ON ŁUKASIEWICZ'S Ł-MODAL SYSTEM

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I want to provide a proof of Łukasiewicz's assertion that his Ł-modal system[1] is characterised by a four-valued matrix. The need for such a proof was pointed out to me by Dr. R. Harrop.

The theorems of the Ł-modal system are the formulae that can be derived from the axioms

1. $C\delta pC\delta Np\delta q$,

2. $Cp\Delta p$,

by use of the rules of substitution (for propositional and functorial variables) and detachment.

The matrix in question (call it **M**) is the cross-product of a pair of two-valued matrices:

$\mathbf{M}_1$:

	C	t	f	N	Δ
*	t	t	f	f	t
	f	t	t	t	f

$\mathbf{M}_2$:

	C	t	f	N	Δ
*	t	t	f	f	t
	f	t	t	t	t

If for convenience we number the elements of the product, writing $\langle t, t \rangle = 1$, $\langle t, f \rangle = 2$, $\langle f, t \rangle = 3$, $\langle f, f \rangle = 4$, we can then describe **M** as follows:

	C	1	2	3	4	N	Δ
*	1	1	2	3	4	4	1
	2	1	1	3	3	3	1
	3	1	2	1	2	2	3
	4	1	1	1	1	1	3

It is easy to show that every theorem is verified by (never takes an undesignated value in) **M**. We want to prove the converse, that every formula verified by **M** is a theorem. We first observe that the truth-table of Δ in the matrix $\mathbf{M}_1$ is such that a formula $\Delta\alpha$ always takes the same truth-value as α itself. If then β is any formula, and β_1 is got from it by replacing

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every part of the form $\Delta\,\alpha$ by plain α, the formulae β and β_1 will take identical values in $\mathbf{M}_1$. In particular, if β is verified by $\mathbf{M}_1$, so is β_1. In the same way, if β_2 is got from β by replacing each part of the form $\Delta\alpha$ by $C\alpha\,\alpha$, then if β is verified by $\mathbf{M}_2$ so is β_2. But by the way $\mathbf{M}$ is constructed β is verified by $\mathbf{M}$ if and only if it is verified by both $\mathbf{M}_1$ and $\mathbf{M}_2$. Hence we have:

Lemma 1: *If β is verified by $\mathbf{M}$ then β_i is verified by $\mathbf{M}_i$ ($i = 1,2$).*

Both the $\mathbf{M}_i$ are got by adding a table for Δ to the ordinary two-valued *C-N* matrix, but since Δ no longer occurs in either of the formulae β_i its table will not enter into their evaluation. (This remains true even when the β_i contain functorial variables, for the functional completeness of the *C-N* matrix means that it alone already contains all possible functions which the functorial variables might take as values.) Hence if β_i is verified by $\mathbf{M}_i$ it is verified by the *C-N* matrix alone. But all such formulae can be proved from Lukasiewicz's axiom *1.*[2] Hence we have:

Lemma 2: *If β_i is verified by $\mathbf{M}_i$* then $\vdash \beta_i$, (i=*1,2*).

At this point we need to establish some theorems in the Ł-modal system. References here to Łukasiewicz are to the chain of theorems at the end of his paper, while "PC" indicates that the formula is a tautology and hence provable via Łukasiewicz's theorems *10,20,22*.

1	$\vdash CEpqC\delta p\,\delta q$	Łukasiewicz, theorem *73*.
2	$\vdash CEp\Delta pC\delta p\,\delta\Delta p$	From *1* by substitution.
3	$\vdash CECpp\Delta pC\delta Cpp\,\delta\Delta p$	From *1* by substitution.
4	$\vdash CCp\Delta pAEp\Delta pECpp\Delta p$	PC.
5	$\vdash Cp\Delta p$	Axiom *1*.
6	$\vdash AEp\Delta pECpp\Delta p$	From *4*, *5* by detachment.
7	$\vdash C\delta pC\delta Cpp\,\delta\Delta p$	From *2*, *3*, *6* by PC.[3]
8	$\vdash CCpqC\Delta p\Delta q$	Łukasiewicz, theorem *78*.
9	$\vdash CCNCqq\alpha C\Delta NCqq\Delta\alpha$	From *8* by substitution.
10	$\vdash CNCqq\alpha$	PC.
11	$\vdash C\Delta NCqq\Delta\alpha$	From *9*, *10* by detachment.
12	$\vdash C\alpha\Delta\alpha$	From axiom *1* by substitution.
13	$\vdash CA\alpha\Delta NCqq\Delta\alpha$	From *11*, *12* by PC.
14	$\vdash CN\Delta qC\Delta pp$	Łukasiewicz, theorem *101*.
15	$\vdash CN\Delta NCqqC\Delta\alpha\,\alpha$	From *14* by substitution.
16	$\vdash C\Delta\alpha A\alpha\Delta NCqq$	From *15* by PC.
17	$\vdash E\Delta\alpha A\alpha\Delta NCqq$	From *13*, *16* by PC.
18	$\vdash CEpqE\delta p\,\delta q$	From *1* by PC.
19	$\vdash E\delta\Delta\alpha\,\delta A\alpha\Delta NCqq$	From *17* by substitution in *18* and detachment.
20	$\vdash E\alpha A\alpha NCqq$	PC.
21	$\vdash E\delta\alpha\delta A\alpha NCqq$	From *20* by substitution in *18* and detachment.

22	$\vdash$	$EC\alpha\alpha A\alpha CNCqqNCqq$	PC.
23	$\vdash$	$E\delta C\alpha\alpha\delta A\alpha CNCqqNCqq$	From *22* by substitution in *18* and detachment.

Let the formula γ be got from β by replacing every part of the form $\Delta\alpha$ by $A\alpha\Delta NCqq$. By successive substitution in *19* above we obtain a chain of provable equivalences (one for each step in the chain of replacements by which γ is got from β) leading from β to γ. Consequently by PC we have $\vdash E\beta\gamma$. Now let γ_1 and γ_2 be got from γ just as β_1 and β_2 were got from β. In view of the way in which γ itself is got from β, this means that γ_1, for example, comes from β by replacing each part $\Delta\alpha$ by $A\alpha NCqq$, whereas to get β_1 we replace $\Delta\alpha$ by plain α. Hence by successive substitutions in *21* above we get $\vdash E\beta_1\gamma_1$. Similarly γ_2 comes from β by replacing each part $\Delta\alpha$ by $A\alpha CNCqqNCqq$, whereas to get β_2 we replace $\Delta\alpha$ by $C\alpha\alpha$. Hence by successive substitution in *23* above we get $\vdash E\beta_2\gamma_2$.

The point about the construction of γ is that in it Δ always occurs prefixed to the same formula, viz. $NCqq$. Hence the implication $C\gamma_1 C\gamma_2\gamma$ is —as $C\beta_1 C\beta_2\beta$ in general is not—a straightforward substitution instance of $C\delta pC\delta Cpp\,\delta\Delta p$, and so by substitution in 7 above we have $\vdash C\gamma_1 C\gamma_2\gamma$. From this and the equivalences just established we get by PC $\vdash C\beta_1 C\beta_2\beta$, and thus have:

Lemma 3: *If* $\vdash \beta_i$ $(i = 1{,}2)$, *then* $\vdash \beta$.

The desired result follows as soon as the three lemmas are put together, but it may be worth illustrating the various constructions by an example. Let β be Łukasiewicz's theorem *92*, $\Delta C\Delta pp$. Then $\beta_1 = Cpp$; $\beta_2 = CCCpppCCppp$; $\gamma = ACAp\Delta NCqqp\Delta NCqq$; $\gamma_1 = ACApNCqqpNCqq$; $\gamma_2 = ACApCNCqqNCqqpCNCqqNCqq$. The tautology β_1 is Łukasiewicz's theorem *13* and β_2 is a substitution instance of it. To prove $E\beta\gamma$ we need two applications of *19*, taking δ to be $C'p$ and $'$ in order to get $E\Delta C\Delta pp\Delta CAp\Delta NCqqp$ and $E\Delta CAp\Delta NCqqpACAp\Delta NCqqp\Delta NCqq$ respectively; and similarly for the (tautologous) equivalences $E\beta_i\gamma_i$. Finally $C\gamma_1 C\gamma_2\gamma$ comes from 7 by substituting $NCqq$ for p and $ACAp'p'$ for δ.

If we compare this proof with Łukasiewicz's own treatment we find that he begins by showing that his axioms and rules are verified by **M** and then goes on to say (p. 127), "It follows from this consideration that all the formulae of our modal logic based on the axioms *1-4* are verified by the matrix **M9**.[4] It also follows that no other formulae besides can be verified by **M9**; this results from the fact that the classical *C-N* propositional calculus, to which all the formulae of our modal logic are matrically reducible, is 'saturated', i.e., any formula must be either asserted on the grounds of its asserted axioms, or rejected on the ground of the axiom of rejection $\dashv p$, which easily follows by substitution from our axiom *3* or *4*. **M9**, therefore, is an adequate matrix of the Ł-modal logic."

Although this is the only place in his paper at which the phrase "matrically reducible" occurs, I take it that Łukasiewicz had in mind something like my own reduction of β to β_1 and β_2, but carried a stage further by the elimination not only of Δ but of functorial variables as well (so that a formula with n functorial variables would have not two but 2.16^n formulae associated with it). It is possible too that the reference to the "saturatedness" of the C-N calculus is intended to link the matrix verification of the various associated formulae with their provability in the system, though it is not an immediately obvious reference to make, since saturatedness is not a matrix property of a calculus at all. On the other hand (and quite crucially) there is nothing at all in Łukasiewicz's account that would link the provability of a formula with the provability of the associated C-N formulae.

There is a further way, not touched on by Łukasiewicz, in which the matrix **M** is characteristic for his system. As well as providing for the assertion of formulae the Ł-modal system also allows for their rejection, by means of two rejected 'axioms':

3. $C\Delta pp$,

4. Δp ,

together with counterparts of the rules of detachment and substitution. We shall show that a formula is rejected in the system if and only if it is falsified by (sometimes takes an undesignated value in) **M**.

It is easy to see that every rejected formula is falsified by **M**. To prove the converse we first need to carry out some rejections in the system:

24	$\vdash$	$CN\Delta NCqqC\Delta pp$	From 14 by substitution.
25	$\dashv$	$C\Delta pp$	Axiom 3.
26	$\dashv$	$N\Delta NCqq$	From 24, 25 by detachment.
27	$\vdash$	$C\Delta NCqq\Delta p$	From 11 by substitution.
28	$\dashv$	Δp	Axiom 4.
29	$\dashv$	$\Delta NCqq$	From 27, 28 by detachment.
30	$\vdash$	$CNCqq\Delta p$	PC
31	$\dashv$	$NCqq$	From 28, 30 by detachment.

Let $\phi_1 = Cqq$, $\phi_2 = N\Delta NCqq$, $\phi_3 = \Delta NCqq$, $\phi_4 = NCqq$. We see that each formula ϕ_i takes exclusively the value i in the matrix, and we see from 26, 29, and 31 above that if i is undesignated ϕ_i is rejected. We recall too that the matrix functions eligible as values for the functorial variables are exactly those singulary functions that can be built up by composition out of the identity function, the four constant functions (the functions whose values are identically 1, 2, 3, or 4), and the functions C, N, and Δ. If f is such a function let ϕ_f be the expression got by carrying out the corresponding composition on ', C'', $N\Delta NC$'', ΔNC'', NC'', and the functors C, N, and Δ.

Suppose now that β is a formula which is falsified by **M**; i.e., suppose that there is a 'falsifying' assignment of values to the propositional and functorial variables of β under which it receives an undesignated value. Let γ be got from β by substituting ϕ_i for each propositional variable which is given the value i in the falsifying assignment, and by substituting ϕ_f for each functorial variable which is given the function f as value in the falsifying assignment. γ has no functorial variables and only the one propositional variable, and its construction is designed so that it always takes the same value, namely the particular value taken by β in the falsifying assignment. Let this value be i, ($i = 2, 3$, or 4). Then both the antecedent and consequent of $C\gamma\phi_i$ always take the value i, so that the implication itself is verified by **M** and thus, by our earlier result, is a theorem. But if $\vdash C\gamma\phi_i$ and $\dashv \phi_i$ then by the rejection-rule of detachment we have $\dashv \gamma$, and from this —since γ is a substitution-instance of β—by the rejection-rule of substitution we get $\dashv \beta$, as was to be shown.

Since we have shown that every formula verified by **M** is asserted in the Ł-modal system, and that every formula falsified by **M** is rejected, and since every formula is either verified or falsified, it follows as a corollary that every formula is either asserted or rejected—i.e. that the Ł-modal system is "saturated".

NOTES

1. Jan Łukasiewicz, "A system of modal logic," *The Journal of Computing Systems* vol. 1 no. 3 (1953), pp. 111-149.

2. Łukasiewicz, *op. cit.* p. 123, referring to an unpublished paper by C. A. Meredith. This paper is still unpublished, but a sufficiently close result is to be found in Meredith's "On an extended system of the propositional calculus," *Proceedings of the Royal Irish Academy*, vol. 54 section A no. 3 (1951), pp. 37-47.

3. This short proof of 7 is due to Professor A. N. Prior.

4. **M9** is what I have called **M**, though Łukasiewicz constructs it in a rather different way, forming the cross-product of the *C-N* matrix with itself and only afterwards inserting a table for Δ as part of the 'multiplication' process.

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