ON A RECENT ALLOTMENT OF PROBABILITIES TO OPEN AND CLOSED SENTENCES

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Probabilities, though frequently allotted to closed sentences, have rarely been allotted to open ones. The recent scheme by Kemeny, Mirkil, Snell, and Thompson in *Finite Mathematical Structures* for allotting probabilities to sentences of the form '$f(x) = a$' is therefore of considerable interest.[1] It has, however, a shortcoming which I should like to discuss here and, possibly, remedy.

Let 'f' be a functional constant, 'x' an individual variable, and 'a' an individual constant; let U be the (finite) set of values of 'x'; and let A be the subset of U whose members satisfy '$f(x) = a$'. Kemeny *et al.* then take the probability of '$f(x) = a$' to be $m(A)$, where $m(A)$ is the measure (in some appropriate sense of the word 'measure') of A.[2] Their scheme is attractive enough and mirrors to some extent what mathematicians understand by the probability of a set.[3] Kemeny *et al.* are careful, of course, to restrict it to open sentences of the form '$f(x) = a$'. Consider, however, a closed sentence of the kindred form '$f(b) = a$', where 'b' is an individual constant. Since '$f(b) = a$' does not contain any occurrence of 'x', it would normally be held to be satisfied by every member of U when true, by none when false. One would accordingly expect Kemeny *et al.* to take the probability of '$f(b) = a$' to be *1* when '$f(b) = a$' is true, *0* when '$f(b) = a$' is false. Yet in their scheme for allotting probabilities to closed sentences, a scheme I shall go into below, they let the probability of a closed sentence equal *1* only when the sentence is logically true, *0* only when it is logically false.[4] '$f(b) = a$' being neither logically true nor logically false, its probability must therefore differ by that scheme from either one of *1* and *0*, a disturbing enough result.

The difficulty becomes even more acute when the calculus, call it C, to whose sentences probabilities are allotted is a simple applied predicate calculus of the first order with identity.

Assume indeed that a set D of individuals has been singled out as the domain of C, a member of D paired with each individual constant W of C as the individual designated by W, and a class of ordered n-tuples of members of D paired with each n-adic predicate constant F of C as the extension of

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F. Assume next that an assignment *Asst* of members of *D* to the individual variables of *C* is said to satisfy a sentence *S* of *C* under the following circumstances:

D1. (*a*) Let *S* be of the form $F(W_1, W_2, \ldots, W_n)$, where *F* is an *n*-adic predicate constant and $W_1, W_2, \ldots,$ and W_n are *n* individual constants or variables. If the ordered *n*-tuple made up of the members of *D* respectively designated by or assigned by *Asst* to $W_1, W_2, \ldots,$ and W_n belongs to the extension of *F*, then *Asst* satisfies *S*;

(*b*) Let *S* be of the form $W_1 = W_2$, where W_1 and W_2 are two individual constants or variables. If the members of *D* respectively designated by or assigned by *Asst* to W_1 and W_2 are the same, then *Asst* satisfies *S*.

(*c*) Let *S* be of the form $\sim (S')$. If *Asst* does not satisfy *S'*, then *Asst* satisfies *S*.

(*d*) Let *S* be of the form $(S') \supset (S'')$. If *Asst* does not satisfy *S'* or *Asst* satisfies *S''*, then *Asst* satisfies *S*.

(*e1*) Let *S* be of the form $(\forall W)\ (S')$, where *W* is an individual variable and *W* is not free in *S'*. If *Asst* satisfies *S'*, then *Asst* satisfies *S*.

(*e2*) Let *S* be of the form $(\forall W)\ (S')$, where *W* is an individual variable and *W* is free in *S'*. If *Asst* satisfies *S'* and every assignment of members of *D* to the individual variables of *L* which is like *Asst* except for the member of *D* it assigns to *W* also satisfies *S'*, then *Asst* satisfies *S*.[5] Assume finally that probabilities are allotted to the sentences of *C* as follows:

D2. Let *S* be a sentence of *C*. Then the probability of *S* equals $m(Asst_S)$, where $Asst_S$ is the set of assignments of members of *D* to the individual variables of *C* which satisfy *S* and $m(Asst_S)$ is the measure (in some appropriate sense of the word 'measure') of $Asst_S$.

It is clear that if the sentence *S* in *D1*-*D2* is allowed to be closed as well as open, if a closed sentence *S* of *C* is taken, as usual, to be true when *S* is satisfied by all assignments of members of *D* to the individual variables of *C*, and if a closed sentence *S* of *C* is taken, as usual again, to be false when *S* is not true, then the probability of a closed sentence *S* of *C* will automatically equal *1* or *0*, when *S* is true, *0* when *S* is false. If, on the other hand, the sentence *S* in *D1*-*D2* is presumed to be open, then other probabilities besides *1* and *0* may consistently be allotted to the closed sentences of *C*. Note, however, that with the sentence *S* in *D1*-*D2* thus presumed to be open, the probability of $(W = W) \supset (S)$, where *W* is an individual variable of *C* and *S* is a closed sentence of *C*, will nonetheless equal *1* or *0*, *1* when *S* is true, *0* when *S* is false.[6] One could therefore not allot probabilities other than *1* and *0* to *S* without thereby allotting different probabilities to *S* and $(W = W) \supset (S)$. But *S* and $(W = W) \supset (S)$ are logically equivalent. One could therefore not allot probabilities other than *1* and *0* to *S* without thereby allotting different probabilities to logically equivalent sentences, a distressing result.[7]

I would accordingly recommend that (*1*) the Kemeny *et al.* scheme for allotting probabilities to open sentences of the form '$f(x) = a$' be made to cover as well closed sentences of the form '$f(b) = a$'. I would also recommend, when it comes to allotting probabilities to the sentences of *C*, that (*2*) the sentence *S* in *D1-D2* be suffered to be open as well as closed, or that (*3*) the probability of a closed sentence *S* of *C* be taken to be that of the open sentence $(W = W) \supset (S)$ we just considered.[8] In all three cases the probability of a closed sentence, be it of the form '$f(b) = a$' or any other, would equal *1* or *0*, and in the last two cases the requirement that equivalent sentences be allotted equal probabilities would be met.

The scheme Kemeny *et al.* used to allot probabilities to closed sentences is roughly as follows. A (finite) set *U* of so-called logical possibilities is assumed to be given; a subset *A* of *U*, consisting of all the members of *U* which are not precluded (so to speak) by a closed sentence *S*, is then paired with *S* as the so-called truth-set of *S*; the probability of *S* is finally taken to be $m(A)$, where $m(A)$ is the measure (in some appropriate sense of the word 'measure') of the truth-set *A* of *S*. It is clear from my brief description of the scheme that the probability of a closed sentence *S*, where *S* is neither logically true nor logically false, must be other than *1* or *0*.[9]

This allotment of probabilities, reminiscent of Carnap's *Logical Foundations of Probabilities*, is at variance with the one I just recommended.[10] It might nonetheless be retained alongside mine if a distinction which Kemeny *et al.* ignore were drawn, that between the truth-value of a sentence and an estimate of the truth-value of a sentence. The probability I have just allotted to a closed sentence *S* coincides with the truth-value of *S*. The one Kemeny *et al.* allot, on the other hand, to such a sentence has all the earmarks of an estimate of the truth-value of *S*. But there is room in inductive logic for estimates of truth-values as well as for truth-values. The Kemeny *et al.* allotment of probabilities to closed sentences might thus be retained—*subject to the above reinterpretation*—alongside mine. It might even be extended in one interesting direction.

Whereas, indeed, the probability Kemeny *et al.* allot to a closed sentence has all the earmarks of an estimate of a truth-value, the one they allot to an open sentence has all these of a truth-value in a generalized sense of the word 'truth-value'. As a matter of fact, when the members of the domain *D* of such a calculus as *C* are paired in a one-to-one fashion with the individual constants of *C*, the probability allotted by *D2* to an open sentence *S* of *C* proves to be a weighted average of the truth-values (in the usual sense of the word) of the so-called instances of *S*.[11] There being room in inductive logic for estimates of the truth-values of open sentences as well as of closed ones, the Kemeny *et al.* allotment of probabilities to closed sentences might therefore be extended to cover open sentences as well.

I cannot recount here the various steps to be taken in carrying out that extension.[12] Once they are taken, however, a sentence *S* (be it closed or open) comes to be allotted two probabilities: one, the truth-value of *S* either in the traditional sense or in a generalized sense of the word, has a statistical flavor of its own and, for that reason, might be dubbed *the statistical probability of S*; the other, an estimate of the truth-value of *S*, has an inductive flavor of its own and, for that reason, might be dubbed *the inductive probability of S*.

Kemeny *et al.* ignore, I said, the distinction I urged above between a truth-value and an estimate of a truth-value. They also ignore as a result the distinction I urge here between a statistical probability and an inductive one.[13] They do so, however, at the price, as I hope to have shown, of allotting different probabilities to the two equivalent sentences S and $(W = W) \supset (S)$.[14]

NOTES

[1] See John G. Kemeny, Hazelton Mirkil, J. Laurie Snell, and Gerald L. Thompson, *Finite Mathematical Structures*, Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1959, chapter 2, section 4, and chapter 3, section 1.

[2] For a definition of the word 'measure', see *Finite Mathematical Structures*, p. 113.

[3] Kemeny *et al.* transfer indeed to the defining condition of a set, understood here as an open sentence, the probability which mathematicians normally allot to the set of elements satisfying that condition.

[4] See *Finite Mathematical Structures*, chapter 2, section 3, and chapter 3, sections 1 and 2.

[5] Some additions, which the reader can easily supply, should be made to the text if C contained functional constants as well as individual and predicate constants.

[6] A new definition of the phrase 'S is true', which the reader can easily supply, would of course be required here.

[7] The same result would hold if C were a simple applied predicate calculus of the first order without identity. The open sentence $((F(W)) \supset (F(W))) \supset (S)$, where F is, say, the alphabetically first predicate constant of C, could then serve in lieu of $(W = W) \supset (S)$.

[8] A different scheme for allotting what I shall call below statistical probabilities will be found in the author's "On chances and estimated chances of being true," *Revue Philosophique de Louvain*, vol. 57 (Mai 1959), pp. 225-239.

[9] See references in footnote 4.

[10] See Rudolph Carnap, *Logical Foundations of Probability*, Chicago: The University of Chicago Press, 1950, chapter V.

[11] This point is made in the author's "On chances and estimated chances of being true," theorem *T3.12*, for a special weighting of the truth-values of the instances of S. It can be made for all weighting of the truth-values in question once *D3.1(b)*, *(c1)*, and *(c2)* are suitably generalized.

[12] See the author's "On chances and estimated chances of being true," where the extension in question is carried out for a family of calculi *C*.

[13] In *A Philosopher Looks at Science*, Princeton, N. J.: D. Van Nostrand Company, Inc., 1959, chapter 4, Kemeny distinguishes between two kinds of probabilities. The distinction he draws there, however, is not reproduced in *Finite Mathematical Structures*.

[14] This paper was given by title at the 1960 International Congress for Logic, Methodology and Philosophy of Science, Stanford University, Stanford, California.

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