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SOME THEOREMS ON THE STRUCTURE OF MUTANT SETS AND THEIR APPLICATIONS TO GROUP AND RING THEORIES

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The author dedicates this paper to the memory of Alan Mathison Turing on the occasion of the 50th anniversary of that logician's birthday, 23rd June, 1912.

§0. Introduction. The purpose of this paper is to present the results of two announcements [1], [2] and to generalize the results of two other papers [3], [4]. The results of §1 are applicable to any general algebraic system, i.e., nonempty set on which there is defined a nonempty index set of closed binary composition laws. For the sake of concreteness some applications to group and ring theories are given in §2 and §3, respectively. Some applications of the general theory to elementary number theory are given elsewhere [5], [6]. Applications in the context of the algebra and logic of relations of abstract mathematical biology are also given elsewhere [4].

§1. General Theory. Recall [6] that a (λ, T)-*mutant* set of an algebraic system $(S, *)$ is a subset M of S that satisfies the condition $M_1 * M_2 * \ldots * M_\lambda \subseteq \overline{M} \cap T$, where $M = M_i$ for all i, λ is an integer ≥ 2 and T together with $*$ forms an algebraic subsystem of $(S, *)$. A (λ, T)-mutant set M of a system $(S, *)$ is said to be a *maximal* (λ, T)-mutant set of $(S, *)$ provided there is no (λ, T)-mutant set of $(S, *)$ which properly contains M.

Theorem 1.1. Every subset of a (λ, T)-mutant set M of $(B, *)$ is a (λ, T)-mutant set of $(B, *)$.

Proof: Put $A = A_i$ and $M = M_i$ for all i. Then $A_1 * \ldots * A_\lambda \subseteq M_1 * \ldots * M_\lambda \subseteq \overline{M} \cap T \subset \overline{A} \cap T$ for every $A \subseteq M$.

Theorem 1.2. Let φ be a homomorphism from $(A, *)$ into $(B, \circ)$. Let M be a (λ, T)-mutant set of $(A, *)$. If $\varphi(\overline{M} \cap T) \subseteq \overline{\varphi(M)} \cap S$ then $\varphi(M)$ is a (λ, S)-mutant set of $(B, \circ)$.

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Proof: Put $M = M_i$ for all i. Then $\varphi(M_1) \circ \ldots \circ \varphi(M_\lambda) = \varphi(M_1 * \ldots * M_\lambda) \subseteq \varphi(\overline{M} \cap T) \subseteq \overline{\varphi(M)} \cap S$.

Theorem 1.3. Let φ be a homomorphism from $(A, *)$ onto $(B, \circ)$. Let M be a (λ, S)-mutant set of $(B, \circ)$. Let C be the inverse image of M under φ. Then (under no hypotheses) C is a (λ, A)-mutant set of $(A, *)$.

Proof: Clearly C is a nonempty set. Let $a_i \in C$. Then $\varphi(a_1 * \ldots * a_\lambda) = \varphi(a_1) \circ \ldots \circ \varphi(a_\lambda) \in \overline{M} \cap S$. Hence $a_1 * \ldots * a_\lambda \notin C$, for otherwise, $\varphi(a_1 * \ldots * a_\lambda) \in M$; a contradiction.

Theorem 1.4. Let φ be an isomorphism from $(A, *)$ onto $(B, \circ)$. Then each (λ, A)-mutant set of $(A, *)$ induces a (λ, B)-mutant set of $(B, \circ)$ and each (λ, B)-mutant set of $(B, \circ)$ induces a (λ, A)-mutant set of $(A, *)$.

Proof: Apply theorems 1.2 and 1.3, observing that the condition $\varphi(\overline{M} \cap A) \subseteq \overline{\varphi(M)} \cap B$ holds since φ is a one-to-one mapping.

Lemma 1.1. If N is a proper subset of a (λ, T)-mutant set of $(B, \circ)$ then N is not a maximal (λ, T)-mutant set of $(B, \circ)$.

Proof: Assuming the contrary contradicts the definition of a maximal (λ, T)-mutant set.

Theorem 1.5. Let φ be a homomorphism from $(A, *)$ onto $(B, \circ)$. Let M be a maximal (λ, A)-mutant set of $(A, *)$. If $\varphi(\overline{M} \cap A) \subseteq \overline{\varphi(M)} \cap S$ then $\varphi(M)$ is a maximal (λ, S)-mutant set of $(B, \circ)$.

Proof: By putting $T = A$ in theorem 1.2 $\varphi(M)$ becomes a (λ, S)-mutant set. Suppose $\varphi(M)$ is not a maximal (λ, S)-mutant set of $(B, \circ)$. Then there exists a (λ, S)-mutant set P of $(B, \circ)$ such that $P \supset \varphi(M)$. By the definition of a mapping $\{x \in A : \varphi(x) \in P\} \supset M$. But by theorem 1.3 and lemma 1.1 there is a contradiction.

Theorem 1.6. Let φ be an isomorphism from $(A, *)$ onto $(B, \circ)$. Then each maximal (λ, A)-mutant set of $(A, *)$ induces a maximal (λ, B)-mutant set of $(B, \circ)$ and conversely.

Proof: By theorem 1.5 and the one-to-one nature of φ each maximal (λ, A)-mutant set of $(A, *)$ gives a maximal (λ, B)-mutant set of $(B, \circ)$. Let M be a maximal (λ, B)-mutant set of $(B, \circ)$. Suppose $\{x \in A : \varphi(x) \in M\} = N$ is not a maximal (λ, A)-mutant set of $(A, *)$. Then there exists a (λ, A)-mutant set V of $(A, *)$ such that $V \supset N$. Since φ is one-to-one $\varphi(V) \supset M$, and by theorem 1.2 and the one-to-one nature of φ, $\varphi(V)$ is a (λ, B)-mutant set. But by lemma 1.1 there is a contradiction.

Theorem 1.7. There exists a unique minimal subsystem (called the *closed hull*) $(S, \circ)$ of $(B, \circ)$ containing a nonempty (λ, T)-mutant set of a system $(B, \circ)$.

Proof: Consider the $\{(S_i, \circ) : i \in I\}$ of all systems containing M. This set is nonempty, since $(B, \circ)$ is one of its elements. Put $S = \bigcap_{i \in I} S_i$. Clearly $S \circ S \subseteq S$.

§2. Groups. In this case we assume that systems have the algebraic structure of groups. E.g., in the additive group of all integers the set E of all even integers is a subgroup and the set of all odd integers is a maximal (λ, E)-mutant set for all strictly positive even integers λ.

Theorem 2.1. Let $(\mathbf{G}, \circ)$ be a group. Then every left coset $a \circ \mathbf{H}$ (modulo a normal subgroup $(\mathbf{H}, \circ)$), with the exception of $\mathbf{H}$ and $(\lambda, \mathbf{H} \circ a^{-1})$-mutant cosets, is a $(\lambda + 2, \mathbf{G})$-mutant set; in addition a similar theorem exists for *right* cosets.

Proof: Let $a \in \mathbf{G}$. Put $a = a_i$ for all i. If there exist $h_i \in \mathbf{H}$ such that $(a_1 \circ h_1) \circ \ldots \circ (a_{\lambda+2} \circ h_{\lambda+2}) = a \circ h_o$, then cancelling the left-hand and right-hand ends of the left-hand side of the equality $(a_2 \circ h_2) \circ \ldots \circ (a_{\lambda+1} \circ h_{\lambda+1}) = h_o \circ a^{-1}$. Thus, by normality of $(\mathbf{H}, \circ)$ which makes the composition of cosets yield a coset, it follows that $(a_2 \circ \mathbf{H}) \circ \ldots \circ (a_{\lambda+1} \circ \mathbf{H}) = \mathbf{H} \circ a^{-1}$. Now generate the contrapositive sequence of implications from the above implications and use the hypotheses that $a \circ \mathbf{H} \neq \mathbf{H}$ and that $(a_2 \circ \mathbf{H}) \circ \ldots \circ (a_{\lambda+1} \circ \mathbf{H}) \not\subseteq \mathbf{H} \circ a^{-1}$ to arrive at the conclusion that $a \circ \mathbf{H}$ is a $(\lambda + 2, \mathbf{G})$-mutant set. The proof is similar for the case of right cosets.

Corollary 2.1. Let $(\mathbf{G}, \circ)$ be a group. Then every left (right or two-sided) coset (modulo a normal subgroup $(\mathbf{H}, \circ)$), except $\mathbf{H}$ and $(\lambda, \mathbf{H})$-mutant cosets, is a $(\lambda + 1, \mathbf{G})$-mutant set.

Proof: In the proof of theorem 2.1, putting $a = a_i$ for all i, observe that $(a_1 \circ \mathbf{H}) \circ \ldots \circ (a_{\lambda+1} \circ \mathbf{H}) = \mathbf{H}$. Then apply the hypotheses of the present corollary to the contrapositive sequence of implications of the proof of theorem 2.1.

Corollary 2.2. Let $(\mathbf{G}, \circ)$ be a group. Then every coset (modulo a subgroup $(\mathbf{H}, \circ)$) except $\mathbf{H}$ is a $(2, \mathbf{G})$-mutant set.

Proof: First note that $(\mathbf{H}, \circ)$ need *not* be a *normal* subgroup. If there exist $h_i \in \mathbf{H}$ such that $(a \circ h_1) \circ (a \circ h_2) = a \circ h_o$, then cancelling, $a = h_1^{-1} \circ h_o \circ h_2^{-1}$ or $a \circ \mathbf{H} = \mathbf{H}$.

Corollary 2.3. Let $(\mathbf{G}, \circ)$ be a group. Then the only idempotent element of $(\mathbf{G}, \circ)$ is its identity element.

Proof: Consider the quotient group $(\mathbf{G}, \circ)/(1, \circ)$ and apply corollary 2.2.

Theorem 2.2. Let $(\mathbf{G}, \circ)$ be a group with identity 1. Then every (λ, T)-mutant set M has a closed hull $\mathbf{H}_M$.

Proof: If M is nonempty apply theorem 1.7 to $(\mathbf{G}, \circ)$. If M is empty put $\mathbf{H}_M = \{1\}$ and apply corollary 2.3.

§3. Rings. In this section we will consider an arbitrary skew field, $(\mathbf{R}, +, .)$.

Theorem 3.1. Consider a skew field $(\mathbf{R}, +, \cdot)$. Let $\mathbf{I}$ be an ideal of $(\mathbf{R}, +, \cdot)$. Consider a coset decomposition of $\mathbf{I}$ viewed as an additive group.

Define addition of cosets in the usual manner so that it coincides with addition + of the elements of **R**. Define multiplication $*$ of cosets by the relation $(a + \mathbf{I}) * (b + \mathbf{I}) = (a \cdot b) + \mathbf{I}$ for $a \in \mathbf{R}$, $b \in \mathbf{R}$. Then every coset with the exception of **I** and $1 + \mathbf{I}$ is a doubly $(2, \mathbf{R})$-mutant set.

Proof: We may and do apply theorem 2.1 to the additive abelian group of the field, since all of its subgroups are *normal* ones. Then we apply corollary 2.2 to the group $(\{a + \mathbf{I} : a \in \mathbf{R}\}, *)$ which is not necessarily an abelian one.

Corollary 3.1. The only doubly idempotent element of a skew field is the additive identity element. The only other idempotent element is the multiplicative identity element.

Proof: Consider $(\mathbf{R}, +, \cdot)/(\mathbf{O}, +, \cdot)$, where **O** is the additive identity. Then apply theorem 3.1.

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