STUDIES IN THE AXIOMATIC FOUNDATIONS OF BOOLEAN ALGEBRA

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Introduction

One way of characterizing Boolean Algebra would be to say that it consists of all those theses which can be deduced from the following axiom-system due to Schröder:

S1. $[a] . a \subset a$
S2. $[abc] : a \subset b . b \subset c . \supset . a \subset c$
S3. $[a] . \wedge \subset a$
S4. $[a] . a \subset \vee$
S5. $[abc] : c \subset a \cap b . \equiv . c \subset a . c \subset b$
S6. $[abc] : a \cup b \subset c . \equiv . a \subset c . b \subset c$
S7. $[abc] . a \cap (b \cup c) \subset (a \cap b) \cup (a \cap c)$
S8. $[a] . a \cap \sim(a) \subset \wedge$
S9. $[a] . \vee \subset a \cup \sim(a)$1)

The totality of theses as deduced from *S1* – *S9* will be referred to as System $\mathfrak{S}$.

As is well known, Boolean Algebra lends itself to various interpretations. For the purpose of the present enquiry I propose to adhere to what may be styled as the *ontological* interpretation. It is important to realize as clearly as possible what this interpretation presupposes and what it implies.

Let us begin with a few introductory remarks on names and name-like expressions in general. Names and name-like expressions of ordinary language can be divided into two classes: the class of referential names, which subdivide into unshared names and shared names, and the class of non-referential or fictitious names.2) A referential name names or designates at least one object. A fictitious name behaves, as regards its syntax, like a referential one but it fails to name or designate anything at all. In accordance with this classification names or name-like expressions such as 'Socrates', 'the husband of Xanthippe', 'philosopher', 'inhabitant of London', are all referential names, whereas 'Pegasus', 'mermaid', 'object which does not exist,' *etc.*, are examples of non-referential names.

Now, the ontological interpretation of Boolean Algebra demands that the variables, '*a*', '*b*', '*c*', *etc.*, should be regarded as *naminal* variables, *i.e.*, as variables for which names, referential or non-referential, could be substituted.

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Further stipulations implied by the ontological interpretation concern the meaning of the constant terms of the Algebra.

The symbol '$\subset$' is to be construed as a proposition forming functor for two nominal arguments. We call it the functor of weak inclusion. With constant names as arguments an expression of the type '$a \subset b$' (to be read: *all a is b*) is a true proposition if and only if for all c, – if c is named by the name for which the 'a' stands than it is also named by the name for which the 'b' stands.

The symbol '$\wedge$' is to be regarded as a constant name. It is a non-referential name as it is meant to be a name that does not designate anything. It may be read: *object which does not exist.*

The symbol '$\vee$' is also a constant name but it names every object. It thus means the same as '*object*', or '*thing*', or '*entity*'.

The symbol '$\sim$' is a name-forming functor for one nominal argument. With a constant name as argument an expression of the type '$\sim(a)$' (to be read: *non-a*) names every object, if there is any, which is not named by the name for which the 'a' stands in the expression.

The symbol '$\frown$' is a name-forming functor for two nominal arguments. With constant names as arguments an expression of the type '$a \frown b$' (to be read: *a and b*) names every object if there is any, which is named by both the names for which the 'a' and the 'b' stand in the expression.[3]

The symbol '$\smile$' is again a name-forming functor for two nominal arguments. With constant names as arguments an expression of the type '$a \smile b$' (to be read: *a or b*) names every object, if there is any, which is named by at least one of the names for which the 'a' and the 'b' stand in the expression.[3]

The remaining symbolism in the axioms *S1* – *S9* is that of the logic of propositions and the theory of quantification.

It is not difficult to see that the ontological interpretation approximates what is sometimes described as class-interpretation.[4] Under the ontological interpretation Boolean Algebra becomes a general theory of objects.

The present essay will consist of four sections. In section I an outline of the proof will be given that the axiom-system comprising *S1* – *S9* is inferentially equivalent to the one consisting of the following six theses:

A1. $[a\, b] \therefore a \subset b \,.\equiv:\, [c\, d\, e]: \sim(c \subset d) \,.\, c \subset e \,.\, c \subset a \,.\supset.\, [\exists f\, g] \,.\sim (f \subset g) \,.\, f \subset e \,.\, f \subset b$

A2. $[a] \therefore a \subset \wedge \,.\equiv:\, [b\, c]: \sim(b \subset c) \,.\, b \subset a \,.\supset.\, [\exists d\, e] \,.\sim(d \subset e) \,.\, d \subset b \,.\sim(d \subset d)$

A3. $[a] \therefore a \subset \vee \,.\equiv:\, [b\, c]: \sim(b \subset c) \,.\, b \subset a \,.\supset.\, [\exists d\, e] \,.\sim(d \subset e) \,.\, d \subset b \,.\, d \subset d$

A4. $[a\, b] \therefore a \subset \sim(b) \,.\equiv:\, [c\, d]: \sim(c \subset d) \,.\, c \subset a \,.\supset.\, [\exists e\, f] \,.\sim(e \subset f) \,.\, e \subset c \,.\sim(e \subset b)$

A5. $[a\, b\, c] \therefore a \subset b \frown c \,.\equiv:\, [d\, e]: \sim(d \subset e) \,.\, d \subset a \,.\supset.\, [\exists f\, g] \,.\sim(f \subset g) \,.\, f \subset d \,.\, f \subset b \,.\, f \subset c$

A6. $[a\,b\,c] :: a \subset b \cup c \,.\equiv .\therefore [d\,e] \therefore \sim(d \subset e) \,.\, d \subset a \,.\supset : [\exists f\,g] : \sim (f \subset g) \,.\, f \subset d : f \subset b \,.\vee .\, f \subset c$

In section II the problem of definitions will be discussed at some length, and in addition to the usual rule for introducing *propositional* definitions a rule for framing *nominal* definitions will be suggested. It will then become evident that a system of Boolean Algebra equipped with an appropriate rule for nominal definitions can be based on *A1* as a sole axiom. In Section III systems of Boolean Algebra based on functors other than that of weak inclusion will be presented in outline. Section IV will be devoted to the discussion of the relationship between Boolean Algebra and Leśniewski's Ontology. It is intended that this discussion should provide a convincing justification of the rule for introducing nominal definitions as proposed in Section II.

SECTION I

The following theses can be derived from the axioms of System $\mathfrak{S}$:

T1. $[a] \,.\, a \subset a \cap \mathrm{V}$ [follows from *S5*, *S1*, and *S4*]

T2. $[a\,b] \,.\, a \cap b \subset a$ [from *S5* and *S1*]

T3. $[a\,b] \,.\, a \cap b \subset b$ [*S5*, *S1*]

T4. $[a\,b] \,.\, a \subset a \cup b$ [*S6*, *S1*]

T5. $[a\,b] \,.\, a \subset b \cup a$ [*S6*, *S1*]

T6. $[a\,b] \,.\, a \cap b \subset b \cap a$ [*S5*, *T3*, *T2*]

T7. $[a\,b] \,.\, a \cup b \subset b \cup a$ [*S6*, *T5*, *T4*]

T8. $[a\,b\,c] : a \subset b \,.\supset .\, c \cap a \subset c \cap b$

Proof:

$[a\,b\,c]:$

(1) $a \subset b \,.\supset .$

(2) $c \cap a \subset b \,.$ [*S2*, *T3*, 1]

$c \cap a \subset c \cap b$ [*S5*, *T2*, 2]

T9. $[a\,b\,c] : a \subset b \,.\supset .\, c \cup a \subset c \cup b$

Proof:

$[a\,b\,c]:$

(1) $a \subset b \,.\supset .$

(2) $a \subset c \cup b \,.$ [*S2*, 1, *T5*]

$c \cup a \subset c \cup b$ [*S6*, *T4*, 2]

T10. $[a\,b] \,.\, a \cap \mathrm{V} \subset a \cap (b \cup \sim(b)\,)$ [*T8*, *S9*]

T11. $[a\,b] \,.\, a \subset a \cap (b \cup \sim(b)\,)$ [*S2*, *T1*, *T10*]

T12. $[a\,b] \,.\, a \subset (a \cap b) \cup (a \cap \sim(b)\,)$ [*S2*, *T11*, *S7*]

T13. $[a\,b] : a \cap \sim(b) \subset \Lambda \,.\supset .\, a \subset b$

Proof:
$[a b]:$
(1) $a \cap \boldsymbol{\sim}(b) \subset \wedge . \supset .$
(2) $a \cap \boldsymbol{\sim}(b) \subset a \cap b$ $[S2, 1, S3]$
(3) $(a \cap b) \cup (a \cap \boldsymbol{\sim}(b)) \subset a \cap b$ $[S6, S1, 2]$
(4) $a \subset a \cap b$ $[S2, T12, 3]$
$a \subset b$ $[S5, 4]$

T14. $[a b c d]: a \subset b . a \subset c \cap \boldsymbol{\sim}(b) . \supset . a \subset d$
Proof:
$[a b c d]:$
(1) $a \subset b .$
(2) $a \subset c \cap \boldsymbol{\sim}(b) . \supset .$
(3) $a \subset \boldsymbol{\sim}(b)$ $[S2, 2, T3]$
(4) $a \subset b \cap \boldsymbol{\sim}(b) .$ $[S5, 1, 3]$
(5) $a \subset \wedge .$ $[S2, 4, S8]$
$a \subset d$ $[S2, 5, S3]$

T15. $[a b c d e] : a \subset b . \sim(c \subset d) . c \subset e . c \subset a . \supset . [\exists f g] . \sim(f \subset g)$
$. f \subset e . f \subset b$
Proof:
$[a b c d e] \therefore$
(1) $a \subset b .$
(2) $\sim(c \subset d) .$
(3) $c \subset e .$
(4) $c \subset a . \supset :$
(5) $c \subset b :$ $[S2, 4, 1]$
$[\exists f g] . \sim(f \subset g) . f \subset e . f \subset b$ $[2, 3, 5]$

T16. $[a b] :: \sim(a \subset b) . \supset \therefore [\exists c d e] \therefore \sim(c \subset d) . c \subset e . c \subset a :$
$[f g] : \sim(f \subset g) . f \subset e . \supset . \sim(f \subset b)$
Proof:
$[a b] :\cdot:$
(1) $\sim(a \subset b) . \supset ::$
(2) $\sim(a \cap \boldsymbol{\sim}(b) \subset \wedge) .$ $[T13, 1]$
(3) $a \cap \boldsymbol{\sim}(b) \subset a \cap \boldsymbol{\sim}(b) .$ $[S1]$
(4) $a \cap \boldsymbol{\sim}(b) \subset a \therefore$ $[T2]$
(5) $[f g] : \sim(f \subset g) . f \subset a \cap \boldsymbol{\sim}(b) . \supset . \sim(f \subset b) ::$ $[T14]$
$[\exists c d e] \therefore \sim(c \subset d) . c \subset e . c \subset a : [f g] : \sim(f \subset g) . f \subset e .$
$\supset . \sim(f \subset b)$ (2, 3, 4, 5)

T17. $[a b c]: a \subset \wedge . \sim(b \subset c) . b \subset a . \supset . [\exists d e] . \sim(d \subset e) . d \subset b . \sim(d \subset d)$

Proof:

$[a b c] \therefore$

(1) $a \subset \wedge .$

(2) $\sim(b \subset c) .$

(3) $b \subset a . \supset .$

(4) $b \subset \wedge .$ [S2, 3, 1]

(5) $b \subset c :$ [S2, 4, S3]

$[\exists d e] . \sim(d \subset e) . d \subset b . \sim(d \subset d)$ [5, 2]

T18. $[a] : : [b c] : (b \subset c) . b \subset a . \supset . [\exists d e] . \sim(d \subset e) . d \subset b . \sim(d \subset d) \therefore \sim(a \subset \wedge) \therefore \supset . a \subset \wedge$

Proof:

$[a] : :$

(1) $[b c] : \sim(b \subset c) . b \subset a . \supset . [\exists d e] . \sim(d \subset e) . d \subset b . \sim(d \subset d) \therefore$

(2) $\sim(a \subset \wedge) \therefore \supset :$

(3) $[\exists d] . \sim(d \subset d) :$ [1, 2, S 1]

$a \subset \wedge$ [S1, 3]

T19. $[a] \therefore [b c] : \sim(b \subset c) . b \subset a . \supset . [\exists d e] . \sim(d \subset e) . d \subset b . \sim(d \subset d) : \supset . a \subset \wedge$ [*T18*]

T20. $[b c] : \sim(b \subset c) . \supset . [\exists d e] . \sim(d \subset e) . d \subset b . d \subset d$ [*S1*]

T21 $[a] \therefore [b c] : \sim(b \subset c) . b \subset a . \subset . [\exists d e] . \sim(d \subset e) . d \subset b . d \subset d : \supset . a \subset \vee$ [*S4*]

T22. $[a b] \therefore [c] : c \subset a . \supset . c \subset b : \supset . a \subset b$ [*S1*]

T23. $[a b c d] : a \subset \sim(b) . \sim(c \subset d) . c \subset a . c \subset b . \supset . \sim(c \subset b)$

Proof:

$[a b c d] :$

(1) $a \subset \sim(b).$

(2) $\sim(c \subset d) .$

(3) $c \subset a .$

(4) $c \subset b . \supset .$

(5) $c \subset \sim(b) .$ [S2, 3, 1]

(6) $c \subset b \frown \sim(b) .$ [S5, 4, 5]

(7) $c \subset \wedge ..$ [S2, 6, S8]

(8) $c \subset d .$ [S2, 7, S3]

$\sim(c \subset b)$ [8, 2]

T24. $\lfloor a\ b\ c\ d \rceil : a \subset \boldsymbol{\sim}(b) \,.\, \sim(c \subset d) \,.\, c \subset a \,.\, \supset . \lfloor \exists\ e\ f \rceil . \sim(e \subset f) \,.\, e \subset c \,.\, \sim(e \subset b)$

Proof:

$\lfloor a\ b\ c\ d \rceil \therefore$

(1) $a \subset \boldsymbol{\sim}(b).$

(2) $\sim(c \subset d) \,.$

(3) $c \subset a. \supset :$

(4) $\sim(c \subset b) :$ $\lfloor T23,\ 1,\ 2,\ 3 \rceil$

(5) $\lfloor \exists\ d \rceil . d \subset c \,.\, \sim(d \subset b) :$ $\lfloor T22, 4 \rceil$

$\lfloor \exists\ e\ f \rceil . \sim(e \subset f) \,.\, e \subset c \,.\, \sim(e \subset b)$ $\lfloor 5 \rceil$

T25. $\lfloor a\ b \rceil \,.\, a \subset b \cup \boldsymbol{\sim}(b)$ $\lfloor S2,\ S4,\ S\,9 \rceil$

T26. $\lfloor a\ b \rceil \,.\, a \cup (\boldsymbol{\sim}(a) \cap b) \subset a \cup b$ $\lfloor T9,\ T3 \rceil$

T27. $\lfloor a\ b \rceil \,.\, a \cup (\boldsymbol{\sim}(a) \cap b) \subset a \cup \boldsymbol{\sim}(a)$ $\lfloor T25 \rceil$

T28. $\lfloor a\ b \rceil \,.\, a \cup (\boldsymbol{\sim}(a) \cap b) \subset (a \cup \boldsymbol{\sim}(a)) \cap (a \cup b)$ $\lfloor S5,\ T27, T26 \rceil$

T29. $\lfloor a\ b \rceil \,.\, a \cap \boldsymbol{\sim}(a) \subset b$ $\lfloor S2, S8,\ S3 \rceil$

T30. $\lfloor a\ b \rceil . \boldsymbol{\sim}(a) \cap a \subset b$ $\lfloor S2, T6, T29 \rceil$

T31 $\lfloor a\ b \rceil : \boldsymbol{\sim}(a) \subset b \,.\, \supset . \boldsymbol{\sim}(b) \subset a$

Proof:

$\lfloor a\ b \rceil :$

(1) $\boldsymbol{\sim}(a) \subset b \,.\, \supset .$

(2) $\boldsymbol{\sim}(a) \subset \boldsymbol{\sim}(a) \cap b.$ $\lfloor S5,\ S1,\ 1 \rceil$

(3) $\boldsymbol{\sim}(a) \subset a \cup (\boldsymbol{\sim}(a) \cap b) \,.$ $\lfloor S2, 2,\ T5 \rceil$

(4) $a \cup \boldsymbol{\sim}(a) \subset a \cup (\boldsymbol{\sim}(a) \cap b).$ $\lfloor S6,\ T4,\ 3 \rceil$

(5) $\boldsymbol{\sim}(b) \subset a \cup (\boldsymbol{\sim}(a) \cap b) .$ $\lfloor S2,\ T25,\ 4 \rceil$

(6) $\boldsymbol{\sim}(b) \subset (a \cup \boldsymbol{\sim}(a)) \cap (a \cup b) .$ $\lfloor S2,\ 5,\ T28 \rceil$

(7) $\boldsymbol{\sim}(b) \subset a \cup b \,.$ $\lfloor S2,\ 6,\ T3 \rceil$

(8) $\boldsymbol{\sim}(b) \subset b \cup a \,.$ $\lfloor S2,\ 7,\ T7 \rceil$

(9) $\boldsymbol{\sim}(b) \subset \boldsymbol{\sim}(b) \cap (b \cup a) .$ $\lfloor S5,\ S1,\ 8 \rceil$

(10) $\boldsymbol{\sim}(b) \subset (\boldsymbol{\sim}(b) \cap b) \cup (\boldsymbol{\sim}(b) \cap a) .$ $\lfloor S2,\ 9,\ S7 \rceil$

(11) $(\boldsymbol{\sim}(b) \cap b) \cup (\boldsymbol{\sim}(b) \cap a)) \subset \boldsymbol{\sim}(b) \cap a \,.$ $\lfloor S6, T30, S1 \rceil$

(12) $\boldsymbol{\sim}(b) \subset \boldsymbol{\sim}(b) \cap a$ $\lfloor S2,\ 10,\ 11 \rceil$

$\boldsymbol{\sim}(b) \subset a$ $\lfloor S2,\ 12,\ T3 \rceil$

T32. $\lfloor a \rceil . \boldsymbol{\sim}(\boldsymbol{\sim}(a)) \subset a$ $\lfloor T31, S1 \rceil$

T33. $\lfloor a\ b \rceil :: \sim(a \subset \sim(b)) \,.\, \supset \therefore \lfloor \exists\ c\ d \rceil \therefore \sim(c \subset d) \,.\, c \subset a : \lfloor e \rceil : e \subset c \,.\, \supset . e \subset b$

Proof:

$[a\ b] : :$

(1) $\sim(a \subset \sim(b)) . \supset .\therefore$

(2) $\sim(a \cap \sim(\sim(b)) \subset \Lambda) .$ $[T13, 1]$

(3) $a \cap \sim(\sim(b)) \subset a .$ $[T2]$

(4) $a \cap \sim(\sim(b)) \subset b .$ $[S2, T3, T32]$

(5) $[e] : e \subset a \cap \sim(\sim(b)) . \subset . e \subset b .\therefore$ $[S2, 4]$

$[\exists c\ d] .\therefore \sim(c \subset d) . c \subset a : [e] : e \subset c . \supset . e \subset b$ $[2, 3, 5]$

T34 $[a\ b\ c\ d] : a \subset \sim(b) . \sim(c \subset d) . c \subset a . \supset . \sim(c \subset b)$ $[T23]$

T35. $[a\ b\ e\ f] : : [c\ d] : \sim(c \subset d) . c \subset a . \supset . \sim(c \subset b) .\therefore \sim(e \subset f) . e \subset a .\therefore \supset . [\exists g] . g \subset e . \sim(g \subset b)$

Proof:

$[a\ b\ e\ f] : :$

(1) $[c\ d] : \sim(c \subset d) . c \subset a . \supset . \sim(c \subset b) .\therefore$

(2) $\sim(e \subset f) .$

(3) $e \subset a .\therefore \supset :$

(4) $\sim(e \subset b) .$ $[1, 2, 3]$

$[\exists g] . g \subset e . \sim(g \subset b)$ $[S1, 4]$

T36. $[a\ b] .\therefore [c\ d] : \sim(c \subset d) . c \subset a . \supset . \sim(c \subset b) : \supset . a \subset \sim(b)$

Proof:

$[a\ b] : :$

(1) $[c\ d] : \sim(c \subset d) . c \subset a . \supset . \sim(c \subset b) : \supset .\therefore$

(2) $[e\ f] : \sim(e \subset f) . e \subset a . \supset . [\exists g] . g \subset e . \sim(g \subset b) .\therefore$ $[T35, 1]$

$a \subset \sim(b)$ $[T33, 2]$

T37. $[a\ b\ c\ d\ e] : a \subset b \cap c . \sim(d \subset e) . d \subset a . \supset . [\exists f\ g] . \sim(f \subset g) . f \subset d . f \subset b . f \subset c$

Proof:

$[a\ b\ c\ d\ e] .\therefore$

(1) $a \subset b \cap c .$

(2) $\sim(d \subset e) .$

(3) $d \subset a . \supset :$

(4) $d \subset b \cap c :$ $[S2, 3, 1]$

(5) $d \subset b . d \subset c :$ $[S5, 4]$

$[\exists f\ g] . \sim(f \subset g) . f \subset d . f \subset b . f \subset c$ $[2, S1, 5]$

T38. $[a\ b\ c\ h\ i\ j] : : [d\ e] : \sim(d \subset e) . d \subset a . \supset . [\exists f\ g] . \sim(f \subset g) . f \subset d . f \subset b . f \subset c .\therefore \sim(h \subset i) . h \subset j . h \subset a .\therefore \supset . [\exists f\ g] . \sim(f \subset g) . f \subset j . f \subset b \cap c$

Proof:

$[a\ b\ c\ h\ i\ j]::$

(1) $[d\ e]:\sim(d \subset e).d \subset a.\supset.[\exists f\ g].\sim(f \subset g).f \subset d.f \subset b.f \subset c \therefore$

(2) $\sim(h \subset i).$

(3) $h \subset j.$

(4) $h \subset a \therefore \supset:$

$[\exists f\ g].$

(5) $\sim(f \subset g).$

(6) $f \subset h.$ [1, 2, 3]

(7) $f \subset b.$

(8) $f \subset c.$

(9) $f \subset j.$ [S2, 6, 3]

(10) $f \subset b \cap c:$ [S6, 7, 8]

$[\exists f g].\sim(f \subset g).f \subset j.f \subset b \cap c$ [5, 9, 10]

T39. $[a\ b\ c] \therefore [d\ e]:\sim(d \subset e).d \subset a.\supset.[\exists f\ g].\sim(f \subset g).f \subset d.f \subset b.f \subset c:\supset.a \subset b \cap c$

Proof:

$[a\ b\ c]::$

(1) $[d\ e]:\sim(d \subset e).d \subset a.\supset.[\exists f g].\sim(f \subset g).f \subset d.f \subset b.f \subset c:\supset \therefore$

(2) $[h\ i\ j]:\sim(h \subset i).h \subset j.h \subset a.\supset.[\exists f\ g].\sim(f \subset g).f \subset j.f \subset b \cap c \therefore$ [T38, 1]

$a \subset b \cap c$ [T16, 2]

T40. $[a\ b]:a \subset \sim(b).\supset.b \subset \sim(a)$

Proof:

$[a\ b]::$

(1) $a \subset \sim(b).\subset \therefore$

(2) $[c\ d]:\sim(c \subset d).c \subset b.\supset.\sim(c \subset a) \therefore$ [T34, 1]

$b \subset \sim(a)$ [T36, 2]

T41. $[a\ b\ c\ d\ e]::a \subset b \cup c.\sim(d \subset e).d \subset a \therefore [f\ g]:\sim(f \subset g).f \subset d.\supset.\sim(f \subset b).\sim(f \subset c) \therefore \supset:[\exists f\ g]:\sim(f \subset g).f \subset d:f \subset b.\vee.f \subset c$

Proof:

$[a\ b\ c\ d\ e]::$

(1) $a \subset b \cup c.$

(2) $\sim(d \subset e).$

(3) $d \subset a \therefore$

(4) $[f\ g] \therefore \sim(f \subset g).f \subset d.\supset.\sim(f \subset b).\sim(f \subset c) \therefore \supset \therefore$

(5) $d \subset \sim(b)\,.$ [T36, 4]

(6) $d \subset \sim(c)\,.$ [T36. 4]

(7) $b \subset \sim(d)\,.$ [T40, 5]

(8) $c \subset \sim(d)\,.$ [T40, 6]

(9) $b \cup c \subset \sim(d)\,.$ [S6, 7, 8]

(10) $a \subset \sim(d)\,.$ [S2, 1, 9]

(11) $d \subset \sim(a)\,.$ [T40, 10]

(12) $\sim(d \subset a) \therefore$ [T34, 11, 2, S1]

$[\exists f g] : \sim(f \subset g) \cdot f \subset d : f \subset b \,.\vee.\, f \subset c$ [3, 12]

T42. $[a\,b\,c\,d\,e] \therefore a \subset b \cup c \,.\sim(d \subset e) \cdot d \subset a \,.\supset : [\exists f g] : \sim(f \subset g) \cdot f \subset d : f \subset b \,.\vee.\, f \subset c$ [T41]

T43. $[a\,b\,c] :\cdot: [d\,e] \therefore \sim(d \subset e) \cdot d \subset a \,.\supset : [\exists f g] : \sim(f \subset g) \cdot f \subset d : f \subset b \,.\vee.\, f \subset c :: \sim(a \subset b \cup c) :: \supset . a \subset b \cup c$

Proof:

$[a\,b\,c]$ ::

(1) $[d\,e] \therefore \sim(d \subset e) \cdot d \subset a \,.\supset : [\exists f g] : \sim(f \subset g) \cdot f \subset d : f \subset b \,.\vee'\, . f \subset c ::$

(2) $\sim(a \subset b \cup c) :: \supset :\cdot:$

$[\exists d\,e\,f] ::$

(3) $\sim(d \subset e)\,.$

(4) $d \subset f\,.$

(5) $d \subset a \therefore$ [T16, 2]

(6) $[g\,h] : \sim(g \subset h) \cdot g \subset f \,.\supset . \sim(g \subset b \cup c) \therefore$

$[\exists g\,h] :$

(7) $\sim(g \subset h)\,.$

(8) $g \subset d :$ [1, 3, 5]

(9) $g \subset b \,.\vee.\, g \subset c :$

(10) $g \subset f\,.$ [S2, 8, 4]

(11) $g \subset b \cup c\,.$ [9, S2, T4, T5]

(12) $\sim(g \subset b \cup c) ::$ [6, 7, 10]

$a \subset b \cup c$ [11, 12]

T44. $[a\,b\,c] :: [d\,e] \therefore \sim(d \subset e) \cdot d \subset a \,.\supset : [\exists f g] : \sim(f \subset g) \cdot f \subset d : f \subset b \,.\vee.\, f \subset c \therefore \supset . a \subset b \cup c$ [T43]

T45 = A1. $[a\,b] \therefore a \subset b \,.\equiv : [c\,d\,e] : \sim(c \subset d) \cdot c \subset e \cdot c \subset a \,.\supset . [\exists f g] \,.\sim(f \subset g) \cdot f \subset e \cdot f \subset b$ [T15, T16]

T46 = A2. $[a] \therefore a \subset \wedge \,.\equiv : [b\,c] : \sim(b \subset c) \cdot b \subset a \,.\supset . [\exists d\,e] . \sim(d \subset e) \cdot d \subset b \,.\sim(d \subset d)$ [T17, T19]

T47 – A3. $[a] \therefore a \subset \vee \,.\equiv : [b\,c] : \sim(b \subset c) \cdot b \subset a \,.\supset . [\exists d\,e] . \sim(d \subset e) \cdot d \subset b \cdot d \subset d$ [T20, T21]

$T48 = A4.\ [a\ b] \therefore a \subset \sim(b) \,.\!\equiv: [c\ d] : \sim(c \subset d) \,.\, c \subset a \,.\supset .\, [\exists\, e f]$
$.\ (e \subset f) \,.\, e \subset c \,.\sim(e \subset b)$ [T24, T33]

$T49 = A5.\ [a\ b\ c] \therefore a \subset b \cap c \,.\!\equiv: [d\ e] : \sim(d \subset e) \,.\, d$
$\subset a \,.\supset .\, [\exists\, f g] \,.\sim(f \subset g) \,.\, f \subset d \,.\, f \subset b \,.\, f \subset c$ [T37, T3 9]

$T50 = A6.\ [a\ b\ c] :: a \subset b \cup c. \equiv \therefore [d\ e] \therefore \sim(d \subset e) \,.\, d \subset a \,.\supset:$
$[\exists\, f g] : \sim(f \subset g) \,.\, f \subset d : f \subset b \,.\vee .\, f \subset c$ [T42, T44]

It is evident from *T45 – T50* that *A1 – A6* can be deduced from *S1 – S9*. Our next task will be to show that, conversely, *S1 – S9* follow from *A1 – A6*.

$T50^{*}1 = S1\quad [a] \,.\, a \subset a$ [T45]

$T50^{*}2\quad [a\ b\ c\ d\ e\ f] : a \subset b \,.\, b \subset c \,.\sim(d \subset e) \,.\, d \subset f \,.\, d \subset a \,.\supset .$
$[\exists\, g\ h] \,.\sim(g \subset h) \,.\, g \subset f \,.\, g \subset c$

Proof:

$[a\ b\ c\ d\ e\ f] ::$

(1) $a \subset b\,.$

(2) $b \subset c\,.$

(3) $\sim(d \subset e)\,.$

(4) $d \subset f\,.$

(5) $d \subset a \,.\supset \therefore$

(6) $[g\ h\ i] : \sim(g \subset h) \,.\, g \subset i \,.\, g \subset a \,.\supset .\, [\exists\, j\ k] \,.\sim(j \subset k) \,.\, j \subset i$
$j \subset b \therefore$ [T45,1]

(7) $[g\ h\ i] : \sim(g \subset h) \,.\, g \subset i \,.\, g \subset b \,.\supset .\, [\exists\, j\ k] \,.\sim(j \subset k) \,.\, j \subset i.$
$j \subset c \therefore$ [T45, 2]

$[\exists\, j\ k]\,.$

(8) $\sim(j \subset k)\,.$

(9) $j \subset f\,.$

(10) $j \subset b \therefore$ } [6, 3, 4, 5]

$[\exists\, g\ h] \,.\sim(g \subset h) \,.\, g \subset f \,.\, g \subset c$ [7, 8, 9, 10]

$T50^{*}3 = S2\quad [a\ b\ c] : a \subset b \,.\, b \subset c \,.\supset .\, a \subset c$

Proof:

$[a\ b\ c] ::$

(1) $a \subset b\,.$

(2) $b \subset c \,.\supset \therefore$

(3) $[d\ e\ f] : \sim(d \subset e) \,.\, d \subset f \,.\, d \subset a \,.\supset .\, [\exists\, g\ h] \,.\sim(g \subset h) \,.\, g \subset f$
$.\, g \subset c \therefore$ [T50*2, 1, 2]

$a \subset c$ [T45, 3]

$T50^{*}4.\ [a\ b] : a \subset \wedge \,.\supset .\, a \subset b$ [T46, T50*1]

$T50^{*}5 = S3.\ [a] \,.\, \wedge \subset a$ [T50*4, T50*1]

$T50^{*}6.\ [a\ b] : \sim(a \subset b) \,.\supset .\, [\exists\, c\ d] \,.\sim(c \subset d) \,.\, c \subset a \,.\, c \subset c$ [T50*1]

*T50*7 = S4.* $[a]\,.\ a \subset V$ $[T47,\ T50{*}6]$

*T50*8.* $[a\,b\,c\,d\,e\,f] : a \subset b \cap c\ .\ \sim(d \subset e)\ .\ d \subset f\ .\ d \subset a\ .\ \supset .\ [\exists\, g\, h]\ .\ \sim(g \subset h)\ .\ g \subset f\ .\ g \subset b$

Proof:

$[a\,b\,c\,d\,e\,f] \therefore$

(1) $a \subset b \cap c\ .$

(2) $\sim(d \subset e)\ .$

(3) $d \subset f\ .$

(4) $d \subset a\ .\ \supset :$

$[\exists\, g\, h]\ .$

(5) $\sim(g \subset h)\ .$

(6) $g \subset d\ .$ $\Big\}$ $[T49,\ 1,\ 2,\ 4]$

(7) $g \subset b\ .$

(8) $g \subset f :$ $[T50{*}3,\ 6,\ 3]$

$[\exists\, g\, h]\ .\ \sim(g \subset h)\ .\ g \subset f\ .\ g \subset b$ $[5,\ 8,\ 7]$

*T50*9.* $[a\,b\,c] : a \subset b \cap c\ .\ \supset .\ a \subset b$

Proof:

$[a\,b\,c] ::$

(1) $a \subset b \cap c\ .\ \supset \therefore$

(2) $[d\,e\,f] : \sim(d \subset e)\ .\ d \subset f\ .\ d \subset a\ .\ \supset .\ [\exists\, g\, h]\ .\ \sim(g \subset h) .\ g \subset f\ .$
$g \subset b \therefore$ $[T50{*}8,\ 1]$

$a \subset b$ $[T45,\ 2]$

*T50*10.* $[a\,b\,c] : a \subset b \cap c\ .\ \supset .\ a \subset c \cap b$ $[T\,49]$

*T50*11.* $[a\,b\,c] : a \subset b \cap c\ .\ \supset .\ a \subset c$ $[T50{*}10,\ T50{*}9]$

*T50*12.* $[a\,b\,c\,d\,e] : a \subset b\ .\ a \subset c\ .\ \sim(d \subset e)\ .\ d \subset a\ .\ \supset .\ [\exists\, f\, g]\ .\ \sim(f \subset g)\ .\ f \subset d\ .\ f \subset b\ .\ f \subset c$

Proof:

$[a\,b\,c\,d\,e] \therefore$

(1) $a \subset b\ .$

(2) $a \subset c\ .$

(3) $\sim(d \subset e) .$

(4) $d \subset a\ .\ \supset :$

$[\exists\, f\, g]\ .$

(5) $\sim(f \subset g)\ .$

(6) $f \subset d\ .$ $\Big\}$ $[T45,\ 4,\ 3,\ T50{*}1]$

(7) $f \subset a\ .$

(8) $f \subset b\ .$ $[T50{*}3,\ 7,\ 1]$

(9) $f \subset c :$ $[T50{*}3,\ 7,\ 2]$

$[\exists fg] . \sim(f \subset g) . f \subset d . f \subset b . f \subset c$ [5, 6, 8, 9]

T50*13 $[a\,b\,c] : a \subset b . a \subset c . \supset . a \subset b \cap c$

Proof:

$[a\,b\,c] ::$

(1) $a \subset b .$

(2) $a \subset c . \supset \therefore$

(3) $[d\,e] : \sim(d \subset e) . d \subset a . \supset . [\exists fg] . \sim(f \subset g) . f \subset b \therefore$

$f \subset c \therefore$ [T50*12, 1, 2]

$a \subset b \cap c$ [T49, 3]

T50*14 = S5. $[a\,b\,c] : c \subset a \cap b . \equiv . c \subset a . c \subset b$

[T50*9, T50*11, T50*13]

T50*15. $[a\,b\,c\,d] \therefore \sim(c \subset d) . c \subset a . \supset : [\exists e f] : \sim(e \subset f) . e \subset c : e \subset a . \vee . e \subset b$

Proof:

$[a\,b\,c\,d] ::$

(1) $\sim(c \subset d) .$

(2) $c \subset a . \supset \therefore$

(3) $c \subset a . \vee . c \subset b \therefore$ [2]

$[\exists e f] : \sim(e \subset f) . e \subset c : e \subset a . \vee . e \subset b$ [1, T50*1, 3]

T50*16 $[a\,b] . a \subset a \cup b$ [T50, T50*15]

T50*17. $[a\,b\,c] : a \cup b \subset c . \supset . a \subset c$ [T50*3, T50*16]

T50*18. $[a\,b] . a \subset b \cup a$ [T50, T50*15]

T50*19 $[a\,b\,c] : a \cup b \subset c . \supset . b \subset c$ [T50*3, T50*18]

T50*20. $[a\,b\,c\,d\,e\,f] : a \subset c . b \subset c . \sim(d \subset e) . d \subset f . d \subset a \cup b . \supset . [\exists g h] . \sim(g \subset h) . g \subset f . g \subset c$

Proof:

$[a\,b\,c\,d\,e\,f] ::$

(1) $a \subset c .$

(2) $b \subset c .$

(3) $\sim(d \subset e) .$

(4) $d \subset f .$

(5) $d \subset a \cup b . \supset \therefore$

$[\exists g h] :$

(6) $\sim(g \subset h) .$

(7) $g \subset d :$ [T50, 5, 3, T50*1]

(8) $g \subset a . \vee . g \subset b :$

(9) $g \subset f .$ [T50*3, 7, 4]

(10) $g \subset c \therefore$ [8, T50*3, 1, 2]

$[\exists gh] . \sim (g \subset h) . g \subset f . g \subset c$ [6, 9, 10]

T50*21 $[abc] : a \subset c . b \subset c . \supset . a \cup b \subset c$

Proof:

$[abc] ::$

(1) $a \subset c .$

(2) $b \subset c . \supset \therefore$

(3) $[def] : \sim (d \subset e) . d \subset f . d \subset a \cup b . \supset . [\exists gh] . \sim (g \subset h) . g$
$\subset f . g \subset c \therefore$ [T50*20, 1, 2]

$a \cup b \subset c$ [T45, 3]

T50*22 = S6. $[abc] : a \cup b \subset c . \equiv . a \subset c . b \subset c$
[T50*17, T50*19, T50*21]

T50*23 $[abcdef] \therefore d \subset a \cap (b \cup c) . \sim (e \subset f) . e \subset d . \supset : [\exists gh] . \sim$
$(g \subset h) . g \subset e : g \subset a \cap b . \vee . g \subset a \cap c$

Proof:

$[abcdef] ::$

(1) $d \subset a \cap (b \cup c) .$

(2) $\sim (e \subset f) .$

(3) $e \subset d . \supset \therefore$

(4) $e \subset a \cap (b \cup c) .$ [T50*3, 3, 1]

(5) $e \subset a .$ [T50*9, 4]

(6) $e \subset b \cup c \therefore$ [T50*11, 4]

$[\exists gh] :$

(7) $\sim (g \subset h) .$

(8) $g \subset e :$ [T50, 6, 2, T50*1]

(9) $g \subset b . \vee . g \subset c :$

(10) $g \subset a :$ [T50*3, 8, 5]

(11) $g \subset a \cap b . \vee . g \subset a \cap c \therefore$ [9, T50*13, 10]

$[\exists gh] : \sim (g \subset h) . g \subset e : g \subset a \cap b . \vee . g \subset a \cap c$ [7, 8, 11]

T50*24 $[abcd] : d \subset a \cap (b \cup c) . \supset . d \subset (a \cap b) \cup (a \cap c)$

Proof:

$[abcd] :\cdot:$

(1) $d \subset a \cap (b \cup c) . \supset ::$

(2) $[ef] \therefore \sim (e \subset f) . e \subset d . \supset : [\exists gh] : \sim (g \subset h) . g \subset e : g \subset$
$a \cap b . \vee . g \subset a \cap c ::$ [T50*23, 1]

$d \subset (a \cap b) \cup (a \cap c)$ [T50, 2]

T50*25 = S7. $[abc] . a \cap (b \cup c) \subset (a \cap b) \cup (a \cap c)$ [T50*24, T50*1]

*T50*26* $[a\,b] : a \subset b \cap \backsim(b) \,.\, \sim(a \subset \wedge) \,.\, \supset \,.\, a \subset \wedge$

Proof:

$[a\,b] ::$

(1) $a \subset b \cap \backsim(b) \,.$

(2) $\sim(a \subset \wedge) \,.\, \supset \therefore$

(3) $a \subset b \,.$ [*T50*9*, 1]

(4) $a \subset \backsim(b) \therefore$ [*T50*11*, 1]

(5) $[c\,d] : \sim(c \subset d) \,.\, c \subset a \,.\, \supset \,.\, c \subset b \therefore$ [*T50*3*, 3]

(6) $\sim(a \subset \backsim(b)) \therefore$ [*T48*, 2, *T50*1*, 5]

$a \subset \wedge$ [4, 6]

*T50*27* = *S8*. $[a] \,.\, a \cap \backsim(a) \subset \wedge$ [*T50*26*, *T50*1*]

*T50*28*. $[a] : \sim(\vee \subset a \cup \backsim(a)) \,.\, \supset \,.\, \vee \subset a \cup \backsim(a)$

Proof:

$[a] \vdots\vdots$

(1) $\sim(\vee \subset a \cup \backsim(a)) \,.\, \supset :\cdot:$

(2) $[\exists\, b\, c] ::$

(2) $\sim(b \subset c) \therefore$ [*T50*, 1]

(3) $[d\,e] : \sim(d \subset e) \,.\, d \subset b \,.\, \supset \,.\, \sim(d \subset a) \,.\, \sim(d \subset \backsim(a)) \therefore$

(4) $\sim(b \subset \backsim(a)) ::$ [3, 2, *T50*1*]

$[\exists\, d\, e] \therefore$

(5) $\sim(d \subset e).$

(6) $d \subset b \therefore$ [*T48*, 4]

(7) $[f\,g] : \sim(f \subset g) \,.\, f \subset d \,.\, \supset \,.\, f \subset a \therefore$

(8) $d \subset a \,.$ [7, 5, *T50 *1*]

(9) $\sim d \subset a) :\cdot:$ [3, 5, 6]

$\vee \subset a \cup \backsim(a)$ [8, 9]

*T50*29* = *S9*. $[a] \,.\, \vee \subset a \cup \backsim(a)$ [*T50* 28*]

Theses *T50*1* – *T50*29* have been deduced from *T45* – *T50*, *i. e.*, from *A1* – *A6*. The totality of theses deducible from *A1* – *A6*, will be referred to as System $\mathfrak{A}^*$. Since *S1* – *S9* are included in this totality as can be seen from *T50*1*, *T50*3*, *T50*5*, *T50* 7*, *T50*14*, *T50*22*, *T50*25*, *T50*27*, and *T50*29*, we can regard System $\mathfrak{S}$ and System $\mathfrak{A}^*$ as inferentially equivalent.

In developing the two systems we made no explicit use of any rules of inference. In our proofs, which strictly speaking are only outlines of proofs, appeal is made to intuition or 'obviousness' rather than to formal conditions which an expression must satisfy if it is to be added to the system as a new thesis. There is, however, no theoretical difficulty in so recasting our deductions as to make it evident that the only rules of inference that are involved, are the following:

R1 *the rule of substitution*
R2 *the rules concerning the use of the quantifiers*
R3 *the rule of detachment*

So far no use has been made of definitions or laws of extensionality.

It is to be noted that axioms *A2* – *A6* exhibit certain uniformity of structure. The significance of this uniformity will become clear in the light of what is going to be said in Section II, which as we have already mentioned will be devoted to the problem of definitions in systems of Boolean Algebra.

SECTION II

Definitions in deductive systems are often explained away as typographical abbreviations. In many cases, however, it is only too obvious that these so called definitional abbreviations are expected to satisfy a number of implicit or even explicit conditions which appear to be much stronger than the conditions required solely for the purpose of streamlining our symbolism. In what follows we propose to adhere to the view that in principle definitions are not abbreviations but rather serve the purpose of expanding the vocabulary of the system. In accordance with this view, which is that of Leśniewski, a definition is, as it were, a single axiom for the constant term it introduces into the system.[5] The rules for writing definitions lay down a number of formal conditions which must be satisfied by an expression if it is to be regarded as a well constructed definition, and if it is to be added to the system as a new thesis without the risk of generating a contradiction. The form of definitions varies from system to system depending on the primitive terms at our disposal. Thus expressions which in one system can be regarded as definitions, may have to be proved as theorems in another system of the same theory.

The constant terms of Boolean Algebra fall into two classes. First we have the class of proposition-forming functors, of which the functor of weak inclusion is an example. Secondly, we have names, like '$\wedge$' and '$\vee$', and the class of name-forming functors represented in $\mathfrak{S}$ and $\mathfrak{B}^*$ by '$\sim$', '$\cap$', and '$\cup$'. If we want to expand the vocabulary of our system by introducing constant terms of either type, we need rules of definition.

Proposition-forming functors will be introduced into $\mathfrak{S}$ (or $\mathfrak{A}^*$) by means of what we might call *propositional* definitions. They are all of the form

$$\text{I} \qquad [\ldots]\,.\,\alpha \equiv \beta$$

and the corresponding rule can be outlined as follows.

R4 *the rule for writing propositional definitions*. On the assumption that a thesis *T* is the last thesis in the system, an expression *E* of type I can be added to the system as a new thesis provided the following conditions are fulfilled: 'α' in *E, i. e.*, the definiendum, is a simple propositional function:[6] its functor is a constant term which does not occur in *T* or in any thesis preceding *T* in the system; the arguments of the function are all variables; none of these variables occurs in 'α' more than once; variables which are of the same semantical category (logical type) as expressions obtainable within the logic of propositions are not allowed in 'α', 'β' in *E*, *i. e.*, the definiens, is, with respect to T, a meaningful propositional expression; every constant term occurring in 'β' occurs in T or in a thesis preceding T in the system, or it occurs in the logic of propositions, and every variable occurring in 'β' belongs to a semantical category (logical type) already available in the system; every variable occurring in 'β' occurs as a free variable in 'β' and every free variable in 'β' occurs in 'α'; the universal quantifier preceding '$\alpha \equiv \beta$' binds all the free variables in '$\alpha \equiv \beta$'.

R4 is a powerful means for expanding our ontological vocabulary within the framework of $\mathfrak{S}$ (or $\mathfrak{A}^*$). It enables us, for instance, to add to the system the following theses:

D1. $[a] : \mathrm{ex}(a) . \equiv . [\exists b] . \sim(a \subset b)$

D2. $[a] :: \mathrm{sol}(a) . \equiv .\therefore [b\, c] .\therefore b \subset a . \supset : a \subset b . \vee . b \subset c$

D3. $[a] :: \mathrm{ob}(a) . \equiv :: [\exists b] . \sim(a \subset b) :: [b\, c] .\therefore b \subset a . \supset : a \subset b . \vee . b \subset c$

D4. $[a\, b] .\therefore a \sqsubset b . \equiv : a \subset b : [\exists c] . \sim(a \subset c)$

D5. $[a\, b] : a \triangle b . \equiv . [\exists c\, d] . \sim(c \subset d) . c \subset a . c \subset b$

D6. $[a\, b] :\cdot: a \in b . \equiv :: [\exists c] . \sim(a \subset c) . a \subset b :: [c\, d] .\therefore c \subset a . \supset : a \subset c . \vee . c \subset d$

D7. $[a\, b] : a \circ b . \equiv . a \subset b . b \subset a$

D8. $[a\, b] : a \square b . \equiv . [\exists c] . \sim(a \subset c) . a \subset b . b \subset a$

D9. $[a\, b] :: a = b . \equiv :: [\exists c] . \sim(a \subset c) . a \subset b :: [c\, d] .\therefore c \subset b . \supset : b \subset c . \vee . c \subset d$ 7)

Definitions D1, D2, and D3 introduce what might be called functors of *existence*. It is evident from their respective definientia that

'ex(a)' means the same as '*there exists at least one a*'
'sol(a)' means the same as '*there exists at most one a*', and
'ob(a)' means the same as '*there exists exactly one a*'.

Definitions D4, D5, and D6 introduce further functors of *inclusion*, namely the functor of *strong inclusion*, the functor of *partial inclusion*, and the functor of *singular inclusion*. Propositions of the type '$a \sqsubset b$', to be read: *every a is b*, correspond exactly to the universal affirmative propositions of syllogistic. Similarly, propositions of the type '$a \triangle b$', to be read: *some a is b*, can be equated with the particular affirmative propositions. Propositions of the type '$a \in b$', to be read: *a is b*, are not to be confused with those set-theoretical propositions which give expression to class-membership, and imply that their arguments belong to different semantical categories (logical types). There is no categorical difference between the arguments required by the functor of singular inclusion.

We shall see in the sequel that our three functors of inclusion, which appear to be shunned by students of Boolean Algebra, give rise to interesting theorems.

Definitions *D7*, *D8*, and *D9*, introduce three different functors of *identity:* the functor of *weak identity*, the functor of *strong identity*, and the functor of *singular identity*. Propositions of the types '$a \mathrm{O} b$', $a \square b$', and '$a = b$' can be read respectively: *only all a is b*, *only every a is b*, and *a is the same object as b*. Of these three functors the functor of weak identity is very familiar. In some systems of Boolean Algebra it is used for the purpose of constructing definitions which introduce constant names of constant name-forming functors, and which we propose to call *nominal* definitions in contra-distinction to propositional definitions.

In System $\mathfrak{S}$ (or $\mathfrak{A}^*$) all nominal definitions will have the following form:

$$\text{II} \quad [a \ldots] \therefore a \subset x . \equiv : [b\, c] : \sim(b \subset c) . b \subset a . \supset . [\exists d\, e] . \sim(d \subset e) . d \subset b . \varphi(d)$$

The rule for writing these definitions can be outlined thus:

R5 *the rule for writing nominal definitions.* On the assumption that a thesis T is the last thesis in the system, an expression E of type II can be added to the system as a new thesis provided the following conditions are fulfilled: 'x' in E is a constant name which does not occur in T or in any thesis preceding T in the system, or it is a simple nominal function; if the latter is the case then the functor of this function is a constant term which does not occur in T or in any thesis preceding T in the system, while the arguments of the function are all variables; none of the variables in '$a \subset x$' occurs in that expression more than once; variables which are of the same semantical category (logical type) as expressions obtainable within the logic of propositions are not allowed in 'x'. '$\varphi(d)$' in E is, with respect to T, a meaningful propositional expression , i.e., every constant in '$\varphi(d)$' occurs in T or in a thesis preceding T in the system or in the logic of propositions, and every variable occurring in '$\varphi(d)$' belongs to a semantical category (logical type) already available in the system; every variable occurring in 'x' occurs as a free variable in '$\varphi(d)$' and every free variable in '$\varphi(d)$' occurs in 'x' or is equiform with(d); there are no free variables in E.

By applying R5 we can add to $\mathfrak{S}$ (or $\mathfrak{A}^*$), among others, the following theses:

D10. $[a\, b\, c] \therefore a \subset b | c . \equiv : [d\, e] : \sim(d \subset e) . d \subset a . \supset . [\exists f\, g] . \sim(f \subset g) . f \subset d . \sim(f \subset b) . \sim(f \subset c)$

D11. $[a\, b\, c] \therefore a \subset b - c . \equiv : [d\, e] : \sim(d \subset e) . d \subset a . \supset . [\exists f\, g] . \sim(f \subset g) . f \subset d . f \subset b . \sim(f \subset c)$

D12. $[a\, b\, c] :: a \subset b \div c . \equiv \therefore [d\, e] \therefore \sim(d \subset e) . d \subset a . \supset : [\exists f\, g] . \sim(f \subset g) . f \subset d : f \subset b . \vee . \sim(f \subset c)$

It can be shown that within the framework of $\mathfrak{S}$ (or $\mathfrak{A}^*$) *D10* is inferentially equivalent to

T51. $[a\, b] . a | b \ \mathrm{O} \sim(a) \cap \sim(b)$

It thus introduces the functor of *rejection.*

D11, which can be shown to be inferentially equivalent to

T52. $[a\ b] .\ a - b\ \mathrm{O}\ a \cap \sim(b)$,

introduces the functor of *exception*.

D12 introduces the functor of *adjunction* as it can be shown to be inferentially equivalent to

T53. $[a\ b] .\ a \div b\ \mathrm{O}\ a \cup \sim(b)$

It is obvious that in many respects our rule for writing nominal definitions departs from the rules more commonly accepted for the purpose. It is therefore, desirable to compare the traditional rules with the stipulations of R5.

More often than not nominal definitions in systems of Boolean Algebra are constructed as weak identities of the form

III $[\ldots] .x\ \mathrm{O}\ y$

If instead of the functor of weak identity we want to make use of the functor of weak inclusion then of course our definitions can be given the following form:

IIIa $[\ldots] . x \subset y . y \subset x$[8])

The corresponding rule for writing this type of nominal definitions could be outlined as follows:

R5a *the rule for writing nominal definitions as identities.* On the assumption that a thesis *T* is the last thesis in the system, an expression *E* of type III (or IIIa) can be added to the system as a new thesis provided the following conditions are fulfilled: 'x' in *E*, *i.e.*, the definiendum is a simple nominal function; its functor is a constant term which does not occur in *T* or in any thesis preceding *T* in the system; its arguments are all variables; none of these variables occurs in 'x' more than once; 'y' in *E*, *i.e.*, the definiens, is, with respect to *T*, a meaningful nominal expression; this means that every constant term occurring in 'y' occurs in *T* or in a thesis preceding *T* in the system, and every variable occurring in 'y' belongs to a semantical category (logical type) already available in the system; every variable in 'x' occurs in 'y' and vice versa; the universal quantifier preceding '$x\ \mathrm{O}\ y$' (or '$x \subset y . y \subset x$') binds all the variables in that expression.

If we equipped $\mathfrak{S}$ (or $\mathfrak{B}^*$) with R5a instead of R5 then theses *T51*, *T52*, and *T53* above could be used as possible definitions.

R5a, which in a sense is analogous to R4, provides for very simple and and intuitive definitions but with its aid we can introduce into our system only those name-forming functors which have names as arguments.

No such restriction applies if we write definitions as equivalences of the form

IV $[a \ldots] : a\ \mathrm{O}\ x . \equiv \varphi(a)$

or IVa $[a \ldots] : a \subset x . x \subset a . \equiv . \varphi(a)$

The corresponding rule for writing nominal definitions of this type could be outlined as follows:

R5b *the rule for writing nominal definitions.* On the assumption that a thesis T is the last thesis in the system, and on a further assumption that an expression of the form

$$\text{V}\quad [\ldots] :: [\exists a] .\therefore \varphi(a) .\therefore [b] : \varphi(b) . \subset . a \mathrm{O} b$$

$$\text{or}\quad \text{Va}\ [\ldots] :: [\exists a] .\therefore \varphi(a) .\therefore [b] : \varphi(b) . \subset . a \subset b . b \subset a$$

is a thesis of the system and precedes T or is identical with T, a corresponding expression E of type IV (or IVa) can be added to the system as a new thesis provided the following conditions are fulfilled: 'x' in E is a constant name which does not occur in T or in any thesis preceding T in the system, or is it a simple nominal function; if the latter is the case then the functor of this function is a constant term which does not occur in T or in any thesis preceding T in the system while the arguments of the function are all variables; none of the variables in '$a \mathrm{O} x$' (or in '$a \subset x$') occurs in that expression more than once; variables which are of the same semantical (logical type) category as expressions obtainable in the logic of propositions are not allowed in 'x'; '$\varphi(a)$' in E, *i.e.*, the definiens, is, with respect to T, a meaningful propositional expression: thus every constant term in '$\varphi(a)$' occurs in T or in a thesis preceding T in the system, or in the logic of propositions, and every variable in '$\varphi(a)$' belongs to a semantical category (logical type) already available in the system; every variable in '$a \mathrm{O} x$' (or in '$a \subset x$') occurs in '$\varphi(a)$' as a free variable in '$\varphi(a)$' occurs in '$a \mathrm{O} x$' (or in '$a \subset x$'); there are no free variables in E.

It is not difficult to see that with the aid of R5b we can define anything that can be defined by making use of R5a.

Nominal definitions which satisfy the requirements of R5b are still very simple and quite intuitive but R5b itself is more complicated than R5 or R5a as it contains an extra condition which makes the application of the rule depend on the availability of certain theses of type V (or Va).

We can remove this extra condition from the formulation of the rule and include it in the definitions themselves. If we do this, and if we express weak identity in terms of weak inclusion then our nominal definitions will have the following form:

$$\text{VI}\ [a\, b \ldots] :: \varphi(b) .\therefore [c] : \varphi(c) . \supset . b \subset c . c \subset b .\therefore \supset : a \subset x . x \subset a . \equiv . \varphi(a)$$

The corresponding rule, which in the sequel will be referred to as R5c, is analogous to R5b except that the condition concerning the availability of theses of type V or (Va) is dropped altogether. It is clear that with the aid of R5c we can define anything that can be defined with the aid of R5b. The converse, however, does not hold. For R5c allows us to add to $\mathfrak{S}$ (or $\mathfrak{B}^*$) theses of type VI with antecedents which, irrespective of the value of the variable represented in VI by 'b', cannot be proved within the system. Such theses still introduce new constant terms into the system but they open no possibilities of employing these terms in theorems of any interest.

As a final preliminary to the discussion of R5 let us note that as regards extensional functions for which theses of the form

VII $[a\,b\,\ldots]:a\subset b\,.\,b\subset a\,.\,\varphi(a)\,.\supset.\,\varphi(b)$

can be proved in the system, R5c is equivalent to corresponding rule, – we shall call it R5d –, which stipulates definitions of the following form:

VIII $[a\,b\,\ldots]::\varphi(b)\therefore[c]\therefore\varphi(c)\,.\supset.\,b\subset c\,.\,c\subset b\therefore\supset:a\subset x.$
$x\subset a\,.\equiv.\,[\exists d]\,.\,a\subset d\,.\,d\subset a\,.\,\varphi(d)$

Now, it can be shown that any thesis added to $\mathfrak{S}$ (or $\mathfrak{A}^*$) in virtue of R5d can be derived within the system by making use of R5 and R1 – R3. Here is an outline of the proof.

In order to derive a thesis of type VIII we begin with the corresponding thesis

E1. $[b\,\ldots]::b\subset x\,.\equiv\therefore[c\,d]:\sim(c\subset d)\,.\,c\subset b\,.\supset:[\exists e\,f]:$
$\sim(e\subset f)\,.\,e\subset c:[\exists g]\,.\,e\subset g\,.\,\varphi(g),$

which we obtain by applying R5. We then proceed as follows:

E2. $[a\,\ldots]:\cdot:[b]::b\subset a\,.\equiv\therefore[c\,d]\therefore\sim(c\subset d)\,.\,c\subset b\,.\supset:[\exists e$
$f]:\sim(e\subset f)\,.\,e\subset c:[\exists g]\,.\,e\subset g\,.\,\varphi(g)::\supset.\,a\subset x\,.\,x\subset a$ [S1, E1]

E3. $[a\,b\,c\,d\,e\,h\,\ldots]\therefore a\subset h\,.\,\varphi(h)\,.\,b\subset a\,.\sim(c\subset d)\,.\,c\subset b\,.\supset:[\exists e$
$f]:\sim(e\subset f)\,.\,e\subset c:[\exists g]\,.\,e\subset g\,.\,\varphi(g)$ [S2, S1]

E4. $[a\,b\,h\,\ldots]::a\subset h\,.\,\varphi(h)\,.\,b\subset a\,.\supset\therefore[c\,d]\therefore\sim(c\subset d)\,.\,c\subset$
$b\,.\subset:[\exists e\,f]:\sim(e\subset f)\,.\,e\subset c:[\exists g]\,.\,e\subset g\,.\,\varphi(g)$ [E3]

E5. $[a\,h\,\ldots]:\cdot:a\subset h\,.\,\varphi(h)\,.\supset::[b]::b\subset a\,.\supset\therefore[c\,d]\therefore\sim(c\subset d)\,.$
$c\subset b\,.\supset:[\exists e\,f]:\sim(e\subset f)\,.\,e\subset c:[\exists g]\,.\,e\subset g\,.\,\varphi(g)$ [E4]

E6. $[a\,b\,h\,i\,j\,k\,l\,\ldots]:\cdot:[c]:\varphi(c)\,.\supset.\,h\subset c\,.\,c\subset h\therefore i\subset a\,.\,\varphi(i)$
$::[c\,d]\therefore\sim(c\subset d)\,.\,c\subset b\,.\supset:[\exists e\,f]:\sim(e\subset f)\,.\,e\subset c:[\exists g]\,.\,e$
$\subset g\,.\,\varphi(g)::\sim(j\subset k)\,.\,j\subset l\,.\,j\subset b::\supset.\,[\exists e\,f]\,.\sim(e\subset f)\,.\,e\subset l$
$.\,e\subset a$ [S2]

E7. $[a\,b\,h\,i\,\ldots]:\cdot:[c]:\varphi(c)\,.\supset.\,h\subset c\,.\,c\subset h\therefore i\subset a\,.\,\varphi(i)\therefore$
$[c\,d]\therefore\sim(c\subset d)\,.\,c\subset b\,.\supset:[\exists e\,f]:\sim(e\subset f)\,.\,e\subset c:[\exists g]$
$.\,e\subset g\,.\,\varphi(g)::\subset.\,b\subset a$ [E6, T16]

E8. $[a\,d\,h\,\ldots]::[c]:\varphi(c)\,.\supset.\,d\subset c\,.\,c\subset d\therefore a\subset h\,.\,h\subset a\,.\,\varphi(h)$
$\therefore\supset.\,a\subset x\,.\,x\subset a$ [E5, E7, E2]

E9. $[a\,b\,\ldots]::\varphi(b)\therefore[c]:\varphi(c)\,.\supset.\,b\subset c\,.\,c\subset b\therefore a\subset x\,.\,x\subset$
$a\therefore\supset.\,[\exists d]\,.\,a\subset d\,.\,d\subset a\,.\,\varphi(d)$ [E8, S2]

E10. $[a\,b\,\ldots]::\varphi(b)\therefore[c]:\varphi(c)\,.\supset.\,b\subset c\,.\,c\subset b\therefore\supset:a\subset x\,.\,x$
$\subset a\,.\equiv.\,[\exists d]\,.\,a\subset d\,.\,d\subset a\,.\,\varphi(d)$ [E9, E8]

E10 being of type VIII our proof that with the aid of R5 we can define anything that can be defined by applying R5d, has been completed. The converse, however, does not hold. R5 proves to be a stronger rule and the problem arises to find out in what way it is. so.

The success of Rb, Rc, and Rd depends on the availability of theses of type V (or Va). A number of such theses can be derived within 𝔖 (or 𝔎*); others could be added axiomatically. Since, however, in view of R4 (and R5b, R5c, or R5d) the syntactical variety of the system is not static but admits of unlimited extension, a special rule is required. This rule, which we propose to call *the rule of univocal functions*, will be formulated below on the presupposition that it is to be used in some systems which have the functor of weak inclusion among their primitive terms.

R6 *the rule of univocal functions.* On the assumption that a thesis T is the last thesis in the system an expression E of the form

$$\text{IX}\quad [\ldots] :\cdot: [\exists b] :: [a] \therefore a \subset b . \equiv : [c\,d] : \sim(c \subset d) . c \subset a . \supset . [\exists e\,f] . \sim(e \subset f) . e \subset c . \varphi(e)$$

can be added to the system provided the following conditions are fulfilled: '$\varphi(e)$' in E is, with respect to T, a meaningful propositional expression, *i.e.*, every constant term in '$\varphi(e)$' occurs in T or in a thesis preceding T in the system or in the logic of propositions, and every variable occurring in '$\varphi(e)$' belongs to the semantical category (logical type) already available in the system; variables of which are the same semantical category (logical type) as expressions obtainable in the logic of propositions are not allowed in '$\varphi(e)$' as free variables. 9)

It can easily be shown that any thesis added to 𝔖 (or 𝔄*) in virtue of R6 can be derived within the system by making use of R5, and R1 – R3. Here is an outline of the proof.

In order to derive a thesis of type IX we begin with the corresponding thesis

$$F1.\quad [a\ldots] \therefore a \subset x . \equiv : [c\,d] : \sim(c \subset d) . c \subset a . \supset . [\exists e\,f] . \sim(e \subset f) . e \subset c . \varphi(e) ,$$

which we obtain by applying R5. We then proceed as follows:

$$F2.\quad [\ldots] :\cdot: [a] \therefore a \subset x . \equiv : [c\,d] : \sim(c \subset d) . c \subset a . \supset . [\exists e\,f] . \sim(e \subset f) . e \subset c . \varphi(e) :: \supset :: [\exists b] :: [a] \therefore a \subset b . \equiv : [c\,d] : \sim(c \subset d) . c \subset a . \supset . [\exists e\,f] . \sim(e \subset f) . e \subset c . \varphi(e)$$

(follows from the logic of propositions by R1 and R2)

$$F3.\quad [a\ldots] \therefore a \subset x . \equiv : [c\,d] : \sim(c \subset d) . c \subset a . \supset . [\exists e\,f] . \sim(e \subset f) . e \subset c . \varphi(e) \therefore \supset :\cdot: [\ldots] :\cdot: [\exists b] :: [a] \therefore a \subset b . \equiv : [c\,d] : \sim(c \subset d) . c \subset a . \supset . [\exists e\,f] . \sim(e \subset f) . e \subset c . \varphi(e)$$

[from *F2*]

$$F4.\quad [\cdot\ldots]:\cdot:\ [\exists b]::[a]\therefore a\subset b.\equiv:[c\,d]:\sim(c\subset d).c\subset a.\supset.[\exists e\,f].\sim(e\subset f).e\subset c.\varphi(e)\qquad[F3,\ F1]$$

F4. being the thesis we set out to derive, our outline is completed.

We now proceed to show that any thesis which satisfies the conditions of R5, can be derived within 𝔖 (or 𝔄*) by applying R5d, R6, and R1 – R3, We begin with the corresponding thesis

$$G1.\quad [\ldots]:\cdot:\ [\exists b]::[a]\therefore a\subset b.\equiv:[c\,d]:\sim(c\subset d).c\subset a.\supset.[\exists e\,f].\sim(e\subset f).e\subset c.\varphi(e),$$

which we obtain by applying R6. Then we derive

$$G2.\quad [g\,h\ldots]:\cdot:\ [a]\therefore a\subset g.\equiv:[c\,d]:\sim(c\subset d).c\subset a.\supset.[\exists e\,f].\sim(e\subset f).e\subset c.\varphi(e)::[a]\therefore a\subset h.\equiv:[c\,d]:\sim(c\subset d).c\subset a.\supset.[\exists e\,f].\sim(e\subset f).e\subset c.\varphi(e)::\supset.g\subset h.h\subset g\qquad[S1]$$

$$G3.\quad [g\ldots]\vdots\cdot\vdots\ [a]\therefore a\subset g.\equiv:[c\,d]:\sim(c\subset d).c\subset a.\supset.[\exists e\,f]:\sim(e\subset f).e\subset c.\varphi(e)::\supset\vdots\vdots\ [\exists b]\vdots\vdots\ [a]\therefore a\subset b.\equiv:[c\,d]:\sim(c\subset d).c\subset a.\supset.[\exists e\,f].\sim(e\subset f).e\subset c.\varphi(e)\vdots\vdots\ [h]:\cdot:[a]\therefore a\subset h.\equiv:[c\,d]:\sim(c\subset d).c\subset a.\supset.[\exists e\,f].\sim(e\subset f).e\subset c.\varphi(e)\therefore\supset.b\subset h.h\subset b\qquad[G2]$$

$$G4.\quad [\ldots]\vdots\vdots\ [\exists b]:\cdot:\ [a]\therefore a\subset b.\equiv:[c\,d]:\sim(c\subset d).c\subset a.\supset.[\exists e\,f].\sim(e\subset f).e\subset c.\varphi(e):\cdot:\ [h]::[a]\therefore a\subset h.\equiv:[c\,d]:\sim(c\subset d).c\subset a.\supset.[\exists e\,f].\sim(e\subset f).e\subset c.\varphi(e)\therefore\supset.b\subset h.h\subset b\qquad[G3,\ G1]$$

$$G5.\quad [b\,i\ldots]\vdots\cdot\vdots\ [a]\therefore a\subset b.\equiv:[c\,d]:\sim(c\subset d).c\subset a.\supset.[\exists e\,f].\sim(e\subset f).e\subset c.\varphi(e):\cdot:\ [h]::[a]\therefore a\subset h.\equiv:[c\,d]:\sim(c\subset d).c\subset a.\supset.[\exists e\,f].\sim(e\subset f).e\subset c.\varphi(e)\therefore\supset.b\subset h.h\subset b:\cdot:\ \supset\vdots\vdots\ i\subset x.x\subset i.\equiv:\cdot:\ [\exists j]::i\subset j.j\subset i::[a]\therefore a\subset j.\equiv:[c\,d]:\sim(c\subset d).c\subset a.\supset.[\exists e\,f].\sim(e\subset f).e\subset c.\varphi(e)\qquad[\text{obtained by applying }R5d]$$

$$G6.\quad [\ldots]:\cdot:\ [\exists j]::x\subset j.j\subset x::[a]\therefore a\subset j.\equiv:[c\,d]:\sim(c\subset d).c\subset a.\supset.[\exists e\,f].\sim(e\subset f).e\subset c.\varphi(e)\qquad[G5,\ G4,\ S1]$$

$$G7.\quad [g\,j\ldots]:\cdot:\ x\subset j::[a]\therefore a\subset j.\equiv:[c\,d]:\sim(c\subset d).c\subset a.\supset.[\exists e\,f].\sim(e\subset f).e\subset c.\varphi(e)::g\subset x::\supset:[c\,d]:\sim(c\subset d).c\subset g.\supset.[\exists e\,f].\sim(e\subset f).e\subset c.\varphi(e)\qquad[S2]$$

$$G8.\quad [g\,j\ldots]:\cdot:\ j\subset x::[a]\therefore a\subset j.\equiv:[c\,d]:\sim(c\subset d).c\subset a.\supset.[\exists e\,f].\sim(e\subset f).e\subset c.\varphi(e)::[c\,d]:\sim(c\subset d).c\subset g.\supset.[\exists e\,f].\sim(e\subset f).e\subset c.\varphi(e)::\supset.g\subset x\qquad[S2]$$

$$G9.\ [j\ldots]\ :\cdot:\ x\subset j\,.\,j\subset x\,::\,[a]\therefore a\subset j\,.\equiv:\,[c\,d]:\sim(c\subset d).\ c\subset a\,.\supset.\,[\exists e\,f]\,.\sim(e\subset f)\,.\,e\subset c\,.\,\varphi(e)\,::\supset::\,[a]\therefore a\subset x\,.\ \equiv:[c\,d]:\sim(c\subset d)\,.\,c\subset a\,.\supset.\,[\exists e\,f]\,.\sim(e\subset f)\,.\,e\subset c\,.\,\varphi(e)\qquad [G7,\ G8]$$

$$G10.\ [a\ldots]\therefore a\subset x\,.\equiv:\,[c\,d]:\sim(c\subset d)\,.\,c\subset a\,.\supset.[\exists e\,f].\sim(e\subset f)\,.\,e\subset c\,.\,\varphi(e)\qquad [G9,\ G6]$$

G10 satisfying the demands of R5, our task has been completed. To sum up, we have shown that R5 is equivalent to R5d and R6 taken together. This however, constitutes no intuitive justification for accepting R5. Such a justfication cannot be given before we have become acquainted with the properties of certain constant terms which can be introduced into systems of Boolean Algebra with the aid of R4. At this stage it can only be noted that on the ground of *D4* any thesis added to $\mathfrak{S}$ (or $\mathfrak{A}^*$) in virtue of R5 is inferentially equivalent to the corresponding thesis of type

$$\mathrm{X}\quad [a\ldots]\ \therefore a\subset x\,.\equiv:[b]:b\sqsubset a\,.\supset.[\exists d]\ .\,d\sqsubset b\,.\,\varphi(d)$$

And any thesis added to $\mathfrak{S}$ (or $\mathfrak{A}^*$) in virtue of R6 is, on the same ground, inferentially equivalent to the corresponding thesis of the type

$$\mathrm{XI}\quad [\ldots]\ :\cdot:\ [\exists b]::[a]\therefore a\subset b\,.\equiv:[c]\ :c\sqsubset a\,.\supset.[\exists d]\ .\,d\sqsubset c\,.\,\varphi(d)$$

Although the intuitive justification of R5 has to be postponed to a later stage in our enquiry, we can already now indicate certain advantages which result from adopting this type of rule for writing nominal definitions. If we consider the axioms of $\mathfrak{A}^*$ then we see at once that *A2 – A6* satisfy the requirements of R5. Thus in a system equipped with R5 they can be regarded as definitions, and *A1* can be raised to the status of a single axiom of the Algebra. *A1* is a relatively simple thesis. Its meaning becomes apparent once we have realized that on the ground of *D5* it is inferentially equivalent to the thesis which says that

$$[a\,b]\ \therefore a\subset b\,.\ \equiv:[c]\ :c\,\triangle\,a\,.\supset.\,c\,\triangle\,b$$

i.e., that *for all a, for all b, – all a is b if and only if for all c, – if some c is a then some c is b.*

With one exception *A1* meets all the demands set up by Leśniewski for well-constructed axiom systems. In particular it is organic and canonic but, as we shall see in the sequel, it is not adequate for the ontological interpretation.[10)] This does not mean that we were wrong in interpreting Boolean Algebra on the lines suggested in the introduction. *A1* and the system based on it lend themselves to the ontological interpretation without the slightest difficulty. *A1*, however, is adequate for this interpretation because, as we shall see, there are propositions which are meaningful with respect to *A1* but which cannot be deduced from it although under the ontological interpretation their truth appears to be indubitable.

NOTES

1) See Schröder, *Vorlesungen über die Algebra der Logik,* Vol. 1, Leipzig 1890, pp. 168, 169, 188, 196, 293, and 302. Strictly speaking in the axiom-system suggested by Schröder the thesis

$$S7^{*} \quad [a\ b\ c] : b \cap c \subset \Lambda\ .\ \supset .\ a \cap (b \cup c) \subset (a \cap b) \cup (a \cap c)$$

is used instead of S7. Considered separately *S7* is stronger than *S7**, but within the framework of the system these two theses are inferentially equivalent. *S1, S2,* and *S7**, are called principles or axioms while the remaining theses *i.e., S3, S4, S5, S6, S8,* and *S9* are described by Schröder as definitions. See also L. Couturat, L'algèbre de la logique, *Scienta,* No. 24, Paris 1905, principles I to VIII.

In the present paper I shall be using the Peano-Russellian symbolism with a few modifications due to Leśniewski. It will be noted that in this version of the symbolism square brackets indicate the quantifiers. Thus, for instance, the expressions '$[a]$', '$[a\ b\ c]$', '$[\exists\ a]$', and '$[\exists\ a\ b\ c]$' mean the same as *'for all a', 'for all a, for all b, for all c', 'for some a',* and *'for some a, for some b, for some c'.* Concerning the interpretation of the particular quantifier it must be emphasized that it has no existential import. The interpretation of the universal quantifier is to be adapted accordingly. For details see my 'Logic and Existence' in *The British Journal for the Philosophy of Science,* 5 (1954).

2) The terminology 'shared name', 'unshared name' and 'fictitious name' has been suggested by Professor Woodger. See his *Biology and Language,* Cambridge 1952, p. 17, and 'Science without Properties', *The British Journal for the Philosophy of Science,* 2 (1952), p. 196.

3) The 'and' and 'or' as name-forming functors for nominal arguments are to be distinguished from the 'and' and 'or' as used in the Logic of Propositions, where these two words are construed as proposition-forming functors for propositional arguments.

4) See E. Schröder, *op. cit.* p. 217 sq.

5) For the treatment of definitions in the logic developed by Leśniewski see his 'Grundzüge eines neuen Systems der Grundlagen der Mathematik', *Fundamenta Mathematicae* 14 (1929), p. 70 sq., and 'Über die Grundlagen der Ontologie', *Comptes rendus des séances de la Société de Sciences et des Lettres do Varsovie,* Classe III, XXIII Année, Warszawa 1930. See also his 'Über Definitionen in der sogenannten Theorie de Deduktion', *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie,* Class III, XXIV Année, Warszawa 1931. For an informal discussion of definitions in Leśniewski's Ontology see C. Lejewski, 'On Leśniewski's Ontology', *Ratio,* 1 (1958), pp. 172 sq.

6) The functor of a simple function is one word. In this respect a simple function differs from a 'many-link' function, in which the functor is itself a function. Thus, for instance, in the thesis

$$[a\ b] : *\,(-b-)\,(a)\ .\ \equiv\ a \subset b$$

the left hand side of the equivalence is a many-link function. Its functor is the expression '*(–*b*–)', which itself is a function. In '*(–*b*–)' '*' is the functor. Together with one nominal argument it forms a proposition-forming functor for one argument, which again is a name.

7) The constant terms defined in *D1* – *D9* occur in Leśniewski's Ontology. *D2* is due to Sobociński. Se C. Lejewski, *op. cit.*; pp. 157 sq.

8) Definitions of this form do no presuppose *D7*.

9) Theses added to the system in virtue of R6 correspond to what Leśniewski used to call *pseudo-definitions*.

10) For a discussion of the concepts of organicity, canonicity, and adequacy see B. Sobociński, 'On Well Constructed Axiom Systems', *VI Rocznik Polskiego Towarzystwa Naukowego na Obczyźnie, Rok 1955 - 56(The Polish Society of Arts and Sciences Abroad, Yearbook for 1955–56)*, London 1956, 54-64.

To be continued.

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