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A CONTRIBUTION TO THE AXIOMATIZATION OF LEWIS' SYSTEM S5

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In [7] Simons has shown that the following six axiom schemata:[1]

H1 $\vdash [\alpha \prec (\alpha \wedge \alpha)]$.
H2 $\vdash [(\alpha \wedge \beta) \prec \beta]$.
H3 $\vdash \{[(\gamma \wedge \alpha) \wedge \sim (\beta \wedge \gamma)] \prec (\alpha \wedge \sim \beta)\}$.
H4 $\vdash (\sim \Diamond \alpha \rightarrow \sim \alpha)$.
H5 $\vdash (\alpha \prec \Diamond \alpha)$.
H6 $\vdash [(\alpha \prec \beta) \prec (\sim \Diamond \beta \prec \sim \Diamond \alpha)]$.

(in which "$\alpha \rightarrow \beta$" and "$\alpha \prec \beta$" are used as the abbreviations of "$\sim (\alpha \wedge \sim \beta)$" and "$\sim \Diamond (\alpha \wedge \sim \beta)$" respectively) together with the rule of inference:

if α *is provable and* $(\alpha \rightarrow \beta)$ *is provable, then* β *is provable,*

constitute a modal system inferentially equivalent to Lewis' system S3. Moreover, he also proved that by adding the schematic analogue of Lewis' C 10.1, viz.

H7 $\vdash (\Diamond \Diamond \alpha \prec \Diamond \alpha)$

to *H1* - *H6* we obtain an axiomatization inferentially equivalent to S4, and that the axiom schemata *H1* - *H7* are mutually independent. On the other hand he remarked that although, obviously, one can get an axiomatization of S5 by adding to *H1* - *H6* the schematic analogue of C 11, viz.

H8 $\vdash (\Diamond \alpha \prec \sim \Diamond \sim \Diamond \alpha)$

he was unable to prove the mutual independence of *H1* - *H6* and *H8*. In [1] Anderson has shown that an addition of the following axiom schema

S $\vdash [(\sim \Diamond \sim \alpha \prec \Diamond \alpha) \prec (\alpha \prec \sim \Diamond \sim \Diamond \alpha)]$

to Simons' *H1* - *H6* gives a set of mutually independent axiom schemata for S5.

In this paper I shall show that:

1) the Simons' formulas *H1*, *H2*, *H3*, *H4*, *H6* and *H8* imply *H5*.

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2) the same holds, if instead of *H8* we adopt the schematic analogue of the, so called, Brouwerian axiom, i.e. C 12 of Lewis, namely

H9 $\vdash (\alpha \prec\!\!\dashv\, \sim \Diamond \sim \Diamond\, \alpha)$

3) the addition of C 12 to Lewis' axioms A 1, A 2, A 3, A 4, A 6, and A 8 gives A 7, i.e. system S5.
4) the same holds, if we add C 11 to system S1° of Feys.
5) the addition of C 12 to S1° gives a system which contains S2° of Feys.

Some minor problems will also be discussed.

It is clear that the formalization used by Simons and Anderson, i.e. the axiom schemata *H1* - *H8* and *S* with the, above mentioned, single rule of inference, is inferentially equivalent to the formalization in which, instead of axiom schemata, the analogous proper axioms are adopted together with two rules of procedure, namely substitution and detachment. Since personally I dislike the use of axiom schemata when the finite axiom-system can be adopted, and the occurence of defined terms in the axioms, I use here the following formalization: 1) Instead of the original symbols of Lewis I adopt a modification of Łukasiewicz symbolism in which "*C*", "*K*" and "*N*" possess the ordinary meaning and "*M*", "*L*", "$\mathfrak{C}$" and "$\mathfrak{E}$" mean "$\Diamond$", "$\sim \Diamond \sim$", "$\prec\!\!\dashv$" and "$=$" respectively. 2) All formulas discussed here are expressed in the primitive terms of Lewis. Thus, instead of "$\mathfrak{C}pq$" I shall write always "*NMKpNq*". 3) Instead of axiom schemata the proper axioms are given. In the systems connected with the results of Simons and Anderson the following two primitive rules of procedure are adopted:

I) The rule of substitution ordinarily used in the propositional calculus, but adjusted to the primitive functors "*K*", "*N*" and "*M*".

II) The rule of detachment adjusted to the primitive functors "*K*" and "*N*", viz.:

If the formulas "$NK\alpha N\beta$" and "α" are the theses of the system, then formula "β" is also a thesis of this system.

In the systems connected with S1° of Feys the four, well-known, Lewis' rules of procedure are used. 4) In the deductions presented below all substitutions and detachments are indicated carefully. In order to present the proofs in more compact way, in the course of the deductions several metarules of procedure will be established and put to use.

§1. In [3] Feys distinguishes the following two subsystems of S1 and S2. Namely, the following five axioms:

F1	*NMKKpqNp*	(i.e. $\mathfrak{C}Kpqp$)
F2	*NMKpqNKqp*	(i.e. $\mathfrak{C}KpqKqp$)
F3	*NMKKKpqrNKpKqr*	(i.e. $\mathfrak{C}KKpqrKpKqr$)
F4	*NMKpNKpp*	(i.e. $\mathfrak{C}pKpp$)
F5	*NMKKNMKpNqNMKqNrNNMKpNr*	(i.e. $\mathfrak{C}K\mathfrak{C}pq\mathfrak{C}qr\mathfrak{C}pr$)

taken together with four Lewis' rules of procedure constitute system S1°. The addition of a new axiom:

K1 $NMKMKpqNMp$ (i.e. $\mathfrak{S}MKpqMp$)

to S1° gives Feys' system S2°. By addition of the following new axiom:

L1 $NMKNMKpNqNNMKMpMq$ (i.e. $\mathfrak{S}\mathfrak{S}pq\mathfrak{S}MpMq$)

to S1° we obtain a subsystem of S3 which I call S3°. And the addition of:

M1 $NMKMMpNMp$ (i.e. $\mathfrak{S}MMpMp$)

to S1° constructs a system which I call S4°, and which is, obviously, a subsystem of S4. These systems, i.e. S3° and S4°, are not considered by Feys.

A modal system based on the following five axioms which are analogous to axiom schemata *H1*, *H2*, *H3*, *H6* and *H4* of Simons:

Z1	$NMKpNKpp$	(i.e. $\mathfrak{S}pKpp$)
Z2	$NMKpqNq$	(i.e. $\mathfrak{S}Kpqq$)
Z3	$NMKKKrpNKqrNKpNq$	(i.e. $\mathfrak{S}KKrpNKqrKpNq$)
Z4	$NMKNMKpNqNNMKNMqNNMp$	(i.e. $\mathfrak{S}\mathfrak{S}pq\mathfrak{S}NMqNMp$)
Z5	$NKNMpNNp$	(i.e. $CNMpNp$)

taken together with the, above mentioned, rules I and II constitutes a subsystem of S3 which I call S3*.

It is clear that S4° contains S3° which in its turn implies S2°. Obviously, S1° is included in each of these systems. Also, evidently, the addition of an analogue of Simons' *H5*:

G1 $NMKpNMp$ (i.e. $\mathfrak{S}pMp$)

to each of the systems S1°, S2°, S3°, S3*, S4 gives S1, S2, S3, S3 and S4 respectively.

I have to note here that I was unable to establish a relationship between S3° and S3*, since it is not known whether S3° implies or not *Z5*, and whether *F5* follows or not from S3*. Also, I do not know how many modalities the systems S3°, S3* and S4° have. These questions remain open.

§2. In this paragraph I shall show that S3* implies the theses and metarules of procedure which we will need later. For this end as the axiom system of S3* we assume

Z1 $NMKpNKpp$
Z2 $NMKKpqNq$
Z3 $NMKKKrpNKqrNKpNq$
Z4 $NMKNMKpNqNNMKNMqNNMp$
Z5 $NKNMpNNp$

and then adjust the rules of procedure I and II to them. Then, we can proceed as follows:[2]

METARULES OF PROCEDURE **RI** and **RII**

RI *If* $\vdash \alpha$ *and* $\vdash NMK\alpha N\beta$, *then* $\vdash \beta$.

Proof:

a)	$\vdash \alpha$	[The assumption]
b)	$\vdash NMK\alpha N\beta$	[The assumption]
c)	$\vdash NK\alpha N\beta$	[*Z5*, $p/K\alpha N\beta$; b]
d)	$\vdash \beta$	[c; a]

Q. E. D.

RII *If* $\vdash \alpha$ *and* $\vdash NMK\alpha NNMK\beta N\gamma$, *then* $\vdash NMKNM\gamma NNM\beta$

Proof:

a)	$\vdash \alpha$	[The assumption]
b)	$\vdash NMK\alpha NNMK\beta N\gamma$	[The assumption]
c)	$\vdash NMK\beta N\gamma$	[b; a; **RI**]
d)	$\vdash NMKNM\gamma NNM\beta$	[Z4, p/β, q/γ; c; **RI**]

Q. E. D.

Z6	$NMKNMKpNqNNMKKrpNKqr$	[Z4, $p/KKrpNKqr$, $q/KpNq$; Z3; **RI**]
Z7	$NMKNpp$	[Z6, q/Kpp, r/Np; Z1; **RII**; Z2, q/p; **RI**]
Z8	$NMKNMKprNNMKrNNp$	[Z6, p/NNp, q/p; Z7, p/Np; **RII**]
Z9	$NMKpNNNp$	[Z8, p/Np, r/p; Z7; **RI**]
Z10	$NMKNKppNNp$	[Z8, $r/NKpp$; Z1; **RI**]
Z11	$NMKKrpNKNNpr$	[Z6, q/NNp; Z9; **RI**]
Z12	$NMKNKNNprNNKrp$	[Z8, p/Krp, $r/NKNNpr$; Z11; **RI**]
Z13	$NMKNNpNKNNpp$	[Z6, $NKNNpp$, $q/NKpp$, r/NNp; Z12, r/p; **RII**; Z10; **RI**]
Z14	$NMKNpNNp$	[Z6, p/NNp, $q/KNNpp$, r/Np; Z13; **RII**; Z2, p/NNp, q/p; **RI**]
Z15	$NMKpNp$	[Z6, p/Np, q/Np, r/p; Z14; **RII**; Z7; **RI**]
Z16	$NMKKpqNKqp$	[Z6, p/q, r/p; Z15, p/q; **RI**]
Z17	$NMKNMKqpNNMKpq$	[Z4, p/Kpq, q/Kqp; Z16; **RI**]
Z18	$NMKNMqNNMKpq$	[Z4, p/Kpq; Z2; **RI**]
Z19	$NMKNMNNpNNMp$	[Z4, q/NNp; Z9; **RI**]
Z20	$NMKNMpNNMNNp$	[Z4, p/NNp, q/p; Z7, p/Np; **RI**]

METARULES OF PROCEDURE **RIII**, **RIV** and **RV**.

RIII *If* $\vdash NMK\alpha N\beta$ *and* $\vdash NMK\beta N\gamma$, *then* $\vdash NMK\alpha N\gamma$

Proof:

a)	$\vdash NMK\alpha N\beta$	[The assumption]
b)	$\vdash NMK\beta N\gamma$	[The assumption]
c)	$\vdash NMKN\gamma\alpha$	[Z6, p/α, q/β, $r/N\gamma$; a; **RII**; b; **RI**]
d)	$\vdash NMK\alpha N\gamma$	[Z17, p/α, $q/N\gamma$; c; **RI**]

Q. E. D.

RIV *If* $\vdash NMK\alpha N\beta$, *then* $\vdash NMKM\alpha NM\beta$

Proof:

a)	$\vdash NMK\alpha N\beta$	[The assumption]
b)	$\vdash NMKNM\beta NNM\alpha$	[Z4, p/α, q/β; a; **RI**]
c)	$\vdash NMKM\alpha NM\beta$	[Z6, $p/NM\beta$, $q/NM\alpha$, $r/M\alpha$; b; **RII**; Z7, $p/M\alpha$; **RI**]

Q. E. D.

RV *If* $\vdash NMK\alpha N\beta$ *and* $\vdash NMK\alpha N\gamma$, *then* $\vdash NMK\alpha NK\beta\gamma$

Proof:

a)	$\vdash NMK\alpha N\beta$	[The assumption]
b)	$\vdash NMK\alpha N\gamma$	[The assumption]
c)	$\vdash NMKK\alpha\alpha NK\beta\alpha$	[Z6, p/α, q/β, r/α; a; **RI**]

$\mathfrak{d}$)	$\vdash NMKK\beta\alpha NK\gamma\beta$	[$Z6$, p/α, q/γ, r/β; $\mathfrak{d}$; **RI**]
$\mathfrak{e}$)	$\vdash NMKK\alpha\alpha NK\gamma\beta$	[$\mathfrak{c}$; $\mathfrak{d}$; **RIII**]
$\mathfrak{f}$)	$\vdash NMK\alpha NK\gamma\beta$	[$Z1$, p/α; $\mathfrak{e}$; **RIII**]
$\mathfrak{g}$)	$\vdash NMK\alpha NK\beta\gamma$	[$\mathfrak{f}$; $Z16$, p/γ, q/β; **RIII**]

Q. E. D.

Z21	*NMKKpqNp*	[*Z16*; *Z2*, *p/q*, *q/p*; **RIII**]
Z22	*NMKMKpqNMp*	[*Z21*; **RIV**]
Z23	*NMKMKpqNMq*	[*Z2*; **RIV**]
Z24	*NMKMKpqNKMpMq*	[*Z22*; *Z23*; **RV**]
Z25	*NMKKrpNKrNNp*	[*Z11*; *Z16*, *p/NNp*, *q/r*; **RIII**]
Z26	*NMKNMKrNNpNNMKrp*	[*Z4*, *p/Krp*, *q/KrNNp*; *Z25*; **RI**]
Z27	*NMKKrKpqNp*	[*Z2*, *p/r*, *q/Kpq*; *Z21*; **RIII**]
Z28	*NMKKrKpqNq*	[*Z2*, *p/r*, *q/Kpq*; *Z2*; **RIII**]
Z29	*NMKKrKpqNKrp*	[*Z21*, *p/r*, *q/Kpq*; *Z27*; **RV**]
Z30	*NMKKrKpqNKrpq*	[*Z29*; *Z28*; **RV**]
Z31	*NMKKpqrNKKrpq*	[*Z16*, *p/Kpq*, *q/r*; *Z30*; **RIII**]
Z32	*NMKNMKKrpqNNMKKpqr*	[*Z4*, *p/KKpqr*, *q/KKrpq*; *Z31*; **RI**]
Z33	*NMKKrKpqNKrq*	[*Z21*, *p/r*, *q/Kpq*; *Z28*; **RV**]
Z34	*NMKKrKpqNKpKrq*	[*Z27*; *Z33*; **RV**]
Z35	*NMKNMKpKrqNNMKrKpq*	[*Z4*, *p/KrKpq*, *q/KpKrq*; *Z34*; **RI**]
Z36	*NMKNMKpNqNNMKNMNpNNMNq*	[*Z8*, *r/Nq*; *Z4*, *p/Nq*, *q/Np*; **RIII**]
Z37	*NMKNMNqNNMKNMNpNNMNq*	[*Z18*, *q/Nq*; *Z36*; **RIII**]
Z38	*NMKNMNNMNqNNMNNMKNMNpNNMNq*	[*Z36*, *p/NMNq*, *q/NMKNMNpNNMNq*; *Z37*; **RI**]
Z39	*NMKNMNNMNqNNMMKNMNpNNMNq*	[*Z38*; *Z19*, *p/MKNMNpNNMNq*; **RIII**]
Z40	*NMKNMNqNNMKNMNNMNpNNMNNMNq*	[*Z37*; *Z36*, *p/NMNp*, *q/NMNq*; **RIII**]
Z41	*NMKMNMNqNMNMKNMNNMNpNNMNNMNq*	[*Z40*; **RIV**]
Z42	*NMKNMKrpNNMKrNNp*	[*Z17*, *q/r*; *Z8*; **RIII**]

METARULE OF PROCEDURE **RVI**.

RVI *If* $\vdash NMK\alpha N\beta$ *and* $\vdash NMK\beta\gamma$, *then* $\vdash NMK\alpha\gamma$

Proof:

$\mathfrak{a}$)	$\vdash NMK\alpha N\beta$	[The assumption]
$\mathfrak{b}$)	$\vdash NMK\beta\gamma$	[The assumption]
$\mathfrak{c}$)	$\vdash NMK\beta NN\gamma$	[$Z42$, p/γ, r/β; $\mathfrak{b}$; **RI**]
$\mathfrak{d}$)	$\vdash NMK\alpha NN\gamma$	[$\mathfrak{a}$; $\mathfrak{c}$; **RIII**]
$\mathfrak{e}$)	$\vdash NMK\alpha\gamma$	[$Z26$, p/γ, r/α; $\mathfrak{d}$; **RI**]

Q. E. D.

§3. In our formalization we can express the theses C 11 (*H8*) and C 12 (*H9*) of Lewis[3] as follows:

V1 *NMKMpMNMp*

and

W1 *NMKpMNMp*

Since in S3* we have *Z42* and *Z26*, the addition of *V1* or *W1* to S3* gives at once:

V1 *NMKMpNNMNMp* = $\mathfrak{C}$*MpLMp* = $\mathfrak{C}$*MpNMNMp* = C 11

and

W1 *NMKpNNMNMp* = $\mathfrak{C}$*pLMp* = $\mathfrak{C}$*pNMNMp* = C 12

Hence we can use *V1* and *W1* in our proof that the addition of C 11 or C 12 to S3* gives S5.

3.1 The addition of *V1* to S3* implies thesis *G1* (*H5*). We assume system S3* and its consequences already proved in §2. And, we add to this system a new axiom.

V1 *NMKMpMNMp*

Then:

V2	*NMKKrMpMNMp*	[*Z2, p/r, q/Mp*; *V1*; **RVI**]
V3	*NMKKMpMNMpr*	[*Z32*; *p/Mp, q/MNMp*; *V2*; **RI**]
V4	*NMKMKpNMpr*	[*Z24, q/NMp*; *V3*; **RVI**]
V5	*NMKrNNMKpNMp*	[*Z8, p/MKpNMp*; *V4*; **RI**]
G1	*NMKpNMp*	[*V5, r/NMKNpp*; *Z7*; **RI**]

Thus, we have a proof that the axiom-system *Z1*-*Z5* and *V1* together with the rules of procedure I and II constitute system S5 of Lewis. An argumentation given by Simons shows that these axioms are mutually independent.[4]

3.2 The addition of *W1* to S3* implies thesis *V1*, and, therefore, gives S5.[5] We assume system S3* and its consequences already proved in §2. And, we add to this system a new axiom:

W1 *NMKpMNMp*

Then:

W2	*NMKKrpMNMp*	[*Z2, p/r, q/p*; *W1*; **RVI**]
W3	*NMKKpMNMpr*	[*Z32, q/MNMp*; *W2*; **RI**]
W4	*NMKMKpNMMpr*	[*Z24, q/NMMp*; *W3, p/Mp*; **RVI**]
W5	*NMKrNNMKpNMMp*	[*Z8, p/MKpNMMp*; *W4*; **RI**]
W6	*NMKpNMMp*	[*W5, r/NMKNpp*; *Z7*; **RI**]
W7	*NMKMNMpp*	[*Z17, p/MNMp, q/p*; *W1*; **RI**]
W8	*NMKNMMpp*	[*Z17, p/NMMp, q/p*; *W6*; **RI**]
W9	*NMKpNNMNMp*	[*Z42, p/MNMp*; *r/p*; *W1*; **RI**]
W10	*NMKNMNpNNMNNMNMp*	[*Z36, q/NMNMp*; *W9*; **RI**]
W11	*NMNNMNMNKNpp*	[*Z20, p/KNpp*; *W10, p/NKNpp*; **RIII**; *Z7*; **RI**]
W12	*NMKNMNNMNpNNMNNMNNMNMp*	[*Z36, p/NMNp, q/NMNNMNMp*; *W10*; **RI**]
W13	*NMNNMNNMNMMNKNpp*	[*W12, p/MNKNpp*; *W11*; **RI**]
W14	*NMKNMNNMNqKNMNpNNMNq*	[*Z39*; *W8, p/KNMNpNNMNq*; **RVI**]
W15	*NMKNMNpKNMNNMNqNNMNq*	[*Z35, p/NMNNMNq, q/NNMNq, r/NMNp*; *W14*; **RI**]

W16 *NMKNMNpNNKNMNNMNqNNMNq*
[*Z42*, *p/KNMNNMNqNNMNq*, *r/NMNp*; *W15*; **RI**]

W17 *NMNNKNMNNMNpNNMNp*
[*Z36*, *p/NMNMNKNpp*, *q/NKNMNNMNpNNMNp*; *W16*, *p/MNKNpp*, *q/p*; **RI**; *W11*; **RI**]

W18 *NMKNMNNMNpNNMNp* [*Z19*, *p/KNMNNMNpNNMNp*; *W17*; **RI**]

W19 *NMKMNMNqKNMNNMNpNNMNNMNq*
[*Z41*; *W7*, *p/KNMNNMNpNNMNNMNq*; **RVI**]

W20 *NMKNMNNMNpKMNMNqNNMNNMNq*
[*Z35*, *p/MNMNq*, *q/NNMNNMNq*, *r/NMNNMNp*; *W19*; **RI**]

W21 *NMKNMNNMNpNNKMNMNqNNMNNMNq*
[*Z42*, *p/KMNMNqNNMNNMNq*, *r/NMNNMNp*; *W20*; **RI**]

W22 *NMNNKMNMNpNNMNNMNp*
[*Z36*, *p/NMNNMNMMNKNpp*, *q/NKMNMNpNNMNNMNp*; *W21*, *p/MMNKNpp*, *q/p*; **RI**; *W13*; **RI**]

W23 *NMKMNMNpNNMNNMNp* [*Z19*, *p/KMNMNpNNMNNMNp*; *W22*; **RI**]

W24 *NMKMNMNpNNMNp* [*W23*; *W18*; **RIII**]

W25 *NMKMpNMNMNMp* [*W9*; **RIV**]

W26 *NMKMpNNMNMp* [*W25*; *W24*, *p/Mp*; **RIII**]

V1 *NMKMpMNMp* [*Z26*, *p/MNMp*, *r/Mp*; *W26*; **RI**]

Since it was already proved that S3* and *V1* constitute system S5, we showed here that the axiom-system *Z1*-*Z5* and *W1* together with the rules of procedure I and II is also inferentially equivalent to S5. The, above mentioned, argumentation of Simons proves again that the axioms *Z1*-*Z5* and *W1* are mutually independent.

§4. In this paragraph I shall investigate some questions arising by the addition of *V1* or *W1* to the systems S1° - S4°. In order to present the subsequent deductions in a more compact way and at the same time to elucidate the idea of proofs given in §3 in the here discussed formulas symbols *C*, *L*, $\mathfrak{C}$, and $\mathfrak{E}$ will be used as the abbreviations and the theses clearly belonging to S1° will be given without the proofs. Since we will discuss here exclusively the systems containing S1° and having Lewis' rules of procedure, the following known metatheorems about S1°:[6]

FI *If* $\vdash \alpha$ *and* $\vdash C\alpha\beta$ *in* S1°, *then* $\vdash \beta$ *in* S1°

FII *If* $\vdash L\alpha$ *in* S1°, *then* $\vdash \alpha$ *in* S1°

FIII *If* α *is a thesis of the classical propositional calculus, then* $\vdash L\alpha$ *in* S1°

FIV *If* $\vdash \mathfrak{E}\alpha\beta$ *in* S1°, *then* $\vdash \mathfrak{E}L\alpha L\beta$ *in* S1°

will be valid for the systems under consideration. Moreover, we have to note that, obviously, our rule I is the second rule of substitution of Lewis, and that metatheorem **FI** shows that our rule II is also valid. Hence, the deductions given in §3 can be repeated in the systems containing S1°, if the involved initial theses are available in the investigated theory.

4.1 An inspection of the proofs given in §3.2 shows that in order to obtain *G1* from *Z1*-*Z5* and *W1* we used the theses *Z2*, *Z7*, *Z8*, *Z17*, *Z19*, *Z20*, *Z24*, *Z26*, *Z32*, *Z35*, *Z36*, *Z39*, *Z41*, *Z42*, and metarules **RI**, **RIII**, **RIV** and **RVI**. Evidently, metarule **RI** is the rule of detachment of Lewis, and due to *F5* and Lewis' rule of adjunction we have **RIII**. Also, elementary reasoning shows that metarule **RVI** and the theses *Z2*, *Z7*, *Z8*, *Z17*, *Z19*, *Z20*, *Z26*, *Z32*, *Z35* and *Z42* hold in S1°. On the other hand, *Z24* and **RIV** (Becker's rule) are not provable in S1°, but we can easily obtain them in S2°, and, a fortiori, in S3°, since *K1* follows from *F1* and *L1* at once. Because in S1° the following thesis

F6 $\mathfrak{C}\mathfrak{C}pqCLpLq$

is provable, in S4° due to *M1* we can obtain *L1*. Hence, each of the systems S3° and S4° contains *Z36* (due to *L1*), *Z24*, **RIV**, *Z39* and *Z41*.

Therefore, this analysis shows immediately that an addition of *W1* (i.e. Brouwerian axiom C 12) to S3° (or to Lewis' axioms A 1, A 2, A 3, A 4, A 6 and A 8) or to S4° (or to A 1 - A 4, A 6 and C 10) gives S5.

Group II of Lewis-Langford shows that *W1* (C 12) is not deductible from S3° or S4°. On the other hand I have no proof that in so constructed axiom-systems of S5 the axioms *L1* or *M1* are superfluous.

4.2 Now, we shall prove that an addition of C 11 to S1° gives S5. We assume system S1°, and add to it *V1* (i.e. Lewis' C 11) as a new axiom. In accordance with the convention given above we can express here *V1* in the form of C 11, i. e.:

N1 $\mathfrak{C}MpLMp$

Since we have S1°, not only the theses *F1*-*F6*, but also:

F7 $\mathfrak{C}\mathfrak{C}pqCMpMq$
F8 $\mathfrak{C}Kpqq$
F9 $\mathfrak{C}CpqCCprCpKqr$
F10 $\mathfrak{C}CpqCNqNp$
F11 $\mathfrak{C}CpNqNKpq$

are at our disposal. Hence, we can proceed as follows:

F12	*CMKpqMp*	[*F7*, *p/Kpq*, *q/p*; **FI**]
F13	*CMKpqMq*	[*F7*, *p/Kpq*; *F8*]
F14	*CMKpqKMpMq*	[*F9*, *p/MKpq*, *q/Mp*, *r/Mq*; *F12*; *F13*; **FII**, **FI**]
F15	*CNKMpMqNMKpq*	[*F10*, *p/MKpq*, *q/KMpMq*; *F14*]
N2	*CMpLMp*	[*N1*; **FII**]
N3	*NKMpMNMp*	[*F11*, *p/Mp*, *q/MNMp*; *N2*]
G1	*NMKpNMp*	[*F15*, *q/NMp*; *N3*; **FI**]

Since we have *G1*, we have completed a proof that the addition of *V1* (C 11) to S1° gives S5. It shows that in the customary axiomatization of Lewis' system S5 one axiom (A 7 or B 7) is superfluous.[7]

It is evident that the above deductions can be repeated in the axiomatization of S5 given by Gödel,[8] but although *G1* can be obtained there without the use of Gödel's axiom

*G1** *CLpp*

the latter thesis is not deductible, since in that system a counterpart of **FII** can only be established with the aid of *G1**.

4.3 Now, I shall prove that the addition of *W1* (i.e. C 12) to $S1^{\circ}$ constitutes a system which I call $S1^{+}$, and which contains $S2^{\circ}$. Thus, we assume $S1^{\circ}$, and we add to it *W1* as a new axiom. Obviously, using a notation adopted in this paragraph we can express *W1* in the form of C 12, i.e.:

P1 $\mathfrak{C}pLMp$

Due to $S1^{\circ}$ we have not only *F1-F15*, but also:

F16 $\mathfrak{C}pCqp$

and

F17 $\mathfrak{C}LCpq\mathfrak{C}pq$

It allows us to proceed as follows:

F18	$CLp\mathfrak{C}qp$	[*F6*, q/Cqp; *F16*; *F17*]
F19	$\mathfrak{C}rCKpqp$	[*F18*, $p/CKpqp$, q/r; *F17*; **FI**]
F20	$\mathfrak{C}LLCpqL\mathfrak{C}pq$	[*F17*; **FIV**]
F21	$\mathfrak{C}L\mathfrak{C}pq\mathfrak{C}MpMq$	[*F6*, $p/\mathfrak{C}pq$, $q/CMpMq$; *F7*; *F17*]
P2	$CLpLLMp$	[*F6*, q/LMp; *P1*]
P3	$LLMCpCqp$	[*P2*, $p/CpCqp$; *F16*; **FI**; *F17*]
P4	$LMCpCqp$	[*P3*; **FII**]
P5	$\mathfrak{C}rMCpCqp$	[*F18*, $p/MCpCqp$, q/r; *P4*; **FI**]
P6	$\mathfrak{C}MCpCqpCKpqp$	[*F19*, $r/MCpCqp$; *P5*, $r/CKpqp$]
P7	$\mathfrak{C}LMCpCqpLCKpqp$	[*P6*; **FIV**]
P8	$\mathfrak{C}LLMCpCqpLLCKpqp$	[*P7*; **FIV**]
P9	$LLCKpqp$	[*F1*; *P8*; *P3*]
P10	$L\mathfrak{C}Kpqp$	[*F1*; *F20*, p/Kpq, q/p; *P9*]
K1	$\mathfrak{C}MKpqMp$	[*F21*, p/Kpq, q/p; *P10*; **FI**]

Thus, the proper axiom of $S2^{\circ}$ is obtained, and, therefore, $S1^{+}$ contains $S2^{\circ}$. On the other hand I do not know whether $S1^{+}$ implies *G1* or, eventually, C 11. This open question is rather important, since $S1^{+}$ possesses an interesting property. Namely, it is known[9] that an addition of an arbitrary formula which has a form $LL\alpha$ and is such that $L\alpha$ is a thesis of S1, to S1 gives system T of Feys-von Wright.[10] An inspection of the proofs given above, especially *P1-P9*, indicates clearly that an addition of an arbitrary formula $LL\alpha$ to $S1^{\circ}$ gives the following metarule:

PI If formula α is a thesis of this system, then also $L\alpha$ is provable in this system.

Hence, the above considerations not only prove a result more strong than previously known about generation of **PI** by the formulas of the form $LL\alpha$, but also show that: 1) If $S1^{+}$ contains *G1*, it contains also system T. 2) If $S1^{+}$ does not contain *G1*, an addition of *G1* to $S1^{+}$ gives a system, say T^{+}, which, obviously, is stronger than T. The questions concerning systems $S1^{+}$ and T^{+}, as e.g. whether $S1^{+}$ is weaker than T^{+}, their relationship to S5, the number of modalities which they have, remain open.

NOTES

1. In this paper: 1) the symbol "$\vdash \alpha$" means: α is provable in a system under consideration. 2) the term "thesis": a formula which is true in a system under consideration.

2. The proofs of several theorems and metarules given in this paragraph are analogous to the deductions of Simons. *Cf.* [7], pp. 310-314.
3. *Cf.* [5], p. 497.
4. *Cf.* [7], pp. 314-315.
5. In [6], pp. 151-152, Parry has proved that an addition of C 12 to S3 gives system S5. The deductions given below differ in several points from that proof, since the result of Parry depends on the use of Lewis' axiom A 7.
6. *Cf.* [3], pp. 485-488, the theorems **6.13**, **6.2**, **6.11** and **6.641**.
7. *Cf.* [5], p. 501.
8. *Cf.* [4], pp. 39-40.
9. *Cf.* [10], p. 45.
10. System T was proposed by Feys in 1937, *cf.* [2], No. 25 and No. 28.1, also *cf.* [3], p. 500, note 1. In [9], appendix II, pp. 85-90, von Wright constructed a modal system which he called system M. In [8] I have proved that the systems T and M are inferentially equivalent.

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