$A1 \quad \phi(O\ a\ I) = a$

$A2 \quad \phi(a\ b\ a) = a$

$A3 \quad \phi(d\ \phi(a\ b\ c)\ e) = \phi(\phi(d\ c\ e)\ \phi(d\ a\ e)\ b)$

Then:

$A4 \quad \phi(a\ b\ c) = \phi(c\ a\ b)$ [$A3$, d/O, e/I; $A1$, $a/\phi(a\ b\ c)$; $A1$, a/c; $A1$]

$A5 \quad \phi(a\ b\ c) = \phi(b\ c\ a)$ [$A4$; $A4$, a/c, b/a, c/b]

$A6 \quad \phi(a\ a\ b) = a$ [$A2$; $A4$, c/a]

$A7 \quad \phi(b\ a\ a) = a$ [$A2$; $A5$, c/a]

$A8 \quad \phi(a\ b\ c) = \phi(b\ a\ c)$

Dem.: $\phi(a\ b\ c) = \phi(a\ \phi(b\ b\ a)\ c) = \phi(\phi(a\ a\ c)\ \phi(a\ b\ c)\ b) = \phi(a\ \phi(a\ b\ c)\ b) = \phi(\phi(a\ c\ b)\ \phi(a\ a\ b)\ b) = \phi(\phi(a\ c\ b)\ a\ b) = \phi(b\ \phi(a\ c\ b)\ a) = \phi(\phi(b\ b\ a)\ \phi(b\ a\ a)\ c) = \phi(b\ a\ c)$
[$A6$, a/b, b/a; $A3$, a/b, c/a, d/a, e/c; $A6$, b/c; $A3$, d/a, c/b; $A6$; $A4$, $a/\phi(a\ c\ b)$, b/a, c/b; $A3$, b/c, c/b, d/b, e/a; $A6$, a/b, b/a; $A7$]

$A9 \quad \phi(\phi(a\ b\ c)\ d\ e) = \phi(\phi(a\ d\ e)\ b\ \phi(c\ d\ e))$

Dem.: $\phi(\phi(a\ b\ c)\ d\ e) = \phi(d\ \phi(a\ b\ c)\ e) = \phi(\phi(d\ c\ e)\ \phi(d\ a\ e)\ b) = \phi(\phi(d\ a\ e)\ b\ \phi(d\ c\ e)) = \phi(\phi(a\ d\ e)\ b\ \phi(c\ d\ e))$
[$A8$, $a/\phi(a\ b\ c)$, b/d, c/e; $A3$; $A5$, $a/\phi(d\ c\ e)$, $b/\phi(d\ a\ e)$, c/b; $A8$, a/d, b/a, c/e; $A8$, a/d, b/c, c/e]

Since $A8$, $A5$ and $A9$ constitute conditions (iii) and (iv̈), the proof is completed. The following modification of Croisot's argumentation, given in [3], pp. 24-25, shows that the axioms $A1$-$A3$ are mutually independent:

α') Assume that both O and I are Boolean 0 and 1 respectively and that $\phi(a\ b\ c)$ is the Boolean formula such that $\phi(a\ b\ c) = a$. Then $A2$ and $A3$ are verified, but $A1$ becomes a false formula.

β) Assume that both O and I are Boolean 0 and that $\phi(a\ b\ c)$ is the Boolean formula: $a \cup b \cup c$. Then $A1$ and $A3$ are verified, but $A2$ is falsified.

γ) Assume that both O and I are Boolean 0 and 1 respectively and that $\phi(a\ b\ c)$ is the Boolean formula: $(a \cup b) \cap c$. Then $A1$ and $A2$ are verified, but $A3$ is a false Boolean formula.

§2. Assume conditions (ẍi) and (ẍii), i.e. the formulas

$B1 \quad \phi(d\ \phi(a\ b\ c)\ e) = \phi(\phi(d\ c\ e)\ \phi(d\ a\ c)\ b)$

and

$B2 \quad \phi(O\ \phi(b\ a\ a)\ I) = a$

Then:

$B3 \quad \phi(b\ a\ a) = \phi(a\ c\ \phi(b\ a\ a))$

Dem.: $\phi(b\ a\ a) = \phi(O\ \phi(\phi(b\ c\ c)\ \phi(b\ a\ a)\ \phi(b\ a\ a))\ I) = \phi(\phi(O\ \phi(b\ a\ a)\ I)\ \phi(O\ \phi(b\ c\ c)\ I)\ \phi(b\ a\ a)) = \phi(a\ c\ \phi(b\ a\ a))$
[$B2$; $a/\phi(b\ a\ a)$, $b/\phi(b\ c\ c)$; $B1$, $a/\phi(b\ c\ c)$, $b/\phi(b\ a\ a)$, $c/\phi(b\ a\ a)$, d/O, e/I; $B2$; $B2$, a/c]

$B4 \quad \phi(b\ a\ a) = \phi(\phi(b\ a\ a)\ a\ c)$

Dem.: $\phi(b\ a\ a) = \phi(O\ \phi(b\ \phi(b\ a\ a)\ \phi(b\ a\ a))\ I) = \phi(O\ \phi(\phi(b\ a\ a)\ c\ \phi(b\ \phi(b\ a\ a)\ \phi(b\ a\ a)))\ I) = \phi(\phi(O\ \phi(b\ \phi(b\ a\ a)\ \phi(b\ a\ a))\ I)\ \phi(O\ \phi(b\ a\ a)\ I)\ c) = \phi(\phi(b\ a\ a)\ a\ c)$
[$B2$, $a/\phi(b\ a\ a)$; $B3$, $a/\phi(b\ a\ a)$; $B1$, $a/\phi(b\ a\ a)$, b/c, $c/\phi(b\ \phi(b\ a\ a)\ \phi(b\ a\ a))$, d/O, e/I; $B2$, $a/\phi(b\ a\ a)$; $B2$]

$B5 \quad a = \phi(b\ a\ a)$

Dem.: $a = \phi(O\ \phi(b\ a\ a)\ I) = \phi(O\ \phi(\phi(b\ a\ a)\ a\ \phi(b\ b\ b))\ I) = \phi(\phi(O\ \phi(b\ b\ b)\ I)\ \phi(O\ \phi(b\ a\ a)\ I)\ a) = \phi(b\ a\ a)$
[$B2$; $B4$, $c/\phi(b\ b\ b)$; $B1$, $a/\phi(b\ a\ a)$, b/a, $c/\phi(b\ b\ b)$, d/O, e/I; $B2$, a/b; $B2$]

$B6 \quad \phi(O\ a\ I) = a$ [$B2$; $B5$]

$B7 \quad a = \phi(a\ b\ a)$

Dem.: $a = \phi(O\ \phi(b\ a\ a)\ I) = \phi(\phi(O\ a\ I)\ \phi(O\ b\ I)\ a) = \phi(a\ b\ a)$
[$B2$; $B1$, a/b, b/a, c/a, d/O, e/I; $B6$; $B6$, a/b]

Since we obtained $B6$ and $B7$, i.e. conditions (i) and (ii), the present set of postulates implies the axiom-system discussed in §1. Hence, the proof is given. The first and the third interpretations given in §1 show that $B1$ and $B2$ are mutually independent.

§3. Assume now conditions (xi) and (xiii), i.e. the formulas

$C1 \quad \phi(d\ \phi(a\ b\ c)\ e) = \phi(\phi(d\ c\ e)\ \phi(d\ a\ e)\ b)$

and

$C2 \quad \phi(O\ \phi(a\ a\ b)\ I) = a$

Then:

$C3 \quad \phi(a\ a\ b) = \phi(c\ a\ \phi(a\ a\ b))$

Dem.: $\phi(a\ a\ b) = \phi(O\ \phi(\phi(a\ a\ b)\ \phi(a\ a\ b)\ \phi(c\ c\ b))\ I) = \phi(\phi(O\ \phi(c\ c\ b)\ I)\ \phi(O\ \phi(a\ a\ b)\ I)\ \phi(a\ a\ b)) = \phi(c\ a\ \phi(a\ a\ b))$
[$C2$, $a/\phi(a\ a\ b)$, $b/\phi(c\ c\ b)$; $C1$, $a/\phi(a\ a\ b)$, $b/\phi(a\ a\ b)$, $c/\phi(c\ c\ b)$, d/O, e/I; $C2$, a/c; $C2$]

$C4 \quad a = \phi(a\ b\ a)$

Dem.: $a = \phi(O\ \phi(a\ a\ b)\ I) = \phi(O\ \phi(b\ b\ b)\ a\ \phi(a\ a\ b))\ I) = \phi(\phi(O\ \phi(a\ a\ b)\ I)$
$\phi(O\ \phi(b\ b\ b)\ I)\ a) = \phi(a\ b\ a)$
[$C2$; $C3$, $c/\phi(b\ b\ b)$; $C1$, $a/\phi(b\ b\ b)$, b/a, $c/\phi(a\ a\ b)$, d/O, e/I; $C2$; $C2$, a/b]

$C5 \quad a = \phi(a\ a\ b)$

Dem.: $a = \phi(a\ \phi(b\ b\ b)\ a) = \phi(\phi(\phi(a\ b\ a)\ \phi(a\ b\ a)\ b)) = \phi(a\ a\ b)$
[$C4$, $b/\phi(b\ b\ b)$; $C1$, a/b, c/b, d/a, e/a; $C4$; $C4$]

$C6 \quad \phi(O\ a\ I) = a$ [$C2$; $C5$]

Since $C6$, $C4$ and $C1$ constitute the axiom-system given in §1, the proof is completed. The first interpretation presented in §1 proves that $C2$ does not follow from $C1$. Assume now that

δ) Both O and I are Boolean 0 and 1 respectively and $\phi(a\ b\ c)$ is the Boolean formula such that $\phi(a\ b\ c) = b$.

This interpretation shows that $C2$ does not imply $C1$.

§4. Assume now conditions ($\ddot{x}$i) and ($\ddot{x}$i$\ddot{v}$), i.e. the formulas

$D1 \quad \phi(d\ \phi(a\ b\ c)\ e) = \phi(\phi(d\ c\ e)\ \phi(d\ a\ e)\ b)$

and

$D2 \quad \phi(O\ \phi(a\ b\ a)\ I) = a$

Then:

$D3 \quad \phi(a\ b\ a) = \phi(a\ a\ b)$

Dem.: $\phi(a\ b\ a) = \phi(O\ \phi(\phi(a\ b\ a)\ b\ \phi(a\ b\ a))\ I) = \phi(\phi(O\ \phi(a\ b\ a)\ I)\ \phi(O$
$\phi(a\ b\ a)\ I)\ b) = \phi(a\ a\ b)$
[$D2$, $a/\phi(a\ b\ a)$; $D1$, $a/\phi(a\ b\ a)$, $c/\phi(a\ b\ a)$, d/O, C/I; $D2$: $D2$]

$D4 \quad \phi(O\ \phi(a\ a\ b)\ I) = a$ [$D2$; $D3$]

Since we obtained $D4$, we have the axiom-system presented in §3, and, therefore, the proof is given. The first and the third interpretations from §1 show that $D1$ and $D2$ are mutually independent.

§5. Assume conditions (ii) and ($\ddot{x}\ddot{v}$), i.e. the formulas

$E1 \quad \phi(a\ b\ a) = a$

and

$E2 \quad \phi(O\ \phi(d\ \phi(a\ b\ c)\ e)\ I) = \phi(\phi(d\ c\ e)\ \phi(d\ a\ e)\ b)$

Then:

$E3 \quad \phi(O\ a\ I) = a$

Dem.: $\phi(O\ a\ I) = \phi(O\ \phi(a\ \phi(a\ a\ a)\ a)\ I) = \phi(\phi(a\ a\ a)\ \phi(a\ a\ a)\ a) = \phi(a\ a\ a) = a$
[*E1*, $b/\phi(a\ a\ a)$; *E2*, b/a, c/a, d/a, e/a; *E1*, b/a; *E1*, b/a; *E1*, b/a]

E4 $\phi(d\ \phi(a\ b\ c)\ e) = \phi(\phi(d\ c\ e)\phi(d\,a\,e)\,b)$ [*E2*; *E3*, $a/\phi(d\ \phi(a\ b\ c)\ e)$]

Since we proved *E3* and *E4*, we obtained the set of postulates given in §1. Hence, the proof is completed. The first and the second interpretations from §1 show that *E1* and *E2* are mutually independent.

§6. Assume conditions (ii) and (xvi), i.e. the formulas

F1 $\phi(a\ b\ a) = a$

and

F2 $\phi(d\ \phi(a\ b\ c)\ e) = \phi(O\ \phi(\phi(d\ c\ e)\ \phi(d\ a\ e)\ b)\ I)$

Then:

F3 $a = (O\ a\ I)$

Dem.: $a = \phi(a\ \phi(a\ a\ a)\ a) = \phi(O\ \phi(\phi(a\ a\ a)\ \phi(a\ a\ a)\ a)\ I) = \phi(O\ \phi(a\ a\ a)\ I) = \phi(O\ a\ I)$
[*F1*, $b/\phi(a\ a\ a)$; *F2*, b/a, c/a, d/a, e/a; *F1*, b/a; *F1*, b/a; *F1*, b/a]

F4 $\phi(d\ \phi(a\ b\ c)\ e) = \phi(\phi(d\ c\ e)\ \phi(d\ a\ e)\ b)$
[*F2*; *F3*, $a/\phi(\phi(d\ c\ e)\ \phi(d\ a\ e)\ b)$]

Since *F3* and *F4* are obtained, we have the axiom-system given in §1. Therefore, the proof is completed. The same interpretations which are used in §5 show that *F1* and *F2* are also mutually independent.

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