

A SIMPLE DECISION PROCEDURE FOR ONE-VARIABLE  
 IMPLICATION/NEGATION FORMULAE IN  
 INTUITIONIST LOGIC

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Those who have agonized over the intuitionist theory of deduction, as I have, will perhaps welcome a simple decision procedure for implication/negation formulae containing only one variable (' $C-N-p$  formulae'). The procedure consists essentially in showing every such formula to be equivalent to one of six non-mutually-equivalent forms. Since the intuitionist calculus admits of the replacement of equivalents, any  $C-N-p$  formula, or  $C-N-p$  portion of a more complex formula, may be replaced by one of these six forms.<sup>1</sup>

The six forms are the following:

1.  $Cpp$
2.  $NNp$
3.  $CNNpp$
4.  $p$
5.  $Np$
6.  $NCpp$ ,

and the more complex formulae in which they occur as arguments of the functions  $C$  and  $N$  are each equivalent to one of the original six, as indicated in the following table:

TABLE I

	C	1	2	3	4	5	6	N
$Cpp$	* 1	1	2	3	4	5	6	6
$NNp$	2	1	1	3	3	5	5	5
$CNNpp$	3	1	2	1	2	5	6	6
$p$	4	1	1	1	1	5	5	5
$Np$	5	1	2	1	2	1	2	2
$NCpp$	6	1	1	1	1	1	1	1

The table is read in the normal way for truth-functional matrices. Thus  $CCNNppNp = C35 = 5 = Np$ . As a more complicated example,  $CCpNpNCCpCpNpNpNCNpp = CC45NCC4C25NC54 = C5NCC45N2 = C5NC55 = C5N1 = C56 = 2 = NNp$ .

It remains to show that the relationships of equivalence summarized in table I can all be proved in intuitionist logic. We show, for example, that  $C46 = 5$  by showing that the formulae  $CC465$  and  $C5C46$  are intuitionist theses. Of the following, theses 1-15 are presented without proof: they are found in Hilbert and Bernays, *Grundlagen der Mathematik*, I, p. 68 ff., and in Church, *Introduction to Mathematical Logic*, pp. 141-2, 146-7. The remainder are deduced from them using the rules of substitution, *modus ponens* and replacement of equivalents, the last being derivable by repeated employment of theses 1, 2 and 9.

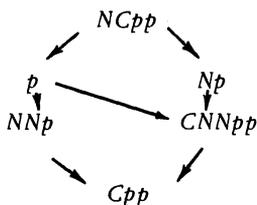
1.  $CCpqCCqrCpr$
2.  $CCqrCCpqCpr$
3.  $CCpCqrCqCpr$
4.  $CpCqp$
5.  $CCpCpqCpq$
6.  $Cpp$
7.  $CCppqq$
8.  $CpCCppqq$
9.  $CCpqCNqNp$
10.  $CCpNqCqNp$
11.  $CpNNp$
12.  $CNNpNp$
13.  $CCpNpNp$
14.  $CNpCpq$
15.  $NNCpNp$
- (4) 16.  $CqCCppq^2$
- 4=C6-17.  $CqCpp$
- 1=C11-18.  $CCNNpNpCpNp$
- (18, 13, 4, RE) 19.  $CCNNpNpNp$
- 2=C11-20.  $CCNppCNpNNp$
- (20, 13, 4, RE) 21.  $CCNppNNp$
- (14, 16, 7, RE) 22.  $CNCppq$
- 3=C6-23.  $CCqpCCCqppp$
- 1=C8-24.  $CCCCqpppCqp$
- 1=C4-25.  $CCCpqrCqr$
- 4=C15-26.  $CCppNNCpNp$
- 10=C26-27.  $CNCpNpNpNCpp$ .

We may now proceed to demonstrate the equivalences of table I. Unbracketed numbers denote formulae of the table, and bracketed numbers refer to theses 1-27 above.

$N1=6$ ;  $N2=5$  (11, 12);  $N3=6$  (22, 27);  $N4=5$ ;  $N5=2$ ;  $N6=1$  (11, 17);  
 $C11=1$ ;  $C12=2$ ;  $C13=3$ ;  $C14=4$ ;  $C15=5$ ;  $C16=6$  (7, 16);  
 $C61=C62=C63=C64=C65=C66=1$  (22, 16, 17);  
 $C21=C31=C41=C51=C22=C33=C44=C55=1$  (6, 16, 17);  
 $C26=C15=5$ ;  $C36=C16=6$ ;  $C46=C15=5$ ;  $C56=C12=2$  (10, 7, 16);  
 $C23=3$  (4, 5);  $C24=3$ ;  $C25=5$  (4, 19);  
 $C42=1$  (11);  $C43=1$  (4);  $C45=5$  (4, 13);  
 $C52=2$  (4, 13);  $C53=1$  (14, 3);  $C54=2$  (21, 14);  
 $C32=C3N5=C5N3$  (10)= $C5N1=C1N5$  (10)= $N5=2$ ;  
 $C34=CC244=CCC5444=C54$  (23, 24) =2;  
 $C35$ . Now  $C5C35=1$  (4), and  $CC355=CCC2455=CCC245C45=1$ (25). Hence  $C35=5$ .

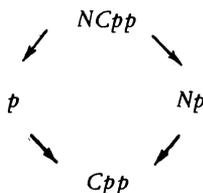
The six non-equivalent forms are related to one another in the following way, where the arrows denote implications:

TABLE II



For comparison, the four non-equivalent  $C-N-p$  forms of two-valued logic are related as follows:

TABLE III



Both the intuitionist and two-valued logics have the same non-equivalent  $C-p$  forms, namely  $p$  and  $Cpp$ .

NOTES

1. The six forms are listed in J. C. C. McKinsey and A. Tarski: 'Some theorems about the sentential calculi of Lewis and Heyting', *The Journal of Symbolic Logic* 13 (1948), p. 12, although no proof is there given that the number of such forms is exactly six.
2. Proof notation is based on that of Łukasiewicz (see e.g. *Aristotle's Syllogistic*, p. 81), with substitutions omitted. ' $(w, x, y, RE)$ ' on line  $z$  means that  $z$  is the result of replacing in  $w$  one expression by another shown to be equivalent to it through the implications  $x$  and  $y$ .

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