THE NUMBER OF MODULI IN N-ARY RELATIONS

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As a consequence of theorem 1, p.p. 385-387, in [1], Tarski has proved that two unary relations and four binary relations are definable by purely logical means, and that in general, "for every natural number n only a specifiable finite number of n-termed relations between individuals can be defined by purely logical means, and each of these relations can be expressed by means of identity and the concepts of the sentential calculus." It is the objective of this note to specify the above mentioned number and to exhibit the relations for $n = 3$.

Let n be a fixed natural number and $x_1, x_2, \ldots, x_n$ n individuals.

D1. If $R(x_1, x_2, \ldots, x_n)$ is an n-ary relation definable in terms of identity and the propositional calculus, it is called a modulus.

D2. If $R(x_1, x_2, \ldots, x_n)$ is a modulus which applies to none of the individuals $x_1, x_2, \ldots, x_n$, it is called the empty modulus and is denoted by Φ.

D3. Let $\mathbf{N} = [(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq n]$

D4. If a set S determines a finite number of propositions, $\bigwedge_S P$ denotes the conjunction of these propositions. In some instances the conjunction of $P_1, P_2, \ldots, P_m$, will be denoted by $\bigwedge_{i=1}^{m} P_i$. In like manner, $\bigvee_S P$ will be used for alternation.

Lemma 1. Any non-empty modulus can be expressed in the form $\bigvee_B (\bigwedge_A T)$; where ($\alpha$) $\phi \subsetneqq A \subseteq \mathbf{N}$, ($\beta$) T is either $x_i = x_j$ or $x_i \neq x_j$ for $(i, j) \in A$, (γ) $\bigwedge_A T \neq \Phi$, (δ) $\{\phi\} \neq B$, $\phi \neq B \subseteq P(\mathbf{N})$ where $P(\mathbf{N})$ is the set of subsets of $\mathbf{N}$.

Proof: Any negation sign can be taken into a T by means of $\sim (p \wedge q) . \equiv . \sim p \vee \sim q$ and $\sim (p \vee q) . \equiv . \sim p \wedge \sim q$. Any Φ can be removed by $p \vee \Phi . \equiv p$.

The conditions (α) through (δ) will apply to the next two definitions.

D5. A Kn-form is a proposition of the form $\bigwedge_{\mathbf{N}} T$.

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D6. An AKn-form is a proposition of the form $\bigvee_{B} (\bigwedge_{\mathbf{N}} T)$.

Lemma 2. Any non-empty modulus can be expressed by an AKn-form.

Proof. By lemma *1* it suffices to show that $\bigwedge_{A} T \equiv \bigvee_{C} (\bigwedge_{\mathbf{N}} T)$. Since $\mathbf{N}$ is finite it suffices to show that if $M \equiv \bigwedge_{A} T$, and $A \subsetneqq \mathbf{N}$, then $\mathbf{M} \equiv \mathbf{M}_1 \vee \mathbf{M}_2$ where $M_k \equiv \bigwedge_{A_k} T$, $A \subsetneqq A_k$ for $k = 1, 2$. Let $(i, j) \in \mathbf{N} - A$. $M \equiv . M \wedge (x_i = x_j . \vee . x_i \neq x_j) . \equiv : M \wedge (x_i = x_j) . \vee . M \wedge (x_i \neq x_j) : \equiv .. M_1 \vee M_2$. Therefore $M_k \equiv \bigwedge_{A_k} T$ with $A_k = A + (i, j)$.

If two forms differ only by the order of their terms they will be considered identical.

Lemma 3. If M and P are distinct Kn-forms then $\sim (M \supset P)$.

Proof. If M and P are distinct Kn-forms there is an $(i, j) \in \mathbf{N}$ such that $x_i = x_j$ is a conjunct of M and $x_i \neq x_j$ is a conjunct of P (or vice versa). Therefore, $M \supset x_i = x_j$ and $P \supset x_i \neq x_j$. $M \supset P . \supset : M \supset : x_i = x_j . \wedge . x_i \neq x_j$. Since $M \neq \Phi$, $\sim (M \supset P)$.

Corollary. Lemma *3* says that distinct Kn-forms are not equivalent and therefore define different relations.

Lemma 4. If $M, P_1, P_2, \ldots, P_k$ are distinct Kn-forms, then $\sim (M \supset \bigvee_{i=1}^{k} P_i)$.

Proof: By lemma *3* we have $\bigwedge_{i=1}^{k} (\sim (M \supset P_i))$. But $\bigwedge_{i=1}^{k} (\sim (M \supset P_i)) . \equiv . \sim (M \supset \bigvee_{i=1}^{k} P_i)$.

Lemma 5. If $M_1, M_2, \ldots, M_m, P_1, P_2, \ldots, P_k$ are distinct Kn-forms, then $\sim (\bigvee_{i=1}^{m} M_i \supset \bigvee_{j=1}^{k} P_j)$.

Proof. By lemma *4* we have $\bigwedge_{i=1}^{m} (\sim (M_i \supset \bigvee_{j=1}^{k} P_j))$. But $\bigwedge_{i=1}^{m} (\sim (M_i \supset \bigvee_{j=1}^{k} P_j)) . \equiv . \sim (\bigwedge_{i=1}^{m} M_i \supset \bigvee_{j=1}^{k} P_k) . \supset . \sim (\bigvee_{i=1}^{m} M_i \supset \bigvee_{j=1}^{k} P_k)$.

Corollary. Lemma *5* says that distinct AKn-forms are not equivalent and therefore define different relations.

By this corollary and lemma *2* we have:

Theorem 1. The number of non-empty moduli equals the number of distinct AKn-forms.

If Kn represents the number of Kn-forms, then by $D6$ and theorem 1, we know that $2^{Kn} - 1$ is the number of non-empty moduli, and consequently that 2^{Kn} is the number of moduli. It therefore suffices to determine Kn.

Since (i, j) varies through all of $\mathbf{N}$ for a Kn-form, it is clear that each Kn-form determines that each pair of the n individuals are either identical or different, i.e. each Kn-form partitions $x_1, x_2, \ldots, x_n$ into classes. Therefore finding Kn is equivalent to the following combinatorial problem: "in how many ways can n distinct objects be placed in n like cells, with no restriction on the emptiness of the cells." This result is well known [2], it being $\sum_{k=1}^{n} \mathbf{S}(n, k)$, where $\mathbf{S}(n, k)$ is the Stirling number of the second kind whose explicit formula is $\mathbf{S}(n, k) = \frac{1}{k!}\left[\sum_{j=0}^{k} \binom{k}{j} (-1)^j (k-j)^n\right]$. Consequently.

Theorem 2. The number of moduli for a given natural number n is 2^a, where $a = \sum_{k=1}^{n} \frac{1}{k!}\left[\sum_{j=0}^{k} \binom{k}{j} (-1)^j (k-j)^n\right]$

For $n = 1, 2, 3, 4$, this number is 2 (the Boolean moduli 0 and 1), 4 (the Schröder moduli $0, 1, 0', 1'$), 32, 32768. The 32 moduli for $n = 3$ are listed below. The list is so arranged that if $\mathbf{M} \supset P$ then $\mathbf{M}$ is listed before P, and if $\mathbf{M}$ is listed in the mth place then $\sim \mathbf{M}$ is listed in the $(33\text{-}m)$th place. The five Kn-moduli are indicated by (*) and the analogs of the four Schröder moduli by (°).

°1. $a \neq a \,.\vee.\, b \neq b \,.\vee.\, c \neq c$
°*2. $a = b \,.\wedge.\, b = c$
*3. $a = b \,.\wedge.\, a \neq c$
*4. $b \neq c \,.\wedge.\, a = c$
*5. $a \neq b \,.\wedge.\, b = c$
°*6. $a \neq b \,.\wedge.\, a \neq c \,.\wedge.\, b \neq c$
7. $a = b \,.\wedge.\, c = c$
8. $a = c \,.\wedge.\, b = b$
9. $b = c \,.\wedge.\, a = a$
10. $a = b \,.\wedge.\, b = c :\vee: a \neq b \,.\wedge.\, b \neq c \,.\wedge.\, a \neq c$
11. $b \neq c \,.\wedge: a = c \,.\vee.\, a = b$
12. $a \neq c \,.\wedge: a = b \,.\vee.\, b = c$
13. $a \neq b \,.\wedge: a = c \,.\vee.\, b = c$
14. $a \neq c \,.\wedge.\, b \neq c$
15. $a \neq b \,.\wedge.\, b \neq c$
16. $a \neq b \,.\wedge.\, a \neq c$
17. $a = b \,.\vee.\, a = c$
18. $a = b \,.\vee.\, b = c$
19. $a = c \,.\vee.\, b = c$
20. $a = b \,.\vee: a \neq c \,.\wedge.\, b \neq c$
21. $a = c \,.\vee: a \neq b \,.\wedge.\, b \neq c$
22. $b = c \,.\vee: a \neq b \,.\wedge.\, a \neq c$
23. $a = b \,.\wedge.\, a \neq c \,.\vee: a = c \,.\wedge.\, b \neq c :\vee: b = c \,.\wedge.\, a \neq b$
24. $b \neq c \,.\wedge.\, a = a$
25. $a \neq c \,.\wedge.\, b = b$
26. $a \neq b \,.\wedge.\, c = c$
27. $a = b \,.\vee.\, a = c \,.\vee.\, b = c$
28. $a = c \,.\vee.\, b \neq c$
29. $a = b \,.\vee.\, b \neq c$
30. $b = c \,.\vee.\, a \neq b$
31. $a \neq b \,.\vee.\, a \neq c$
°32. $a = a \,.\wedge.\, b = b \,.\wedge.\, c = c$

BIBLIOGRAPHY

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[2] Riordan, J. *An Introduction to Combinatorial Analysis*, Wiley and Sons, Inc., 1958.

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