ON CHARACTERIZATIONS OF THE FIRST-ORDER FUNCTIONAL CALCULUS

JULIUSZ REICHBACH

In papers [5] and [7][1] I have presented some characterizations of theses of the first-order functional calculus; in this paper I give a generalization of two characterizations of one.

We consider the first-order functional calculus with the symbolism described in [4][2] and besides signs accepted in the logic literature we use the following ones:

(0,1) $E, F, G, E_1, F_1, G_1 \ldots$ – variables representing expressions,

(0,2) $Sw\{E\}$ – the set of all symbols occurring in the expression E,

(0,3) Skt – the set of all formulas[3] of the form $\Sigma a_1 \ldots \Sigma a_i\, \Pi a_{i+1} \ldots \Pi a_k F$,[4] where F is a quantifierless expression containing no free variables and Πa_j is the sign of the universal quantifier binding the apparent variable a_j, and $\Sigma a_j G = (\Pi a_j G')'$, for $j = 1, \ldots, k$.

(0,4) $C(E)$ – the set of all significant parts of the formula E: $F \in C(E) \cdot \equiv \cdot F = E$ or there exist such G, H that: $(F = G) \wedge (E = G') \vee [(F = G) \vee (F = H)] \wedge (E = G + H) \vee (\exists i)\{F = G(x_i/a)\} \wedge (E = \Pi a G)$.[5]

(0,5) $w(E)$ – the number of different free variables occurring in the expression E,

(0,6) $p(E)$ – the number of different apparent variables occurring in the expression E,

(0,7) $i_1, \ldots, i_{w(E)}$, or $j_1, \ldots, j_{w(E)}$ or $l_1, \ldots, l_{w(E)}$ – different indices of these and only these free variables which occur in the expression E,

(0,8) $i(E) = \max\{i_1, \ldots, i_{w(E)}\}$,

(0,9) $m(E) = w(E) + p(E)$,

(0,10) $n(E) = \max\{m(E), i(E)\}$,

(0,11) $E(x/y)$ – the expression resulting from E by the substitution of x for each occurrence of y in E; if y is an apparent variable, then y does not belong in E to the scope of the quantifier Πy; if x is an apparent variable, then y does not belong to the scope of the quantifier Πx,

(0,12) $\Sigma(F) = 0$, if F is a quantifierless formula; $\Sigma(F + G) = \max\{\Sigma(F), \Sigma(G)\}$; $\Sigma(\Pi a F) = \Sigma\{F(x/a)\}$, where $x \overline{\epsilon} Sw\{F\}$; $\Sigma(\Sigma a F) = w(F) + 1$, if $\Sigma\{F(x/a)\} = 0$;[6] $\Sigma(\Sigma a F) = \Sigma\{F(x/a)\}$, if $x \overline{\epsilon} Sw(F)$ and $\Sigma\{F(x/a)\} \neq 0$;

Received April 7, 1960

If $\Sigma(F)$ is not defined above, then:

$\Sigma(F) = \max\{\Sigma(G)\}$, for each $G \epsilon C(E)$, where here if $G = \Pi a H$ then $\Sigma(G) = w(H) + 1$, and $\Sigma(F') = \Sigma(F)$, $\Sigma(E + F) = \max\{\Sigma(E), \Sigma(F)\}$

For example:

(1°) If $E \epsilon Sks$ and $E = \Sigma a_1 \ldots \Sigma a_i \, \Pi a_{i+1} \ldots \Pi a_k \, F$, for some F, then $\Sigma(E) = i$.
(2°) If $E = \{\Pi a_1 f_t^r(x_{t_1}, , , , x_{t_{r-1}}, a_1) + f_1^1(x_{t_r})\}'$, then $\Sigma(E) = r$.
(3°) $\Sigma(E) \leq m(E) \leq n(E)$.

(0,13) $E^* \epsilon P$ if and only if an arbitrary substitution for free variables in E belong to P; we define also $E^* + F^* = (E + F)^*$; we assume also that if we write E^*, F^*, G^*, . . . , then we consider the same substitution in all formulas E, F, G, . . .
(0,14) M, M_1, M_2, . . . – arbitrary models,
(0,15) T, T_1, T_2, . . . – arbitrary tables of the rank k,
(0,16) Q, Q_k – non-empty sets of tables of the rank k,
(0,17) $\{Q_n\}$ – the sequence of sets Q_k, where Q_k is defined in (0,16),
(0,18) $(\{Q_n\})$ – for every $\{Q_n\}$,
(0,19) $[M \mid s_1, \ldots, s_k]$ – the truncated model of the rank k,
(0,20) $F \epsilon A(E) \cdot \equiv \cdot (\exists F_1) \ldots (\exists F_n) \{E = F_1 + \ldots + F_{i-1} + F + F_{i+1} + \ldots + F_n$, where the arrangement of brackets is respectively$\} \wedge (F_1)(F_2)(F \neq F_1 + F_2)$,[7]

Some notions which we introduced above are defined in the following pages of the paper.

From [4] or [6] we obtain the following rules of constructions of formal theorems of the first-order functional calculus:

(1,1) The formula $F + F'$ is a formal theorem.
(1,2) If $F_1 + F_2 + \ldots + F_n$ is a formal theorem and $k_1, \ldots, k_n$ is an arbitrary permutation of natural numbers $\leq n$, then $F k_1 + F k_2 + \ldots + F k_n$ is a formal theorem (the arrangement of brackets is here arbitrary).
(1,3) If F is a formal theorem and G a formula, then $F + G$ is a formal theorem.
(1,4) If $F + G$ and $F + G'$ are formal theorems, then F is a formal theorem.
(1,5) If $F + G$ is a formal theorem and the free variable $x \,\bar{\epsilon}\, Sw\{F\}$ then $F + \Pi a G(a/x)$ is a formal theorem.
(1,6) If $F + \Pi a G$ is a formal theorem, then $F + G(x/a)$ is a formal theorem.

D.1. The sequence of formulae $E_1, \ldots, E_n$ is a formalized proof of the formula E in the first-order functional calculus with added axioms **U**[8] if and only if $E = E_n$ and for each $t < n$ the following conditions are satisfied:

1. every E_t is an alternative of significant parts of the formula E or of some formulas which belong to **U**.
2. $E_t \epsilon$ **U** or there exists such F that $E_t = F + F'$, or
3. there exist such $i, j < t$ that E_t results from E_i and E_j by applying the rule (1,4), or

4. there exist such $i < t$ that E_t results from E_i by applying one of the rules (1,2) – (1,3), (1,5) and (1,6).

D.2. The formula E is a thesis of the first-order functional calculus with added axioms **U** if and only if there exists at least one formalized proof of the formula E in the first-order functional calculus with added axioms **U**.[9]

D.3. The formula E is a thesis if and only if E is a thesis of the first-order functional calculus with added axioms **U** and **U** is empty.

By the length of a formalized proof $E_1, \ldots, E_n$ we mean the number n.

We notice that because in the proof of Gödel's theorem for E,[10] see [4], we may only consider the significant parts of E, therefore we may replace (1,4) and (1,6) in *D.3.* by:

(1,4') If $F + G$ and $F + G'$ are formal theorems, $G' \epsilon C(F)$ then F is a formal theorem.

(1,6') if $F + \Pi aG$ is a formal theorem, $w(G) < \Sigma (F + \Pi aG)$ and $\Pi aG \epsilon C(F)$, then $F + G(x/a)$ is a formal theorem.

It is known that if $E \epsilon Skt$, then E is a thesis if and only if E may be obtained by means of rules (1,2), (1,5) and the following:[11]

(1,7) If $F + E + E$ is a thesis, then $F + E$ is a thesis.

(1,8) If $F + G(x/a)$ is a thesis, then $F + \Sigma aF$ is a thesis.

It is easy to show:

L.0. If the length of the formalized proof of the formula E is n, then the length of formalized proof of E^* is also n.

L.1. For each formula E it may be written down such a formula $F \epsilon Skt$ that E is a thesis if and only if F is a thesis; we may also assume that $F = \Sigma a_1 \ldots \Sigma a_i \Pi a_{i+1} G$, for some G.[12]

D.4. The sequence $< B, \{F^i_j\} >$ is a model if and only if B is an arbitrary non-empty set and $\{F^i_j\}$ is such an arbitrary doubly infinite sequence of relations that F^m_k is a m-ary relation between elements of B.

In the further consideration we assume that the usual definition of satisfiability is known, see [4] or [10].

D.5. $\mathbf{M}\{E\} = 0 \cdot \equiv \cdot E'$ is true in the model **M**.

D.6. $\mathbf{M}\{E(s_1, \ldots, s_k)\} = 0 \cdot \equiv \cdot$ there exists such model $< B, \{F^i_j\} >$ that $\mathbf{M} = < B, \{F^i_j\} >$, $s_1, \ldots, s_k \epsilon B$, x_i are the names of s_i, $i = 1, \ldots, k$, and $s_i, \ldots, s_k$ do not satisfy E in the model **M**.

The following theorem is known, see for example [4]:

T.1. A formula E is a thesis if and only if it is true.

D.7. The sequence $< B_k, \{F^i_j\} >$ is a table of the rank k if and only if it is a model and B_k has exactly k-elements which are numbers $1, \ldots, k$.

D.8. $[\mathbf{M} | s_1, \ldots, s_k]$ is a truncated model of the rank k with respect to

$s_1, \ldots, s_k$ – briefly: a truncated model of the rank k – if and only if there exists such model $< B, \{F_j^i\} >$ that $\mathbf{M} = < B, \{F_j^i\} >, s_1, \ldots, s_k \in B$ and there exists such table $< B_k, \{\phi_j^i\} >$ of the rank k that: if $r_i \in B_k$, for $i = 1, \ldots, m$, then

$$\phi_t^m (r_1, \ldots, r_m) \cdot \equiv \cdot F_t^m (s_{r_1}, \ldots, s_{r_m}) .$$

We notice that $[\mathbf{M} \mid s_1, \ldots, s_k]$ is a submodel of the model $\mathbf{M}$ in the meaning of homomorphism.

D.9. $N(Q,k)$ if and only if Q is an arbitrary non-empty set of tables of the rank k and for an arbitrary sequence $t_1, \ldots, t_k$ of the natural numbers $\leq k$ we have:

If $T \in Q$, then $[T \mid t_1, \ldots, t_k] \in Q$.

D.10. $Q[\mathbf{M},k] \cdot \equiv \cdot (T) \{T \in Q \cdot \equiv \cdot (\exists s_1) \ldots (\exists s_k) (T = [\mathbf{M} \mid s_1, \ldots, s_k])\}$

D.11. $T^0 \in [Q \mid 1, \ldots, k] \cdot \equiv \cdot (\exists m) (\exists T) \{(m \geq k) \wedge (Q$ is a non-empty set of tables of the rank $m) \wedge (T \in Q) \wedge (T^0 = [T \mid 1, \ldots, k])\}$.

It is easy to prove:[13]

L.2. If $\mathbf{M} = < B, \{F_j^i\} >, s_1, \ldots, s_k \in B, t_1, \ldots, t_k \leq k$, then $[[\mathbf{M} \mid s_1, \ldots, s_k] \mid t_1, \ldots, t_q] = [\mathbf{M} \mid s_{t_1}, \ldots, s_{t_q}]$[14]

L.3. If $Q[\mathbf{M}, k]$, then $N(Q, k)$.

L.4. If T_1, T_2 are two tables of the rank k and $r_1, \ldots, r_i, r_{i+1}, \ldots, r_j, r_{j+1}, \ldots, r_t$ $(t \leq k)$ is a sequence of different natural numbers $\leq k$, then if $[T_1 \mid r_1, \ldots, r_i] = [T_2 \mid r_1, \ldots, r_i]$, then there exists such table T_3 of the rank k that

$$[T_3 \mid r_1, \ldots, r_i, r_{i+1}, \ldots, r_j] = [T_1 \mid r_1, \ldots, r_i, r_{i+1}, \ldots, r_j] ,$$
$$[T_3 \mid r_1, \ldots, r_i, r_{j+1}, \ldots, r_t] = [T_2 \mid r_1, \ldots, r_i, r_{j+1}, \ldots, r_t] .$$

L.5. Let $N(Q^0, k)$ and let

$$T \in Q \cdot \equiv \cdot (\exists T^0) (\exists t_1), \ldots, (\exists t_m) \{ (1 \leq t_i \leq k) \wedge (i = 1, \ldots, m) \wedge (T^0 \in Q^0) \wedge (T = [T^0 \mid t_1, \ldots, t_m])\}$$[15]

Then:

I. $N(Q, m)$.
II. If $k \leq m$, then $Q^0 = [Q \mid 1, \ldots, k]$.
III. If $k \geq m$, then $Q = [Q^0 \mid 1, \ldots, m]$.

D.12. $[Q_m \mid Q_1, \ldots, Q_{m-1}] \cdot \equiv \cdot (k) \{(k \leq m) \to (Q_k = [Q_m \mid 1, \ldots, k])\}$.

Obviously:

L.6. If $Q_1[\mathbf{M}, 1], \ldots, Q_m[\mathbf{M}, m]$, then $Q_m[Q_1, \ldots, Q_{m-1}]$.

L.7. If $Q_m[Q_1, \ldots, Q_{m-1}]$, then $Q_{m-1}[Q_1, \ldots, Q_{m-2}]$.

From *1.5.* we obtain immediately:

L.8. If $Q_m[Q_1, \ldots, Q_{m-1}]$, $N(Q_m, m)$, then for every $k = 1, \ldots, m$, we have $N(Q_k, k)$.

L.9. If $Q_m[Q_1, \ldots, Q_{m-1}]$, $N(Q_m, m)$, then there exists such Q_{m+1} that $Q_{m+1}[Q_1, \ldots, Q_m]$ and $N(Q_{m+1}, m+1)$.

L.10. If $Q_m[Q_1, \ldots, Q_{m-1}]$, $N(Q_m, m)$, $k \leq m$ and $T \in Q_k$, then for arbitrary sequence $i_1, \ldots, i_t$, $t \leq k$, of the natural numbers $\leq k$ we have $[T | i_1, \ldots, i_t] \in Q_t$.

D.13. **M** is a biunique t-model – in symbols $\mathbf{M} \in R_t$ – if and only if there exists such model $< B, \{F_j^i\} >$, that $\mathbf{M} = < B, \{F_j^i\} >$ and for arbitrary $s_1, \ldots, s_t, s'_1, \ldots, s'_t \in B$ we have: if $[\mathbf{M} | s_1, \ldots, s_t] = [\mathbf{M} | s'_1, \ldots, s'_t]$, then $s_1 = s'_1, \ldots, s_t = s'_t$.

The example of $\mathbf{M} \in R_t$ may be easily given, see [5] and [7].

By an extension of a model $\mathbf{M}_1 = < B, \{F_j^i\} >$ we understand here a model $\mathbf{M}_2 = < B, \{F_j^i\}, \{G_k^1\} >$, where $\{G_k^1\}$ is an infinite sequence of co-sets of B.

L.11. Each model $\mathbf{M}_1$ may be extended to model $\mathbf{M} \in R_1$, and therefore to $\mathbf{M} \in R_t$, for every t.

Proof: –Let $\mathbf{M}_1 = < B, \{F_j^i\} >$, let

(0) $(s_1, s_2), (s_1, s_3), (s_2, s_3), \ldots$

be the sequence of all pairs of different elements of B and let

$$G_1^1, G_2^1, G_3^1, \ldots$$

a sequence of relations with the following properties:

(0,1) if $[\mathbf{M}_1 | s_1] = [\mathbf{M}_1 | s_2]$, then $G_1^1(s_1)$ and $\sim G_1^1(s_2)$.

(0,2) if (s_i, s_j) is the m-th pair of the sequence (0) and $[\mathbf{M}_1 | s_i] = [\mathbf{M}_1 | s_j]$, then $G_m^1(s_i)$ and $\sim G_m^1(s_j)$.

Obviously that (0,1) and (0,2) give the construction of this sequence of relation.

Let $\mathbf{M} = < B, \{F_j^i\}, \{G_r^1\} >$.

It is obvious that **M** is an extension of $\mathbf{M}_1$ and $\mathbf{M} \in R_t$, for every t.[16]

D.14. $N(r, Q_1, \ldots, Q_k) \cdot \equiv \cdot (r \leq k) \wedge (i_1) \ldots (i_l)(i_{l+1}) \quad (T) \{ \quad (l < r) \wedge (i_1, \ldots, i_{l+1} \leq k) \wedge ([T | i_1, \ldots, i_l] \in Q_l) \wedge ([T | i_{l+1}] \in Q_1) \rightarrow (\exists T_1) \{([T_1 | i_1, \ldots, i_l, \quad i_{l+1}] \in Q_{l+1})$ $\wedge$ (for each sequence $j_i, \ldots, j_s$ of natural numbers $\leq k$), if $[T | j_1, \ldots, j_s] \in Q_s$, then $[T | j_1, \ldots, j_s] = [T | j_1, \ldots, j_s]\}\}$.[17]

It is easy to show, see [7]:

L.12. If $\mathbf{M} = < B, \{F_j^1\} >$,[18] $Q_1[\mathbf{M}, 1], \ldots, Q_k[\mathbf{M}, k]$, then for every t we have $N(t, Q_1, \ldots, Q_k)$.

L.13. If $N(r, Q_1, \ldots, Q_k)$, $t < r$, then $N(t, Q_1, \ldots, Q_k)$.

L.14. If $Q_1[\mathbf{M}, 1], \ldots, Q_k[\mathbf{M}, k]$, then:

1. if T is an arbitrary table of the rank k, $[T | i] \in Q_1$, $[T | j] \in Q_1$, $i, j \leq k$, then there exists such table T_1 of the rank k that $[T_1 | i, j] \in Q_2$[19] and

$$[T_1 | 1, \ldots, i-1, i+1, \ldots, k] = [T | 1, \ldots, i-1, i+1, \ldots, k],$$
$$[T_1 | 1, \ldots, j-1, j+1, \ldots, k] = [T | 1, \ldots, j-1, j+1, \ldots, k].$$

2. if $k \geq 2$, then $N(2, Q_1, \ldots, Q_k)$:

L.15. If $Q_{k+1}\ [Q_1, \ldots, Q_k]$, $N(Q_{k+1}, k+1)$, $N(r, Q_1, \ldots, Q_{k+1})$, $r \leq k$, then $N(r, Q_1, \ldots, Q_k)$.[20]

L.16. If $Q_k\ [Q_1, \ldots, Q_{k-1}]$, $N(r, Q_1, \ldots, Q_k)$, $N(Q_k, k)$, then there exists such Q_{k+1} that $N(r, Q_1, \ldots, Q_{k+1})$, $Q_{k+1}\ [Q_1, \ldots, Q_k]$ and $N(Q_{k+1}, k+1)$.

L.17. If $\mathbf{M} \in R_1$, $Q_1\ [\mathbf{M}, 1], \ldots, Q_k\ [\mathbf{M}, k]$, then for each r we have $N(r, Q_1, \ldots, Q_k)$.

D.15. $R(T, T_1, Q_1, \ldots, Q_k, i_1, \ldots, i_m, i) \cdot \equiv \cdot (m \leq k) \wedge ([T|i_1, \ldots, i_m] = [T_1|i_1, \ldots, i_m]) \wedge \{(\exists t)(\{1 \leq t \leq m\} \wedge \{i = i_t\}) \rightarrow ([T_1|i_1, \ldots, i_m] \in Q_m)\} \wedge \{(t)(\{1 \leq t \leq m\} \rightarrow \{i \neq i_t\}) \rightarrow ([T_1|i_1, \ldots, i_m, i] \in Q_{m+1})\}$.[21]

For an arbitrary sequence $Q_1, \ldots, Q_k$, where Q_i are non-empty sets of tables of the rank i $(i = 1, \ldots, k)$, for an arbitrary table T of the rank k and for an arbitrary formula E which indices of the free variables occurring in it are $\leq k$, we introduce the following inductive definition of the functional V:

(1d) $V\{T, Q_1, \ldots, Q_k, f_t^m(x_{r_1}, \ldots, x_{r_m})\} = 1 \cdot \equiv \cdot F_t^m(r_1, \ldots, r_m)$,

(2d) $V\{T, Q_1, \ldots, Q_k, F'\} = 1 \cdot \equiv \cdot \sim V\{T, Q_1, \ldots, Q_k, F\} = 1 \cdot \equiv \cdot \equiv V\{T, Q_1, \ldots, Q_k, F\} = 0$.

(3d) $V\{T, Q_1, \ldots, Q_k, F + G\} = 1 \cdot \equiv \cdot V\{T, Q_1, \ldots, Q_k, F\} = 1 \vee V\{T, Q_1, \ldots, Q_k, G\} = 1$.

(4d) $V\{T, Q_1, \ldots, Q_k, \Pi aF\} = 1 \cdot \equiv \cdot (i)(T_1)\{(i \leq k) \wedge R(T, T_1, Q_1, \ldots, Q_k, i_1, \ldots, i_{w(F)}, i) \rightarrow V\{T_1, Q_1, \ldots, Q_k, F(x_i/a)\} = 1\}$.[22]

D.16. $E \in P(Q_1, \ldots, Q_k) \cdot \equiv \cdot (T)\{(H)\{(H \in A\{F\}) \rightarrow$[23]$([T|i_1, \ldots, i_{w(H)}] \in Q_{w(H)})\} \rightarrow V\{T, Q_1, \ldots, Q_k, E\} = 1\}$[24]

D.17. $E \in P(k, r) \cdot \equiv \cdot (Q_1) \ldots (Q_k)\{Q_k[Q_1, \ldots, Q_{k-1}] \wedge N(Q_k, k) \wedge N(r, Q_1, \ldots, Q_k) \rightarrow (E \in P\{Q_1, \ldots, Q_k\})\}$.

D.18. $E \in P \cdot \equiv \cdot E \in P\{n(E), \Sigma(E)\}$.

We explain the meaning of the above definitions:

1. The expression $V\{T, Q_1, \ldots, Q_k, E\} = 1$ may be read: T satisfies E relatively to a sequence $Q_1, \ldots, Q_k$.
2. If $\mathbf{M}$ is a model and $Q_i[\mathbf{M}, i]$, $i = 1, \ldots, k$, then elements of Q_i are submodels of $\mathbf{M}$ (see *D.8.*), the number i in (4d) is a name of an arbitrary element of the domain of $\mathbf{M}$ and in *D.16.* and *D.17.* we assume that we consider only submodels of $\mathbf{M}$; in *D.18.* we associate to each formula a pair of numbers.
3. P is the set of all true formulas (see *T.4.*).

Obviously:

(3d') $V\{T, Q_1, \ldots, Q_k, F + G\} = 0 \cdot \equiv \cdot V\{T, Q_1, \ldots, Q_k, F\} = 0 \wedge V\{T, Q_1, \ldots, Q_k, G\} = 0$.

(4d') $V\{T, Q_1, \ldots, Q_k, \Pi aF\} = 0 \cdot \equiv \cdot (\exists i)(\exists T_1)\{(i \leq k) \wedge R(T, T_1, Q_1, \ldots, Q_k, i_1, \ldots, i_{w(F)}, i) \wedge V\{T_1, Q_1, \ldots, Q_k, F(x_i/a)\} = 0\}$.

(5d) $V\{T, Q_1, \ldots, Q_k, \Sigma aF\} = 1 \cdot \equiv \cdot (\exists i)(\exists T_1)\{(i \leq k) \wedge R(T, T_1, Q_1, \ldots, Q_k, i_1, \ldots, i_{w(F)}, i) \wedge V\{T_1, Q_1, \ldots, Q_k, F(x_i/a)\} = 1\}$.[25]

(5d') $V\{T, Q_1, \ldots, Q_k, \Sigma aF\} = 0 \cdot \equiv \cdot (i)(T_1)\{(i \leq k) \wedge R(T, T_1, Q_1, \ldots, Q_k, i_1, \ldots, i_{w(F)}, i) \rightarrow V\{T_1, Q_1, \ldots, Q_k, F(x_i/a)\} = 0\}$.

(6d) $E \,\bar{\epsilon}\, P \cdot \equiv \cdot (\exists Q_1) \ldots (\exists Q_{n(E)})(\exists T)\{(H)(\{H \epsilon A(E)\} \rightarrow \{[T|i_1, \ldots, i_{w(H)}] \epsilon Q_{w(H)}\}) \wedge N\{Q_{n(E)}, n(E)\} \wedge Q_{n(E)}[Q_1, \ldots, Q_{n(E)-1}] \wedge N(\Sigma(E), Q_1, \ldots, Q_{n(E)}) \wedge V\{T, Q_1, \ldots, Q_{n(E)}, E\} = 0\}$.

T.2. If $E \epsilon Skt$, $F \epsilon C(E)$, $\mathbf{M}\{E\} = 0$, $k \geq n(E)$, $Q_1[\mathbf{M}, 1], \ldots, Q_k[\mathbf{M}, k]$, then:

1. $Q_k[Q_1, \ldots, Q_{k-1}]$, $N(Q_i, i)$, for $i = 1, \ldots, k$, $N(2, Q_1, \ldots, Q_k)$.
2. If $\mathbf{M} \epsilon R_1$, then $N(r, Q_1, \ldots, Q_k)$, for every $r \leq k$.
3. If $[\mathbf{M}|s_{i_1}, \ldots, s_{i_{w(F)}}] = [T|i_1, \ldots, i_{w(F)}]$ and $\mathbf{M}\{F(s_{i_1}, \ldots, s_{i_{w(F)}})\} = 0$, then $V\{T, Q_1, \ldots, Q_k, F\} = 0$.
4. $E' \epsilon P(Q_1, \ldots, Q_k)$ and $E \,\bar{\epsilon}\, P(k, 2)$.
5. If $\mathbf{M} \epsilon R_1$, then $E \,\bar{\epsilon}\, P$.

Proof: —From *L.3*, *L.6*, *L.14.* and *L.17.* we obtain *1* and *2*; conclusions *4* and *5* follow from *1, 2, 3*, $\mathbf{M}\{E\} = 0$, *(5d')*, *D.16*, *D.17.* and *D.18.*

We shall proof *(3)* by induction on the number of quantifiers occurring in F:

If $F \epsilon C(E)$ and F is a quantifierless formula, then *3* holds.

It is left for us to verify that if *3* holds for $F(x_i/a) \epsilon C(E)$, then it holds also for the formulas belonging to $C(E)$ of the form:

(1') ΠaF,
(2') ΣaF.

In the case *(1')* by virtue of the definition of satisfiability, of the assumption, *L.2.* and *(4d')* we obtain:

If $\mathbf{M}\{\Pi aF(s_{i_1}, \ldots, s_{i_{w(F)}})\} = 0$, then $(\exists i)(\exists s_i)\{(x_i \,\bar{\epsilon}\, Sw\{F\}) \wedge (i \leq k) \wedge \mathbf{M}\{F(x_i/a)(s_{i_1}, \ldots, s_{i_{w(F)}}, s_i)\} = 0\}$, then $(\exists i)(\exists s_i)(\exists T_1)\{(x_i \,\bar{\epsilon}\, Sw\{F\}) \wedge (i \leq k) \wedge ([\mathbf{M}|s_{i_1}, \ldots, s_{i_{w(F)}}, s_i] = [T_1|i_1, \ldots, i_{w(F)}, i] \epsilon Q_{w(F)+1}) \wedge (\mathbf{M}\{F(x_i/a)(s_{i_1}, \ldots, s_{i_{w(F)}}, s_i)\} = 0)\}$, then $(\exists i)(\exists T_1)\{(x_i \,\bar{\epsilon}\, Sw\{F\}) \wedge (i \leq k) \wedge ([T_1|i_1, \ldots, i_{w(F)}, i] \epsilon Q_{w(F)+1}) \wedge ([T_1|i_1, \ldots, i_{w(F)}] = [T|i_1, \ldots, i_{w(F)}]) \wedge V\{T_1, Q_1, \ldots, Q_k, F(x_i/a)\} = 0\}$, then $(\exists i)(\exists T_1)\{(i \leq k) \wedge R(T, T_1, Q_1, \ldots, Q_k, i_1, \ldots, i_{w(F)}, i) \wedge V\{T_1, Q_1, \ldots, Q_k, F(x_i/a)\} = 0 \wedge V\{T, Q_1, \ldots, Q_k, \Pi aF\} = 0\}$.

In the case *(2')* by virtue of $\Sigma aF \epsilon C(E)$, $E \epsilon Skt$, of the definition of satisfiability, $\mathbf{M}\{E\} = 0$ and of the assumption we obtain that for an arbitrary $i \leq k$ and for each $[T_1|i_1, \ldots, i_{w(F)}] \epsilon Q_{w(F)}$, if exists such $r \leq W(F)$, that $i = i_r$, and for each $[T_1|i_1, \ldots, i_{w(F)}, i] \epsilon Q_{w(F)+1}$ we have $V\{T_1, Q_1, \ldots, Q_k, F(x_i/a)\} = 0$ and therefore *(5d')* for considered tables.

The above give us the complete inductive proof of *3*; q.e.d.

L.18. Let E^o results from E by replacing free variables with indices $i_1, \ldots,$

$i_{w(E)}$ correspondingly by free variables with indices $j_1, \ldots, j_{w(E^o)}$, $w(E) = w(E^o)$[26], and

$$[T|i_1, \ldots, i_{w(E)}] = [T^o \mid j_1, \ldots, j_{w(E^o)}]^{27}.$$

Then:

$$V\{T, Q_1, \ldots, Q_k, E\} = 1 . \equiv . V\{T^o, Q_1, \ldots, Q_k, E^o\} = 1 .$$

L.19. Let $k \geq n(E)$, T is a table of the rank $k+1$ and $T_o = [T \mid 1, \ldots, k]$; then:

$$V\{T, Q_1, \ldots, Q_{k+1}, E\} = 1 \cdot \equiv \cdot V\{T_o, Q_1, \ldots, Q_k, E\} = 1 .$$

The proofs of *L.18.* and *L.19.* are inductive on the length of the formula E and are analogical to the proofs of *L.12.* and *L.14.* respectively from [5].

It is easy to show:

L.20. (1') $F + F' \epsilon P(Q_1, \ldots, Q_k)$.

(1) $F + F' \epsilon P$.

(2') If $F_1 + \ldots + F_n \epsilon P(Q_1, \ldots, Q_k)$ and $k_1, \ldots, k_n$ is an arbitrary permutation of natural numbers $\leq n$, then $F_{k_1} + \ldots + F_{k_n} \epsilon P(Q_1, \ldots, Q_k)$.

(2) If $F_1 + \ldots + F_n \epsilon P$ and $k_1, \ldots, k_n$ is an arbitrary permutation of natural numbers $\leq n$, then $F_{k_1} + \ldots + F_{k_n} \epsilon P$.

(3') If $F \epsilon P(Q_1, \ldots, Q_k)$, then $F + G \epsilon P(Q_1, \ldots, Q_k)$.

(3) If $F \epsilon P$, then $F + G \epsilon P$.

(4') If $F + G$, $F + G' \epsilon P(Q_1, \ldots, Q_k)$ and $G' \epsilon C(F)$, then $F \epsilon P(Q_1, \ldots, Q_k)$.

(4) If $F + G$, $F + G' \epsilon P$ and $G' \epsilon C(F)$, then $F \epsilon P$.[28]

L.21. If $F^* + G^* \epsilon P(Q_1, \ldots, Q_k)$, $j \leq k$, $x_j \bar{\epsilon} Sw\{F^*\}$, $x_j \epsilon Sw\{G^*\}$, $k \geq n\{F^* + \Pi aG^*(a/x_j)\}$, $N(Q_k, k)$, $[Q_k \mid Q_1, \ldots, Q_{k-1}]$, then $F^* + \Pi aG^*(a/x_j) \epsilon P(Q_1, \ldots, Q_k)$.

Proof: –Let $F^* + G^* \epsilon P(Q_1, \ldots, Q_k)$, $j \leq k$, $N(Q_k, k)$, $[Q_k \mid Q_1, \ldots, Q_{k-1}]$, $x_j \epsilon Sw(G^*)$, $x_j \bar{\epsilon} Sw(F^*)$, $k \geq n\{F^* + \Pi aG^*(a/x_j)\}$, $V\{T, Q_1, \ldots, Q_k, F^* + \Pi aG^*(a/x_j)\} = 0$ and $[T \mid i_1, \ldots, i_{w(H)}] \epsilon Q_{w(H)}$, for each $H \epsilon A\{F^* + \Pi aG^*(a/x_j)\}$.

Therefore in view of (3d') and (4d') we obtain: $V\{T, Q_1, \ldots, Q_k, F^*\} = 0$ and there exist such $i \leq k$ and T_1 that $R(T, T_1, Q_1, \ldots, Q_k, i_1, \ldots, i_{w\{G^*(a/x_j)\}}, i)$ and $V\{T_1, Q_1, \ldots, Q_k, G^*(x_i/x_j)\} = 0$; hence $[T \mid i_1, \ldots, i_{w\{G^*(a/x_j)\}}] = [T_1 \mid i_1, \ldots, i_{w\{G^*(a/x_j)\}}]$ and $[T_1 \mid i_1, \ldots, i_{w\{G^*(a/x_j)\}}] \epsilon Q_{w\{G^*(a/x_j)\}}$ and $[T_1 \mid i_1, \ldots, i_{w\{G^*(a/x_j)\}}, i] \epsilon Q_{w\{G^*(a/x_j)\}+1}$, if $(t)\{(1 \leq t \leq w\{G^*(a/x_j)\}) \rightarrow \{i \neq i_t\}\}$.

We consider here two cases:

1. there exists such $t \leq w\{G^*(a/x_j)\}$ that $i = i_t$.
2. for each $t \leq w\{G^*(a/x_j)\}$, $i \neq i_t$.

In the case *1* – for the shortest writing – we assume $i = i_1$.

From the assumption we obtain: $[T_1 \mid i_1, \ldots, i_{w\{G^*(a/x_j)\}}] = [T \mid i_1, \ldots, i_{w\{G^*(a/x_j)\}}]$, $[T_1 \mid i_1, \ldots, i_{w\{G^*(a/x_j)\}}] \epsilon Q_{w\{G^*(a/x_j)\}}$, $w\{G^*(a/x_j)\} = w\{G^*(x_{i_1}/x_j)\}$, $V\{T_1, Q_1, \ldots, Q_k, G^*(x_{i_1}/x_j)\} = 0$ and it may be assumed that the sequence $i_1, \ldots, i_{w\{G^*(a/x_j)\}}$ and $i_1, \ldots, i_{w\{G^*(x_{i_1}/x_j)\}}$ are identical. Therefore in view of *L. 18.* we obtain: $V\{T, Q_1, \ldots, Q_k, G^*(x_{i_1}/x_j)\} = 0$ and $[T \mid i_1, \ldots, i_{w\{G^*(x_{i_1}/x_j)\}}] \epsilon Q_{w\{G^*(x_{i_1}/x_j)\}}$.

Hence by virtue of the assumption, *(3d')* and *L. 10.* we obtain: $V\{T, Q_1, \ldots, Q_k, F^* + G^*(x_{i_1}/x_j)\} = 0$ and for each $H \epsilon A\{F^* + G^*(x_{i_1}/x_j)\}$, $[T \mid i_1, \ldots, i_{w(H)}] \epsilon Q_{w(H)}$; therefore $F^* + G^*(x_{i_1}/x_j) \overline{\epsilon} P(Q_1, \ldots, Q_k)$, which is inconsistent with the assumption.

Hence in the case *1.* we have $F^* + \Pi\, aG^*(a/x_j) \epsilon P(Q_1, \ldots, Q_k)$.

In the case *2.* from the assumption we obtain: $[T_1 \mid i_1, \ldots, i_{w\{G^*(a/x_j)\}}] = [T \mid i_1, \ldots, i_{w\{G^*(a/x_j)\}}]$, $[T_1 \mid i_1, \ldots, i_{w\{G^*(a/x_j)\}}, i] \epsilon Q_{w\{G^*(a/x_j)\}+1}$, $x_i \overline{\epsilon} Sw\{G^*(a/x_j)\}$ and $V\{T_1, Q_1, \ldots, Q_k, G^*(x_i/x_j)\} = 0$.

Let $i \leq j$ and let

$$T_1^o = \begin{cases} T_1 \text{ if } i = j \\ [T_1 \mid 1, \ldots, i-1, j, i+1, \ldots, j-1, i, j+1, \ldots, k], \text{ if } i < j. \end{cases}$$ [29]

Hence and in view of *L. 2.* we obtain: $[T_1^o \mid i_1, \ldots, i_{w\{G^*(a/x_j)\}}, j] = [[T_1 \mid 1, \ldots, j-1, j, i+1, \ldots, j-1, i, j+1, \ldots, k] \mid i_1, \ldots, i_{w\{G^*(a/x_j)\}}, j] = [T_1 \mid i_1, \ldots, i_{w\{G^*(a/x_j)\}}, i] \epsilon Q_{w(G^*)}$ because $w(G^*) = w\{G^*(a/x_j)\} + 1$; hence $[T_1^o \mid i_1, \ldots, i_{w(G^*)}] = [T_1 \mid i_1, \ldots, i_{w\{G^*(x_i/x_j)\}}] \epsilon Q_{w(G^*)}$, where the order of sequences $i_1, \ldots, i_{w(G^*)}$ and $i_1, \ldots, i_{w\{G^*(x_i/x_j)\}}$ are given above, $w(G^*) = w\{G^*(x_i/x_j)\}$, and $[T_1^o \mid i_1, \ldots, i_{w\{G^*(a/x_j)\}}] = [T \mid i_1, \ldots, i_{w\{G^*(a/x_j)\}}]$.

From the above and by virtue of *L. 18.* we obtain: $V\{T_1^o, Q_1, \ldots, Q_k, G^*\} = 0$, $[T_1^o \mid i_1, \ldots, i_{w(G^*)}] \epsilon Q_{w(G^*)}$ and assuming that $t_1, \ldots, t_r$ are all such different elements of sequences $j_1, \ldots, j_{w(F^*)}$ and $i_1, \ldots, i_{w\{G^*(a/x_j)\}}$ [30] which occur in both sequences we have $[T_1^o \mid t_1, \ldots, t_r] = [T \mid t_1, \ldots, t_r]$; therefore in view of *L. 4.* there exists such T_2 of the rank k that $[T_2 \mid j_1, \ldots, j_{w(F^*)}] = [T \mid j_1, \ldots, j_{w(F^*)}]$ and $[T_2 \mid i_1, \ldots, i_{w(G^*)}] = [T_1^o \mid i_1, \ldots, i_{w(G^*)}]$; hence in view of *L. 18.* we have: $V\{T_2, Q_1, \ldots, Q_k, F^*\} = 0$, $V\{T_2, Q_1, \ldots, Q_k, G^*\} = 0$ and by virtue of *(3d')*, the assumption and *L. 10.* we have: $V\{T_2, Q_1, \ldots, Q_k, F^* + G^*\} = 0$ and $[T_2 \mid i_1, \ldots, i_{w(H)}] \epsilon Q_{w(H)}$, for each $H \epsilon A\{F^* + G^*\}$; therefore $F^* + G^* \overline{\epsilon} P(Q_1, \ldots, Q_k)$, which is inconsistent with the assumption.

Therefore in the second case we have also:

$$F^* + \Pi\, aG^*(a/x_j) \epsilon P(Q_1, \ldots, Q_k); \text{ q.e.d.}$$

L. 21'. If $F^* + G^* \epsilon P$, $x_j \overline{\epsilon} Sw\{F^*\}$, then $F^* + \Pi\, aG^*(a/x_j) \epsilon P$.

Proof: –If $x_j \overline{\epsilon} Sw\{G^*(x_j/a)\}$, then *L. 21'.* follows from *D. 20.* and from some simple considerations.

Let $x_j \in Sw\,\{G^*(x_j/a)\}$, $x_j \bar{\epsilon}\, Sw\,\{F^*\}$, $t = n\,\{F^* + \Pi\, aG^*(a/x_j)\}$, $k = n\,\{F^* + G^*\}$, $Q_t\,[Q_1, \ldots, Q_{t-1}]$, $N(Q_t, t)$, $N\{\Sigma(F + \Pi\, aG^*(a/x_j)), Q_1, \ldots, Q_t\}$, $[T^o \mid i_1, \ldots, i_{w(H)}] \in Q_{w(H)}$ for each $H \in A\,\{F^* + \Pi\, aG^*\}$, and $V\,\{T^o, Q_1, \ldots, Q_t, F^* + \Pi\, aG^*(a/x_j)\} = 0$.

Because $k = n\,\{F^* + G^*\} \geq n\,\{F^* + \Pi\, aG^*(a/x_j)\} = t, \Sigma(F^* + G^*) \leq \Sigma\{F^* + \Pi\, aG^*(a/x_j)\}$, therefore by virtue of *L.13.*, *L.16.* and *L.19.* we obtain: $Q_k\,[Q_1, \ldots, Q_{k-1}]$, $N(Q_{k+1}, k{+}1)$, $N\{\Sigma(F^* + G^*), Q_1, \ldots, Q_k\}$ and $V\,\{T, Q_1, \ldots, Q_k, F^* + \Pi\, aG^*(a/x_j)\} = 0$, where $T^o = [T \mid 1, \ldots, t]$, and for each $H \in A\,\{F^* + \Pi\, aG^*(a/x_j)\}$ we have $[T \mid i_1, \ldots, i_{w(H)}] \in Q_{w(H)}$.

Hence and in view of *D.18:* $F^* + \Pi\, aG^*(a/x_j) \bar{\epsilon}\, P(Q_1, \ldots, Q_k)$ and by virtue of *L.21.* and the assumption $F^* + G^* \bar{\epsilon}\, P(Q_1, \ldots, Q_k)$ and therefore $F^* + G^* \bar{\epsilon}\, P$.

The above consideration prove *L.21'*.

L.22.[31] If $F^* + G^*(x_j/a) \in P(Q_1, \ldots, Q_k)$, $k = N\,\{F^* + \Sigma\, aG^*\}$, $r = \Sigma\,(F^* + \Sigma\, aG^*)$, $N(r, Q_1, \ldots, Q_k)$, $N(Q_k, k)$, $Q_k\,[Q_1, \ldots, Q_{k-1}]$, then $F^* + \Sigma\, aG^* \in P(Q_1, \ldots, Q_k)$.

Proof: —We assume that the assumptions of *L.22.* hold.

If $x_j \bar{\epsilon}\, Sw\,\{G^*\}$, then the proof is obvious.

Let $x_j \in Sw\,\{G^*\}$, $V\,\{T, Q_1, \ldots, Q_t, F^* + \Sigma\, aG^*\} = 0$ and for each $H \in A\,\{F^* + \Sigma\, aG^*\}$, $[T \mid i_1, \ldots, i_{w(H)}] \in Q_{w(H)}$; hence and by virtue of $(3d')$ we obtain:

$$V\,\{T, Q_1, \ldots, Q_t, F^*\} = 0,\ V\,\{T, Q_1, \ldots, Q_t, \Sigma\, aG^*\} = 0.$$

If $x_j \in Sw\,\{\Sigma\, aG^*\}$, then taking $H = \Sigma\, aG^*$ we have $[T \mid i_1, \ldots, i_{w(G^*)}] \in Q_{w(G^*)}$ and from $(5d')$ $V\,\{T, Q_1, \ldots, Q_t, G^*(x_j/a)\} = 0$; hence in view of $(3d')$, the assumption and *L.10.* we obtain $V\,\{T, Q_1, \ldots, Q_t, F^* + G^*(x_j/a)\} = 0$ and $[T \mid i_1, \ldots, i_{w(H)}] \in Q_{w(H)}$; for each $H \in A\,\{F^* + G^*(x_j/a)\}$; therefore $F^* + G^*(x_j/a) \bar{\epsilon}\, P(Q_1, \ldots, Q_k)$, which is impossible, and therefore $F^* + \Sigma\, aG^* \in P(Q_1, \ldots, Q_k)$ in this case.

If $x_j \in Sw\,\{F^*\}$, then analogously to above—using *L.10.*—we have: $[T \mid i_1, \ldots, i_{w(G^*)}] \in Q_{w(G^*)}$ and $[T \mid j] \in Q_1$.

Because $r = \Sigma\,(F^* + \Sigma\, aG^*) \geq \Sigma\,(F^* + G^*\,(x_j/a))$ and $r > w(G^*)$, then in view of the assumption and *D.14.* there exists such T_1 of the rank k that $[T_1 \mid i_1, \ldots, i_{w(G^*)}, j] \in Q_{w(G^*)+1}$ and for each $H \in A\,\{F^* + \Sigma\, aG^*\}$, $[T_1 \mid i_1, \ldots, i_{w(H)}] = [T \mid i_1, \ldots, i_{w(H)}]$.

Hence in view of *L.18.* and $(3d')$ $V\,\{T_1, Q_1, \ldots, Q_k, F^*\} = 0$ and by virtue of $(5d')$ and *D.15.* $V\,\{T_1, Q_1, \ldots, Q_k, G^*(x_j/a)\} = 0$.

From the above and in view of $(3d')$ and *L.10.* we have: $V\,\{T_1, Q_1, \ldots, Q_k, F^* + G^*(x_j/a)\} = 0$ and for each $H \in A\,\{F^* + G^*(x_j/a)\}$ $[T_1 \mid i_1, \ldots, i_{w(H)}] \in Q_{w(H)}$; hence $F^* + G^*(x_j/a) \bar{\epsilon}\, P(Q_1, \ldots, Q_k)$, and therefore $F^* + \Sigma\, aG^* \in P(Q_1, \ldots, Q_k)$ in this case also.

If $x_j \bar{\epsilon}\, Sw\,\{F\}$, $x_j \bar{\epsilon}\, Sw\,\{\Sigma\, aG^*\}$ and $F^* + \Sigma\, aG^*$ has no free variables, then because Q_1 is non-empty, then there exists such T_1 of the rank k that $[T_1 \mid 1] \in Q_1$ and by virtue of *L.18:* $V\,\{T_1, Q_1, \ldots, Q_k, F^*\} = 0$ and $V\,\{T_1, Q_1, \ldots, Q_k, \Sigma\, aG^*\} = 0$; hence in view of $(5d')$ and *D.15.* $V\,\{T_1, Q_1, \ldots,$

Q_k, $G(x_1/a)\} = 0$ and therefore by virtue of (3d') $V\{T_1, Q_1, \ldots, Q_k, F^* + G^*(x_1/a)\} = 0$, which proves that $F^* + G^*(x_1/a) \bar{\epsilon} P(Q_1, \ldots, Q_k)$, and therefore $F^* + \Sigma\, aG^* \epsilon P(Q_1, \ldots, Q_k)$ in the third case.

If $x_j \bar{\epsilon} Sw\{F^*\}$, $x_j \bar{\epsilon} Sw\{G^*\}$ and–for the shortest writing–$x_1 \epsilon Sw\{F^* + \Sigma\, aG^*\}$, then analogously to the second case there exists such T_1 of the rank k that $[T_1 \mid i_1, \ldots, i_{w(G^*)}, 1] \epsilon Q_{w(G^*)+1}$, $V\{T_1, Q_1, \ldots, Q_k, F^*\} = 0$, $V\{T_1, Q_1, \ldots, Q_k, \Sigma\, aG^*\} = 0$ and for each $H \epsilon A\{F^* + \Sigma\, aG^*\}$, $[T_1 \mid i_1, \ldots, i_{w(H)}] = [T \mid i_1, \ldots, i_{w(H)}] \epsilon Q_{w(H)}$; therefore analogously $F^* + G^*(x_1/a) \bar{\epsilon} P(Q_1, \ldots, Q_k)$, and therefore $F^* + \Sigma\, aG^* \epsilon P(Q_1, \ldots, Q_k)$.

The above considerations prove *L.22.*

L.22'. If $F^* + G^*(x_j/a) \epsilon P$, then $F^* + \Sigma\, aG^* \epsilon P$.[32]

Proof: –Because $n\{F^* + G^*(x_j/a)\} \geq \Sigma\{F^* + G^*(x_j/a)\}$, $\Sigma\{F^* + G^*(x_j/a)\} \leq \Sigma\{F^* + \Sigma\, aG\}$, then in view of *L.16*, *L.19.* and *L.22.* we obtain *L.22'*; the whole proof is analogous to the proof of *L.21'*.

L.23. If $F + \Pi\, aG \epsilon P(Q_1, \ldots, Q_k)$, $r = \Sigma\{F + \Pi\, aG\} > w(G)$ and $N(r, Q_1, \ldots, Q_k)$, then $F + G(x_j/a) \epsilon P(Q_1, \ldots, Q_k)$.

Proof: –We assume the assumption of *L.23.* and let $V\{T, Q_1, \ldots, Q_k, F + G(x_j/a)\} = 0$, and for each $H \epsilon A\{F + G(x_j/a)\}$ $[T \mid i_1, \ldots, i_{w(H)}] \epsilon Q_{w(H)}$.

Because $r > w(G)$, then using *D.14.* many times we obtain that there exists such T_1 that $[T_1 \mid i_1, \ldots, i_{w\{G(x_j/a)\}}, j] \epsilon Q_{w(G)+1}$, and for each $H \epsilon A\{F + G(x_j/a)\}$ $[T \mid i_1, \ldots, i_{w(H)}] \epsilon Q_{w(H)}$; therefore in view of *L.18.* and (3d'): $V\{T_1, Q_1, \ldots, Q_k, F\} = 0$, $V\{T_1, Q_1, \ldots, Q_k, G(x_j/a)\} = 0$ and by virtue of (4d') and (3d') $V\{T_1, Q_1, \ldots, Q_k, F + \Pi\, aG\} = 0$ and by virtue of *L.10.* for each $H \epsilon A\{F + \Pi\, aG\}$, $[T_1 \mid i_1, \ldots, i_{w(H)}] \epsilon Q_{w(H)}$; hence $F + \Pi\, aG \bar{\epsilon} P(Q_1, \ldots, Q_k)$, which is inconsistent with the assumption; therefore $F + G(x_j/a) \epsilon P(Q_1, \ldots, Q_k)$; q.e.d.

L.23'. If $F + \Pi\, aG \epsilon P$, $\Pi\, aG \epsilon C(F)$, $\Sigma\,(F + \Pi\, aG) > w(G)$, then $F + G(x_j/a) \epsilon P$.

Proof: –Because here $\Sigma\{F + \Pi\, aG\} \leq \Sigma\{F + G(x_j/a)\}$, therefore in view of *L.16*, *L.19.* and *L.23.* we obtain *L.23'*.; the whole proof is analogous to the proof of *L.21.*

T.3. If E is a thesis, then $E \epsilon P$.

The proof of *T.3.* is inductive on the length of formalized proof of the formula E and this follow from *L.0*, *L.20'*, *L.21'*. and *L.23'*.

A simple conclusion from *L.1*, *T.1*, *T.2*, and *T.3.* is:

T.4. A formula E is a thesis if and only if $E \epsilon P$.[33]

For example:

If $E \epsilon Skt$ and $E = \Sigma\, a_1 \ldots \Sigma\, a_i\, \Pi\, a_{i+1} \ldots \Pi\, a_k F$, then E is a thesis if and only if $E \epsilon P(k, i)$.[34]

Obviously:

$P(k, 1) \subset P(k+1, 2) \subset P(k+2, 3) \subset \ldots.$

From *L.12*, *L.14*, and *T.4* follow some generalization of theorem Gödel-Kalmar, see [1], [3] that the class of theses of the form $\Sigma\, a_1\, \Sigma\, a_2\, \Pi\, a_3 \ldots \Pi\, a_k F$, where F is a quantifierless formula containing no free variables, is decidable:

The classes $P(k, 1)$ and $P(k, 2)$ are decidable, $k = 1, 2, \ldots$

The monadic first-order functional calculus is decidable.

From *T.4.* we obtain the decidability function for the classes $P(k, 1)$ and $P(k, 2)$, $k = 1, 2, \ldots$

From [1] follows that the decidability of the class $P(4, 3)$ is equivalent with the decidability of the class $P(k, m)$, for $m \geq 3$, $k \geq m$; it follows also that the function V defined in *D.18.* for the classes $P(k, m)$, $m \geq 3$, $k \geq m$, is not general recursive.

If we shall add to the considered functional calculus the description of tables, then the above considerations we may write in the domain of those theories, see [9].

Another characterization of theses of the first-order functional calculus we shall obtain from [8] in the following way:

First of all we introduce the function V_1 which is defined for an arbitrary finite sequence $\{Q_n\}$, where Q_i are non-empty sets of tables of the rank i, $i = 1, \ldots, n$, for an arbitrary table $T \in Q_k$ and for an arbitrary formula E whose indices of the free variables are $\leq k$ and $k + p(E) \leq n$:

(*d1*) $V_1\{T, \{Q_n\}, f_t^m(x_{r_1}, \ldots, x_{r_m})\} = 1 \cdot \equiv \cdot F_t^m(r_1, \ldots, r_m)$,

(*d2*) $V_1\{T, \{Q_n\}, F'\} = 1 \cdot \equiv \cdot \sim V_1\{T, \{Q_n\}, F\} = 1 \cdot \equiv \cdot V_1\{T, \{Q_n\}, F\} = 0$,

(*d3*) $V_1\{T, \{Q_n\}, F + G\} = 1 \cdot \equiv \cdot V_1\{T, \{Q_n\}, F\} = 1 \vee V_1\{T, \{Q_n\}, G\} = 1$,

(*d4*) $V_1\{T, \{Q_n\}, \Pi\, aF\} = 1 \cdot \equiv \cdot (i)\,\{(i \leq k) \rightarrow V_1\{T, \{Q_n\}, F(x_i/a)\} = 1\} \wedge (T_1)\,\{(T_1 \in Q_{k+1}) \wedge (T = [T_1 \mid 1, \ldots, k]) \rightarrow V_1\{T_1, \{Q_n\}, F(x_{k+1}/a)\} = 1\}$.

D.19. $F \in P(\{Q_n\}) \cdot \equiv \cdot (T)\,\{(T \in Q_{i(F)}) \rightarrow V_1\{T, \{Q_n\}, F\} = 1\}$.

D.20. $N_1(\{Q_n\}, G) \cdot \equiv \cdot (i)\,(T_1)\,(T_2)\,(i + p(G) < n) \wedge (T_1 \in Q_i) \wedge (T_2 \in Q_{i+1}) \wedge (T_1 = [T_2 \mid 1, \ldots, i]) \wedge V_1\{T_2, \{Q_n\}, G\} = 1 \rightarrow V_1\{T_1, \{Q_n\}, G\} = 1$.

D.21. $F \in P[G, \{Q_n\}] \cdot \equiv \cdot N_1(\{Q_n\}, G) \rightarrow \{F \in P(\{Q_n\})\}$.

D.22. $F \in P\{n, E\} \cdot \equiv \cdot (\{Q_n\})(\exists G)(\{G \in C(E)\} \wedge \{F \in P[G, \{Q_n\}]\})$.

D.23. $F \in P\,|E| \cdot \equiv \cdot (\exists n)\,\{(F \in P\{n, E\}) \wedge (n \geq n(F))\}$.

D.24. $E \in P_1 \cdot \equiv \cdot E \in P\,|E|$.

It may be proved, see [8]:

T.5. A formula E is a thesis if and only if $E \in P_1$.

We note that if we shall replace *D.22.* by:

D.22'. $F \in P\{n, E\} \cdot \equiv \cdot (\{Q_n\})\,(F \in P[E, \{Q_n\}])$, then analogously we may show:

T.6. If $E \in Skt$, then E is a thesis if and only if $E \in P_1$.

T.6. may also be proved in another way.

The function V_1 has interesting properties which may be applicable to the verification of formulas of considered calculus, see [8].

By a simple generalization of the above definitions we may obtain a new characterization of theses of the first-order functional calculus with added axioms **U**, see [4].

NOTES

1. The numbers in the square brackets refer to the bibliography given at the end of this paper.

2. The symbols of this calculus are:
   (a) free individual variables: $x_1, x_2, \ldots$ (or simply x),
   (b) apparent individual variables: $a_1, a_2, \ldots$ (or simply a),
   (c) functional variables with m-arguments: $f_1^m, f_2^m, \ldots,$
   (d) logical constants: ' (the negation), + (the alternative), Π (the general quantifier),

3. Here the formula has the same meaning which has the well formed formula. An expression in which an apparent variable a belongs to the scope of two quantifiers $\Pi\, a$ is not a formula.

4. It is Skolem's normal form for theses.

5. We see that every significant part of the formula E is a formula.

6. We notice here that if $\Sigma\,\{F(x/a)\} = 0$, then $\Sigma\,\{F(x_i/a)\} = 0$, for each i.
   In exactly given cases the number $\Sigma(F)$ may be less than defined above.

7. The dots separate more strongly than parentheses.

8. If **U** is empty, then we say that $E_1, \ldots, E_n$ is a formalized proof of E, or—briefly—a formalized proof.

9. This is a form of Herbrand's theorem.

10. See *T.1.*

11. See [2].

12. *L.1* asserts the existence of Skolem's normal form for theses.

13. The whole proof of these lemmas is given in [5].

14. This lemma is proved by L. Kalmar in [3].

15. (T) we read: for each T (of the respective rank)
    $(\exists T)$ we read: there exists such T that

16. Another extension of model $\mathbf{M}_1$ to model $\mathbf{M} \in R_t$, for every t, is given in [5].

17. See footnote 15. We notice that T_1 is an extension of T with some conditions.

18. We notice that **M** is here a monadic model.

19. If $[T \mid i, j] \in Q_2$, then we assume $T = T_1$.

20. To the proof of *L.15.* and *L.16.* we use also *L.5*, *L.7*, *L.8.* and *L.9.*

21. If $m = 0$, then we write $R(T, T, i)$.

22. See footnote 15.

23. (G) we read: for every G; $(\exists G)$ – there exists G such that

24. We assume that $[T \mid] \in Q_i$, for each i; we have this case when H has no free variables.

25. We notice that $\Sigma\, aF = (\Pi\, aF')'$.

26. Then E results from E^o by replacing the free variables with indices $j_1, \ldots, j_{w(E^o)}$ correspondingly by free variables with indices $i_1, \ldots, i_{w(E)}$.

27. Obviously $i_1, \ldots, i_{w(E)}, j_1, \ldots, j_{w(E^o)} \leq k$

28. It is easy to show:

(5'') If $F + G + G \in P(Q_1, \ldots, Q_k)$, then $F + G \in P(Q_1, \ldots, Q_k)$.
(5'') If $F + G + G \in P$, then $F + G \in P$.

See p. 3, (1,7) and footnote 34.

29. If $i \geq j$, then we assume

$$T_1^o = \begin{cases} T_1, \text{ if } i = j \\ [T_1 \mid 1, \ldots, j-1, i, j+1, \ldots, i-1, j, i+1, \ldots, k], \text{ if } i > j. \end{cases}$$

30. Since $x_j \,\bar{\epsilon}\, Sw\{F^*\}$, then we may write here $i_{w(G^*)}$ for $i_{w\{G^*(a/x_j)\}}$

31. The reader may omit this lemma in the first reading.

32. See footnote 31.

33. We notice here that if the Skolem's normal form for theses does not belong to P, then $E \,\bar{\epsilon}\, P$.

34. Another proof of this theorem we may obtain from *T.1*, *T.2.* and *L.0*, *L.20'*, *L.21'*, *L.22'*, see p. 4, (1,7), (1,8) and footnote 28.

BIBLIOGRAPHY

[1] *K. Gödel:* Zum Entscheidungsproblem des logischen Funktionenkalkül *Monatshefte für Mathematik und Physik*, T.40, 1933.

[2] *D. Hilbert-P. Bernays: Grundlagen der Mathematik,* Bd, II, Berlin 1939.

[3] *L. Kalmar:* Über die Erfülbarkeit derjenigen Zahlausdrücke weche in der Normal-form zwei benachbarte Allzeichen enthalten, *Mathematische Annalen*, Bd. 108, 1933.

[4] *J. Reichbach:* O pełności węższego rachunku funkeyjnego, *Studia Logica*, T.II, 1955.

[5] *J. Reichbach:* On the first-order functional calculus and the truncation of models, *Studia Logica*, T.VII, 1958.

[6] *J. Reichbach:* Rachunek funkcyjny, w którym występują zmienne dowolnych typów i jego modele, *Prace Matematyczne*, in press.

[7] *J. Reichbach:* Some characterizations of theses of the first-order functional calculus, sent to *Fundamenta Mathematica.*

[8] *J. Reichbach:* On theses of the first-order functional calculus, sent to *The Bulletin of the Research Council of Israel. Section F: Mathematics and Physics.*

[9] *J. Reichbach:* Some methods of proofs of formulas of the first-order functional calculus, in preparation.

[10] *A. Tarski: Pojęcie prawdy w językach nauk dedukeyjnych,* Prace Towarzystwa Naukowego Warszawskiego, Wydział III, 1933. Cf.: A. Tarski: *Logic, Semantics, Metamathematics.* Oxford, 1956. P.p. 152-278.

Tel Aviv, Israel