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FOUR TYPES OF GENERAL RECURSIVE WELL-ORDERINGS

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In this paper the term 'g.r. (general recursive) well-ordering' refers to g.r. well-orderings of the set of all the natural numbers. Markwald in [1, Satz 5] gave an example showing that some of his recursively enumerable well-orderings can exhibit non-constructive character in an important aspect. In this paper we shall give more examples of similar kind; they are all g.r. well-orderings and are more or less non-constructive.

The examples suggest a classification of g.r. well-orderings into four types, which is based on the following three conditions:

α) There are two g.r. functions $\mathbf{H}(n)$ and $\mathbf{G}(n)$ such that $\mathbf{H}(n) = 0$, 1 or 2 according as n is the first element, a successor or a limit and $\mathbf{G}(n) = 0$ or 1 according as n is the last element or not.

β) There is an effective method for finding the successor of any element which is not the last one.

γ) There is an effective method for finding the limit of any g.r. increasing bounded sequence $\{a_j\}$.

Here the term 'effective' is used in the sense that, given a set of entities, each of which is associated with a unique number in some specified manner, then a method for finding the associated number for each such entity is effective if, for any effectively given sequence of such entities $\{E_j\}$, the number associated with each E_j can be found by that method and the number found is a g.r. function of j.

The four types of g.r. well-orderings are characterized by the conditions as shown in the following.

Types of g.r. well-ordering	I	II	III	IV
Conditions satisfied		α	α, β	α, β, γ
Conditions not satisfied	α, β, γ	β, γ	γ	

In view of the nature of the conditions α, β, γ, g.r. well-orderings of each of the types II, III, IV can be considered as 'more constructive' than

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those of the preceding type. Examples of g.r. well-ordering for each of the four types are given in this note. Theorem 1 shows that g.r. well-orderings of type IV give only order types $\leqq \omega + \omega$.

The author has used the term 'constructively given g.r. well-ordering' elsewhere [2] informally. Now we find two plausible definitions for this term. We may define it as to refer to g.r. well-orderings of type IV or let it to refer to those that satisfy the following condition.

δ) There is an effective method for finding the first element of any non-empty g.r. set.

The latter definition seems more natural. However, constructively given g.r. well-orderings under this definition give still smaller ordinal. In fact they give only the order type ω as is asserted by Theorem 3. Thus, strictly speaking, there is no constructive theory of ordinal numbers unless it is limited to a very narrow field. From Theorem 3 and Example of Type IV it can be easily seen that for any g.r. well-ordering $\prec$, δ implies α & β & γ but not vice versa.

In connection with the notations for constructive ordinals [3] we need consider two more conditions as follows:

β') There is an effective method for finding the predecessor of any element which is a successor.

γ') There is an effective method for finding, for any element which is a limit, a g.r. increasing bounded sequence which has the element as limit.

By using a suggestion made by Markwald in [1, Satz 9] it can be shown that unlike γ, γ' is satisfied by every g.r. well-ordering. Further we have (i) α and β imply β' and (ii) α and β' imply β. Thus with respect to any one of the g.r. well-orderings of type III or IV the natural numbers form a system S_3 of notations for a segment of the constructive ordinals. (See [3].)

The g.r. well-orderings of type III, though not wholly free from non-constructive characters, give much more order types than those of Type IV. In fact, their order types include all the constructive ordinals. (See [4, Theorem A, *p*. 410].) They also include all the order types of the g.r. well-orderings in general. (See [1, Satz 9].)

Theorem 1. The order types of the g.r. well-orderings of type IV do not exceed $\omega + \omega$.

Proof. Suppose that a g.r. well-ordering $\prec$ is of type IV and that its order type is $> \omega + \omega$. We shall show that these two suppositions are incompatible. By the condition α, we can effectively find the first element in the ordering $\prec$. Let $S(x)$ mean the successor of x. Let a_0 be the first element and $a_{i+1} = S(a_i)$; b_0 be the limit of $\{a_i\}$ and $b_{i+1} = S(b_i)$ and c_0 be the limit of $\{b_i\}$. Since $\prec$ is a linear ordering of all the natural numbers and its order type is $> \omega + \omega$, then by β and γ, the values a_i, b_i and c_0 can all be effectively found. In particular, $\{a_i\}$ and $\{b_i\}$ are both g.r. increasing bounded sequences with b_0 and c_0 as limits respectively.

Let $\mathbf{T}(n, y)$ be an abbreviation for the predicate $\mathbf{T}_1(n, n, y)$ of Kleene [5, p. 283] so that $(Ey)\ \mathbf{T}(n, y)$ is not a g.r. predicate. Let, for each n fixed, $\{k_{n,j}\}$ be a sequence defined by $k_{n,j} = a_j$ if $(i)_{i \leqq j} \overline{\mathbf{T}}(n, i)$ and $k_{n,j} = b_j$ if $(Ei)_{i \leqq j} \mathbf{T}(n, i)$. Evidently $\{k_{n,j}\}$ is g.r., increasing and bounded and it has as limit b_o or c_o according as $(y)\ \overline{\mathbf{T}}(n, y)$ or $(Ey)\ \mathbf{T}(n, y)$. Since $(Ey)\ \mathbf{T}(n, y)$ is not g.r., the limit $\mathbf{L}(n)$ of $\{k_{n,j}\}$ is not a g.r. function of n. This contradicts γ. Hence the theorem is proved.

Example of Type III. Every g.r. well-ordering which satisfies α and β and has an order type $> \omega + \omega$ is one of Type III. It does not satisfy γ because of Theorem 1.

Example of Type IV. A g.r. well-ordering $\prec$ of this type is defined by $0 \prec 2 \prec 4 \prec, \ldots, \prec 1 \prec 3 \prec 5 \prec, \ldots$. That $\prec$ satisfies γ can be seen from the fact that every g.r. increasing bounded sequence in the ordering $\prec$ has 1 as limit.

Example of Type II. Using Markwald's result [1, Satz 5] we can easily find a g.r. well-ordering $\overset{*}{\prec}$, of order type ω, which does not satisfy β. Let $\overset{+}{\prec}$ be a g.r. well-ordering of type III which does not satisfy γ. Then a g.r. well-ordering $\prec$ of type II is defined by

$$x \prec y \longleftrightarrow (x \text{ is even } \&\ y \text{ is odd}) \vee$$
$$(x \text{ is even } \&\ y \text{ is even } \&\ x/2 \overset{*}{\prec} y/2) \vee$$
$$(x \text{ is odd } \&\ y \text{ is odd } \&\ (x-1)/2 \overset{*}{\prec} (y-1)/2).$$

Theorem 2. There is a g.r. well-ordering $\prec$ which does not satisfy α.

Proof. Let $\mathbf{T}(n, y)$ have the same meaning as in the proof of Theorem 1. The ordering $\prec$ is defined by

$$x \prec y \longleftrightarrow ((Ez)_{z \leqq \sigma_2(x)} \mathbf{T}(\sigma_1(x), z)\ \&\ (z)_{z \leqq \sigma_2(y)} \overline{\mathbf{T}}(\sigma_1(y), z)) \vee$$
$$((Ez)_{z \leqq \sigma_2(x)} \mathbf{T}(\sigma_1(x), z)\ \&\ (Ez)_{z \leqq \sigma_2(y)} \mathbf{T}(\sigma_1(y), z)\ \&\ x < y) \vee$$
$$((z)_{z \leqq \sigma_2(x)} \overline{\mathbf{T}}(\sigma_1(x), z)\ \&\ (z)_{z \leqq \sigma_2(y)} \overline{\mathbf{T}}(\sigma_1(y), z)\ \&$$
$$(\sigma_1(x) < \sigma_1(y) \vee (\sigma_1(x) = \sigma_1(y)\ \&\ \sigma_2(x) < \sigma_2(y)))).$$

Example of Type I. Let $\overset{1}{\prec}$ stand for the ordering of Theorem 2 and $\overset{2}{\prec}$ for the ordering of the example of Type II. Then a $\prec$ is defined by

$$x \prec y \longleftrightarrow (x \text{ is even } \&\ y \text{ is odd}) \vee$$
$$(x \text{ is even } \&\ y \text{ is even } \&\ x/2 \overset{1}{\prec} y/2) \vee$$
$$(x \text{ is odd } \&\ y \text{ is odd } \&\ (x-1)/2 \overset{2}{\prec} (y-1)/2).$$

Theorem 3. Given any g.r. well-ordering $\prec$, if there is an effective method for finding the first element of any non-empty g.r. set, then the order type of $\prec$ does not exceed ω.

Proof. It suffices to show that no number is a limit in the ordering $\prec$, for the numbers are then either the first element or the successors and $\prec$ must have order type ω.

Suppose there is a number p which is a limit. Since $\prec$ satisfies y', we can find a g.r. increasing sequence $\{a_i\}$ with p as limit. Let, for each n, a g.r. set k_n be defined by

$$x \in k_n \longleftrightarrow (p \prec x \vee p = x) \vee (Ey)_{y \leq \mu t} \mathbf{T}(n, y)$$

where $\mathbf{T}(n, y)$ is the same predicate as in the proof of Theorem 1 and μt is the least number t such that $(p \prec x \vee p = x \vee x \prec a_t \prec p)$. We can see that if $(y)\ \overline{\mathbf{T}}(n, y)$, then $x \in k_n \longleftrightarrow p \prec x \vee p = x$. Thus $(y)\ \overline{\mathbf{T}}(n, y)$ implies that p is the first element of k_n. On the other hand, if $(Ey)\ \mathbf{T}(n, y)$, then some a_t (where $\mathbf{T}(n, t)$) belong to k_n. In this case, the first element of k_n is $\prec p$. Thus, for each n, $(y)\ \mathbf{T}(n, y)$ or not according as p is the first element of k_n or not. Since $(y)\ \mathbf{T}(n, y)$ is not g.r. then the first element of k_n is not a g.r. function of n. This contradicts the supposition that there is an effective method for finding them. Hence there is no number which is a limit in the ordering $\prec$. This completes the proof.

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