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A SIMPLE DECISION PROCEDURE FOR ONE-VARIABLE IMPLICATION/NEGATION FORMULAE IN INTUITIONIST LOGIC

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Those who have agonized over the intuitionist theory of deduction, as I have, will perhaps welcome a simple decision procedure for implication/negation formulae containing only one variable ('*C-N-p* formulae'). The procedure consists essentially in showing every such formula to be equivalent to one of six non-mutually-equivalent forms. Since the intuitionist calculus admits of the replacement of equivalents, any *C-N-p* formula, or *C-N-p* portion of a more complex formula, may be replaced by one of these six forms.[1]

The six forms are the following:

1. *Cpp*
2. *NNp*
3. *CNNpp*
4. *p*
5. *Np*
6. *NCpp*,

and the more complex formulae in which they occur as arguments of the functions *C* and *N* are each equivalent to one of the original six, as indicated in the following table:

TABLE I

	C	1	2	3	4	5	6	N
Cpp	* 1	1	2	3	4	5	6	6
NNp	2	1	1	3	3	5	5	5
CNNpp	3	1	2	1	2	5	6	6
p	4	1	1	1	1	5	5	5
Np	5	1	2	1	2	1	2	2
NCpp	6	1	1	1	1	1	1	1

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The table is read in the normal way for truth-functional matrices. Thus *CCNNppNp* = *C35* = *5* = *Np*. As a more complicated example, *CCpNpNCCp CNNpNpNCNpp* = *CC45NCC4C25NC54* = *C5NCC45N2* = *C5NC55* = *C5N1* = C56 = 2 = NNp.

It remains to show that the relationships of equivalence summarized in table I can all be proved in intuitionist logic. We show, for example, that *C46* = *5* by showing that the formulae *CC465* and *C5C46* are intuitionist theses. Of the following, theses 1-15 are presented without proof: they are found in Hilbert and Bernays, *Grundlagen der Mathematik,* I, p. 68 ff., and in Church, *Introduction to Mathematical Logic*, pp. 141-2, 146-7. The remainder are deduced from them using the rules of substitution, *modus ponens* and replacement of equivalents, the last being derivable by repeated employment of theses *1*, *2* and 9.

	1.	*CCpqCCqrCpr*
	2.	*CCqrCCpqCpr*
	3.	*CCpCqrCqCpr*
	4.	*CpCqp*
	5.	*CCpCpqCpq*
	6.	*Cpp*
	7.	*CCCppqq*
	8.	*CpCCpqq*
	9.	*CCpqCNqNp*
	10.	*CCpNqCqNp*
	11.	*CpNNp*
	12.	*CNNNpNp*
	13.	*CCpNpNp*
	14.	*CNpCpq*
	15.	*NNCNNpp*
(4)	*16.*	*CqCCppq*[2]
4=C6–	*17.*	*CqCpp*
1=C11–	*18.*	*CCNNpNpCpNp*
(18,13,4,RE)	*19.*	*CCNNpNpNp*
2=C11–	*20.*	*CCNppCNpNNp*
(20,13,4,RE)	*21.*	*CCNppNNp*
(14,16,7,RE)	*22.*	*CNCppq*
3=C6–	*23.*	*CCqpCCCqppp*
1=C8–	*24.*	*CCCCqpppCqp*
1=C4–	*25.*	*CCCpqrCqr*
4=C15–	*26.*	*CCppNNCNNpp*
10=C26–	*27.*	*CNCNNppNCpp.*

We may now proceed to demonstrate the equivalences of table I. Unbracketed numbers denote formulae of the table, and bracketed numbers refer to theses *1-27* above.

N1=6; N2=5 (11,12); N3=6 (22,27); N4=5; N5=2; N6=1 (11,17);
C11=1; C12=2; C13=3; C14=4; C15=5; C16=6 (7,16);
C61=C62=C63=C64=C65=C66=1 (22,16,17);
C21=C31=C41=C51=C22=C33=C44=C55=1 (6,16,17);
C26=C15=5; C36=C16=6; C46=C15=5; C56=C12=2 (10,7,16);
C23=3 (4,5); C24=3; C25=5 (4,19);
C42=1 (11); C43=1 (4); C45=5 (4,13);
C52=2 (4,13); C53=1 (14,3); C54=2 (21,14);
C32=C3N5=C5N3 (10)=C5N1=C1N5 (10)=N5=2;
C34=CC244=CCC5444=C54 (23,24) =2;
C35. Now *C5C35=1 (4),* and *CC355=CCC2455=CCC245C45=1(25).* Hence *C35 =5.*

The six non-equivalent forms are related to one another in the following way, where the arrows denote implications:

TABLE II

$$\begin{tikzcd}
& NCpp \arrow[dl] \arrow[dr] & \\
p \arrow[d] \arrow[drr] & & Np \arrow[d] \\
NNp \arrow[dr] & & CNNpp \arrow[dl] \\
& Cpp &
\end{tikzcd}$$

For comparison, the four non-equivalent *C-N-p* forms of two-valued logic are related as follows:

TABLE III

$$\begin{tikzcd}
& NCpp \arrow[dl] \arrow[dr] & \\
p \arrow[dr] & & Np \arrow[dl] \\
& Cpp &
\end{tikzcd}$$

Both the intuitionist and two-valued logics have the same non-equivalent *C-p* forms, namely *p* and *Cpp*.

NOTES

1. The six forms are listed in J. C. C. McKinsey and A. Tarski: 'Some theorems about the sentential calculi of Lewis and Heyting', *The Journal of Symbolic Logic* 13 (1948), p. 12, although no proof is there given that the number of such forms is exactly six.
2. Proof notation is based on that of Łukasiewicz (see e.g. *Aristotle's Syllogistic*, p. 81), with substitutions omitted. '(*w,x,y,RE*)' on line *z* means that *z* is the result of replacing in *w* one expression by another shown to be equivalent to it through the implications *x* and *y*.

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