FUNCTIONAL COMPLETENESS OF HENKIN'S PROPOSITIONAL FRAGMENTS

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It was shown in [1] that if $\phi(x_1, \ldots, x_2)$ has the defined property of being Tarskian, the addition of schemata $(\phi)*$ as in [2] to the positive logic of implication, *A1-2*, yields the complete system of classical implication. Knowledge of [1] is pre-supposed. We define:

Def. $\mathfrak{T}_1$ For all ϕ, ϕ is Tarskian$_1$ iff ϕ is Tarskian and is valued F when all its arguments are valued T.

Def. $\mathfrak{T}_2$ For all ϕ, ϕ is Tarskian$_2$ iff ϕ is Tarskian and is valued T when all its arguments are valued T.

THEOREM 1. If ϕ is Tarskian$_1$, $\{A1\text{-}2, (\phi)*\}$ is functionally complete.

Proof. If ϕ is Tarskian$_1$, the proof of Lemma 1, case (i)a and the corresponding sub-case of case (ii), in [1], shows that $\{A1\text{-}2, (\phi)*\}$ contains *S1-2* with each i-th argument of ϕ either A or $A \supset A$. Defining the negation of A for ϕ with these arguments, we get from *S1-2*:

$$(1)\quad A \supset . \sim A \supset C$$

$$(2)\quad A \supset C \supset . \sim A \supset C \supset C .$$

Taking C in (2) as A, and detaching $A \supset A$, we get

$$(3)\quad \sim A \supset A \supset A .$$

Since hypothetical syllogism is given by *A1-2*, and this with (1) and (3) constitutes the well known Łukasiewicz base for a full and functionally complete system in implication and negation, the theorem follows.

THEOREM 2. If ϕ is valued T when all its arguments are valued T, negation is not definable in the system $\{A1\text{-}3, (\phi)*\}$.

Proof. The system $\{A1\text{-}3, (\phi)*\}$ is, by [2], complete for tautologies in implication and ϕ. So every expression $A \supset B$ with A and B tautologous is provable, and by the hypothesis on ϕ, $\phi(A \supset A, \ldots, A \supset A)$ is provable. Hence every expression f (imp, ϕ, $A \supset A$) with implication and ϕ as the only functors, and all elementary argument places filled by $A \supset A$, is provable. We suppose now that negation is definable. We should have as provable

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$$(4)\quad \sim A \longleftrightarrow f\,(\text{imp}, \phi, A) \qquad \text{for some } f,$$

$$(5)\quad \sim (A \supset A) \supset A\,.$$

Taking A in (4) as $A \supset A$, we should get from (4) and (5), A, and the system would be inconsistent. As the system is known to be consistent, we conclude to the theorem.

From Theorem 2 and Def. $\mathfrak{T}_2$ there follows:

THEOREM 3. If ϕ is Tarskian$_2$, $\{A1\text{-}3, (\phi)*\}$ is functionally incomplete.

THEOREM 4. If ϕ is not Tarskian and is F for all values of its arguments, (i) $\{A1\text{-}2, (\phi)*\}$ is functionally incomplete; (ii) $\{A1\text{-}3, (\phi)*\}$ is functionally complete.

Proof. (i) follows from Lemma 7 of [1] which states that if ϕ is not Tarskian, $A3$ is independent in $\{A1\text{-}3, (\phi)*\}$. In the system of (ii) we can define 0 for the constant $\phi\ (A \supset A\,, \ldots, A \supset A)$ and prove $0 \supset A$, which with the complete implicational system given by $A1\text{-}3$ yields a functionally complete system, as is well known.

THEOREM 5. If ϕ is not Tarskian and is not F for all values of its arguments, the system $\{A1\text{-}3, (\phi)*\}$ is functionally incomplete.

Proof. If ϕ is as in the hypothesis, it is valued T when all its arguments are T. The conclusion follows by Theorem 2.

We conclude by tabling some results of this paper and [1]. The axiom schemata indicated are in every case independent. ϕ as in Theorem 4 is denoted by $Z\phi$.

	ϕ Tarskian		ϕ not Tarskian	
	Tarskian$_1$	Tarskian$_2$	$Z\phi$	not $Z\phi$
Base complete for	$\{A1\text{-}2, (\phi)*\}$		$\{A1\text{-}3, (\phi)*\}$	
implication	yes	yes	yes	yes
all functions	yes	no	yes	no

It may be worth remarking that if ϕ is Tarskian$_1$ (Tarskian$_2$) and schemata $(\phi)*$ are weakened to just the two required for derivation of $S1\text{-}2$ ($S3\text{-}4$), we shall still have systems complete for all two-valued functions (implication), but they will no longer be categorical, since the value of ϕ itself will be undetermined for certain values of its arguments.

The questions of independence discussed in [2], and occasioning [1] and [3], could of course be circumvented by choosing a sole axiom for the implicational base of Henkin's fragments. And if ϕ be Tarskian, [1] shows that a positive sole axiom is sufficient. But if the version of $(\phi)*$ used in [1] and the present paper - like Henkin's except for having antecedents x in place of $x \supset y \supset y$ - be used, a greater economy still can be effected for all but the two cases of medadic ϕ. Let us denote the briefer schemata by '$(\Phi)*$'. We prove:

THEOREM 6. The implicational schemata

T1. $A \supset B \supset A \supset A$

T2. $A \supset B \supset . B \supset C \supset . A \supset C$

form a sufficient implicational base for Henkin's fragments iff both (1) ϕ is at least unary, and (2) $(\Phi)*$ is used instead of $(\phi)*$.

Proof. If (1), then $(\Phi)*$ contains a schema of the form $A \supset . \alpha \supset \beta$, which, by the result of Łukasiewicz's [4], is sufficient with *T1-2* for full implication; but full implication and $(\Phi)*$ is equivalent to full implication and $(\phi)*$.

Conversely, if either not (1) - in which case ϕ is T or F, and $(\Phi)* = (\phi)*$ - or not (2), then $A \supset . B \supset A$ is unprovable in the system $\{T1\text{-}2, (\phi)*\}$. For L'Abbé's [3], Theorem 1, shows that every ϕ can be valued so as to verify $(\phi)*$ in connexion with the hereditary matrix

$\supset$	0	1	2
*0	0	1	0
1	0	0	0
*2	1	1	0

which verifies *T1*, *T2*, but falsifies $A \supset . B \supset A$ for $A = 0$, $B = 2$.

Lastly, we prove

THEOREM 7. *T1*, *T2*, $(\Phi)*$ are always independent.

Proof. *T1* is falsified by the following (non-regular but hereditary) matrix:

$\supset$	0	1	2
*0	0	1	1
1	0	0	0
2	0	1	1

if we put 0 = T, 1 = F, and evaluate *T1* for $A = B = 2$. But *T2* is verified, and ϕ can be defined so that $(\Phi)*$ is verified. The matrix shows that $(\Phi)*$ is verified if the valuation is confined to the values 0 and 1. Since $A \supset 2 = A \supset 1$, verification is preserved if the auxiliary variable y be allowed to take the value 2. If some argument x_i of ϕ takes the value 2, x_i occurs elsewhere only in the immediate context $x_i \supset y$; since $2 \supset y = 0 \supset y$, verification will be effected generally if $\phi(\ldots 2 \ldots) = \phi(\ldots 0 \ldots)$.

T2 is falsified by the hereditary matrix:

$\supset$	0	1	2
*0	0	1	0
1	0	0	0
2	0	0	0

if we put 0 = T, 1 = F, and evaluate $T2$ for $A = 0$, $B = 2$, $C = 1$. But $T1$ is verified, and so is $(\Phi)*$ if we put $\phi\,(\ldots 2 \ldots) = \phi\,(\ldots 1 \ldots)$, as can be readily shown by argument parallel to that in the previous case.

Finally, schemata $(\Phi)*$ are independent, for which Henkin's proof of the independence of $(\phi)*$ may be taken over unchanged.

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