# A SIMPLE FORMULA EQUIVALENT TO THE AXIOM OF CHOICE

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It is well known that a theorem:

I. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $\mathfrak{m} < \mathfrak{n}$, *then there exists a cardinal number* $\mathfrak{p}$ $(>0)$ *such that* $\mathfrak{n} = \mathfrak{m} + \mathfrak{p}$.

is provable in the set theory (and also in logic) without the aid of the axiom of choice[1]. It can be shown easily that a modification of this theorem, viz.

I°. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\mathfrak{m} < \mathfrak{n}$, *then* $\mathfrak{n} = \mathfrak{m} + \mathfrak{n}$.

is inferentially equivalent to an assumption:

$\mathfrak{A}$. *For any cardinal number* $\mathfrak{m}$ *which is not finite:* $2\mathfrak{m} = \mathfrak{m}$.

This equivalence can be proved e.g. by an elementary application of a known theorem of F. Bernstein, viz.

$\mathfrak{B}$. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $2\mathfrak{m} = 2\mathfrak{n}$, *then* $\mathfrak{m} = \mathfrak{n}$.

which is provable without the aid of the axiom of choice[2].

As far as I know it has not been observed yet that a formula analogous to I but formulated for the multiplication of cardinals:

II. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\mathfrak{m} < \mathfrak{n}$, *then there exists a cardinal number* $\mathfrak{p}$ $(>0)$ *such that* $\mathfrak{n} = \mathfrak{m}\mathfrak{p}$.

is inferentially equivalent to the axiom of choice. From a proof which is presented below of this equivalence it follows obviously that a formula analogous to I°, viz.

II°. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\mathfrak{m} < \mathfrak{n}$, then $\mathfrak{n} = \mathfrak{m}\mathfrak{n}$.

possesses also the same property.

*Proof:* In order to show the discussed equivalence it is sufficient to prove that the axiom of choice is a consequence of the formula II, as it is evident that II follows from that axiom. Having the formula II we can establish the following:

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*Lemma: For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\mathfrak{m} < \mathfrak{n}$ *and* $\mathfrak{m}^2 = \mathfrak{m}$, *then* $\mathfrak{n} = \mathfrak{m}\mathfrak{n}$.

Since $\mathfrak{m} < \mathfrak{n}$, then according to II there exists such cardinal $\mathfrak{p}$ that $\mathfrak{n} = \mathfrak{m}\mathfrak{p}$. But $\mathfrak{m}^2 = \mathfrak{m}$, hence $\mathfrak{n} = \mathfrak{m}\mathfrak{p} = \mathfrak{m}^2\mathfrak{p} = \mathfrak{m}(\mathfrak{m}\mathfrak{p}) = \mathfrak{m}\mathfrak{n}$.

Now having this lemma proved let us assume that

a) $\mathfrak{r}$ is an arbitrary cardinal which is not finite. And

b) Put $\mathfrak{s} = \mathfrak{r}^{\aleph_0}$. Hence $\mathfrak{s} = \mathfrak{r}^{\aleph_0} = \mathfrak{r}^{2\aleph_0} = (\mathfrak{r}^{\aleph_0})^2 = \mathfrak{s}^2$. Therefore: $\mathfrak{s}$ is a cardinal number which is not finite and such that $\mathfrak{s} = \mathfrak{s}^2$.

c) Since $\mathfrak{s}$ is a cardinal then without the aid of the axiom of choice it can be constructed, so called, Hartogs' aleph $\aleph(\mathfrak{s})$, i.e. an aleph which is not smaller or equal to $\mathfrak{s}$ [3].

d) A formula $\mathfrak{s} \leq \mathfrak{s} + \aleph(\mathfrak{s})$ obviously holds in the set theory without the aid of the choice axiom. But since a case $\mathfrak{s} = \mathfrak{s} + \aleph(\mathfrak{s})$ gives $\mathfrak{s} \geq \aleph(\mathfrak{s})$ and this contradicts a property of Hartogs' aleph, we have $\mathfrak{s} < \mathfrak{s} + \aleph(\mathfrak{s})$.

e) Applying our lemma to the formulas $\mathfrak{s} < \mathfrak{s} + \aleph(\mathfrak{s})$ and $\mathfrak{s} = \mathfrak{s}^2$ which we have from the points b) and d) we get at once:

$$\mathfrak{s} + \aleph(\mathfrak{s}) = \mathfrak{s}(\mathfrak{s} + \aleph(\mathfrak{s})) = \mathfrak{s}^2 + \mathfrak{s}\,\aleph(\mathfrak{s}) = \mathfrak{s} + \mathfrak{s}\,\aleph(\mathfrak{s}) = \mathfrak{s}(1 + \aleph(\mathfrak{s})) = \mathfrak{s}\,\aleph(\mathfrak{s})$$

f) From point c), the proved formula:

$$\mathfrak{s} + \aleph(\mathfrak{s}) = \mathfrak{s}\,\aleph(\mathfrak{s})$$

and a theorem (provable without aid of the axiom of choice) of Tarski[4], viz.

$\mathfrak{C}$. *For any cardinal number* $\mathfrak{m}$ *which is not finite and an arbitrary aleph, if* $\mathfrak{m} + \aleph = \mathfrak{m}\aleph$, *then either* $\mathfrak{m} \geq \aleph$ *or* $\mathfrak{m} \leq \aleph$.

it follows immediately that

$$\aleph(\mathfrak{s}) > \mathfrak{s}$$

Hence $\aleph(\mathfrak{s}) > \mathfrak{s} = \mathfrak{r}^{\aleph_0} \geq \mathfrak{r}$, which shows that our arbitrary cardinal $\mathfrak{r}$ is an aleph. Therefore the discussed inferential equivalence is proved.

As a consequence of this proof we can make the following remark. A definition of the difference of two cardinal numbers says that $\mathfrak{p} = \mathfrak{n} - \mathfrak{m}$ if and only if for the cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$ there exists one and only one cardinal number $\mathfrak{p}$ such that $\mathfrak{n} = \mathfrak{m} + \mathfrak{p}$. Analogously the quotient is defined: $\mathfrak{p} = \mathfrak{n} : \mathfrak{m}$ if and only if for the cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$ there exists one and only one cardinal number $\mathfrak{p}$ such that $\mathfrak{n} = \mathfrak{m}\mathfrak{p}$. It is known that each of the following formulas:

III. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $\mathfrak{m} < \mathfrak{n}$, *then the difference* $\mathfrak{n} - \mathfrak{m}$ *exists.*

III$^{\circ}$. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $0 < \mathfrak{m}$, $\mathfrak{n}$ *is not finite and* $\mathfrak{m} < \mathfrak{n}$, *then the quotient* $\mathfrak{n} : \mathfrak{m}$ *exists.*

is inferentially equivalent to the axiom of choice.[5] The given above proof shows that in III$^{\circ}$ a requirement concerning uniqueness of the quotient is superfluous. Contrary, a similar condition in III cannot be dismissed. A fact that the theorem I is obtainable without the aid of the axiom of choice and that the analogous theorem II is inferentially equivalent to it shows that the structure of the multiplication of cardinals differs sharply from the structure of their addition.

## NOTES

[1] Cf. Wacław Sierpiński. Cardinal and Ordinal Numbers. *Monografie Matematyczne*, vol. 34. Warszawa, 1958. Pp. 144-145.

[2] Cf. F. Bernstein. Untersuchungen aus der Mengenlehre. *Math. Annalen* 61 (1905), pp. 117-155, and W. Sierpiński. Sur L'égalité $2\mathfrak{m} = 2\mathfrak{n}$ pour les nombres cadinaux. *Fundamenta Mathematicae,* vol 3 (1922), pp. 1-16.

[3] Cf. F. Hartogs: Über das Problem der Wohlordnung. *Math. Annalen* 76 (1914), pp. 438-443. Cf. also W. Sierpiński. Les Exemples effectives et l'axiome du choix. *Fundamenta Mathematicae*, vol 2 (1921), pp. 112-118. Also W. Sierpiński. Cardinal and Ordinal Numbers, pp. 407-409.

[4] Cf. A. Tarski. Sur quelques théorèmes qui équivalent à l'axiome du choix. *Fundamenta Mathematicae*, vol. 5 (1924), pp. 147-154, lemme 1. Cf. also W. Sierpiński, op. cit., pp. 419-421 and p. 423, Exercises 1.

[5] Cf. a remark of Sierpiński in his *op. cit.*, p. 170 and, *ibidem*, p. 416. Theorem 3.

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