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Notre Dame Journal of Formal Logic
Volume VIII, Numbers 1 and 2, April 1967

A RECURSIVE MODEL FOR THE EXTENDED SYSTEM $\mathcal{A}$ OF B. SOBOCIŃSKI

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In this note we construct a model in the recursive arithmetic of words over the alphabet $\mathcal{J}_2 = \{S_0, S_1\}$ for the extended system $\mathcal{A}$, which was introduced by B. Sobociński in [1], as a complete extension of author's original system **A** from [2]. With this, an error which appeared in [2], as pointed by B. Sobociński in [1], will now be eliminated.

As Sobociński's system $\mathcal{A}$ is not covered by I. Thomas's general construction in [4], we have to construct the model for $\mathcal{A}$ differently as in [3]. However, the principle is the same.

Presupposing the knowledge of our paper [3], we construct the model as follows. Interpret

(1) Cpq as $[1 \dot{-} \alpha(X)] \cdot Y$;
(2) Np as $S_1 \dot{-} X$;
(3) Kpq as $\alpha(S_1 \dot{-} X) \cdot (X + Y) + [1 \dot{-} \alpha(S_1 \dot{-} X)] \cdot S_1$

and

(4) Apq as
$$[1 \dot{-} \alpha(1 \dot{-} X)] \cdot \{[1 \dot{-} \alpha(1 \dot{-} Y)] \cdot S_1 + [1 \dot{-} \alpha(S_1 \dot{-} Y)] \cdot S_0\}$$
$$+ [1 \dot{-} \alpha(S_1 \dot{-} X)] \cdot \{[1 \dot{-} \alpha(1 \dot{-} Y)] \cdot S_0 + [1 \dot{-} \alpha(S_1 \dot{-} Y)] \cdot S_1\}.$$

We show that under this interpretation all axioms of $\mathcal{A}$ become provable equations of **RAW**; as to the rules of inference of $\mathcal{A}$, RI is the rule of substitution of **RAW** and RII is interpreted as (2.22) of [3], i.e. is provable in **RAW**.

We now interpret every axiom. The numeration of axioms is the numeration of [1]; primed numbers denote equations of **RAW** corresponding to axioms of $\mathcal{A}$ with the same unprimed number.

(F1). The corresponding equation in **RAW** is the equation (3.3) of [3], and was proved there.

(F2) $CNpCpq$.
(F2)' $[1 \dot{-} \alpha(S_1 \dot{-} X)] \cdot [1 \dot{-} \alpha(X)] \cdot Y = 0$.

The easy proof of this equation is by recursion in X.

Received March 11, 1965

(F3) *CCNppNNp*
(F3)' $\{1 \dot{-} [1 \dot{-} \alpha(S \dot{-} X)] \cdot \quad)\} \cdot [S_1 \dot{-} (S_1 \dot{-} X)] = 0.$

(We have used the obvious equations

(5) $\alpha(X \cdot Y) = \alpha(X) \cdot \alpha(Y)$

and

(6) $\alpha[1 \dot{-} \alpha(X)] = 1 \dot{-} \alpha(X)$

to simplify (F3)'). To prove (F3)' denote its left side by $F(X)$. Then $F(0) = S_1 \dot{-} S_1 = 0$, $F(S_0X) = S_1 \dot{-} S_1 = 0$ and $F(S_1X) = (1 \dot{-} 1) \cdot S_1 = 0$. So $F(X) = 0$ for all X.

(F4) *CpCNqNCpq*.
(F4)' $[1 \dot{-} \alpha(X)] \cdot [1 \dot{-} \alpha(S_1 \dot{-} Y)] \cdot \{S_1 \dot{-} [1 \dot{-} \alpha(X)] \cdot Y\} = 0.$

$F(0, Y) = \{1 \dot{-} \alpha(S_1 \dot{-} Y)\} \cdot [S_1 \dot{-} Y] = 0$ by the formula (2.20) of [3]. $F(S_\mu X, Y) = 0$ as $1 \dot{-} \alpha(S_\mu X) = 0$ for $\mu = 0,1$.

(F5) *CNCpqNq*.
(F5)' $(1 \dot{-} \alpha\{S_1 \dot{-} [1 \dot{-} \alpha(X)] \cdot Y\}) \cdot (S_1 \dot{-} Y) = 0.$

To shorten the proofs for equations corresponding to the axioms for conjunction we introduce the function

(7) $K(X, Y) = \alpha(S_1 \dot{-} X) \cdot (X+Y) + [1 \dot{-} \alpha(S_1 \dot{-} S)] \cdot S_1$

We note that

(8) $$\begin{cases} K(0, Y) = Y, \\ K(S_0X, Y) = S_0X+Y, \\ K(S_1X, Y) = S_1. \end{cases}$$

(F6) *CKpqp*.
(F6)' $[1 \dot{-} \alpha(K(X, Y))] \cdot X = 0.$

$F(0, Y) = 0$ as the second factor is 0; $F(S_0X, Y) = [1 \div \alpha(S_0X + Y)] \cdot S_0X$. Let $\psi(Y) = [1 \dot{-} \alpha(S_0X + Y)] \cdot S_0X$. Then $\psi(0) = 0$, $\psi(S_\mu Y) = [1 \dot{-} \alpha\{S_0 \cdot (S_0X + Y)\}] \cdot S_0X = 0$. So $F(S_0X, Y) = 0$. At last, $F(S_1X, Y) = [1 \dot{-} \alpha(S_1)] \cdot S_1X = 0$.

(F7) *CKpqq*.
(F7)' $[1 \dot{-} \alpha(K(X, Y))] \cdot Y = 0$.

Here, $F(0, Y) = [1 \dot{-} \alpha(Y)] \cdot Y = 0$. Other cases as for (F6)'.

(F8) *CpCqKpq*
(F8)' $[1 \dot{-} \alpha(X)] \cdot [1 \dot{-} \alpha(Y)] \cdot K(X, Y) = 0.$

$F(0, Y) = [1 \dot{-} \alpha(Y)] \cdot Y = 0$, $F(S_\mu X, Y) = 0$ as $1 \dot{-} \alpha(S_\mu X) = 0$.

(F9) *CNpNKpq*.
(F9)' $[1 \dot{-} \alpha(S_1 \dot{-} X)] \cdot [S_1 \dot{-} K(X, Y)] = 0.$

$F(0, Y) = F(S_0X, Y) = 0$ as the first factor is 0. $F(S_1X, Y) = S_1 \dot{-} S_1 = 0$.

(F10) *CNqNKpq*.

(F10)' $[1 \dot{-} \alpha(S_1 \dot{-} Y)] \cdot [S_1 \dot{-} K(X,Y)] = 0$.

$F(0,Y) = [1 \dot{-} \alpha(S_1 \dot{-} Y)] \cdot [S_1 \dot{-} Y] = 0$; $F(S_0X,Y) = [1 \dot{-} \alpha(S_1 \dot{-} Y)] \cdot [S_1 \dot{-} (S_0X + Y)] = \psi(Y)$. Now $\psi(0) = \psi(S_0Y) = 0$ as the first factor is 0, and $\psi(S_1Y) = 0$ as the second factor is 0. At last, $F(S_1X,Y) = 0$ as the last factor is $S_1 \dot{-} S_1 = 0$.

(F11) *CNNpCNNqNNKpq*.

(F11)' $(1 \dot{-} \alpha\{S_1 \dot{-} [S_1 \dot{-} X]\}) \cdot (1 \dot{-} \alpha\{S_1 \dot{-} [S_1 \dot{-} Y]\}) \cdot \{S_1 \dot{-} [S_1 \dot{-} K(X,Y)]\} = 0$.

$F(0,Y) = [1 \dot{-} \alpha\{S_1 \dot{-} (S_1 \dot{-} Y)\}] \cdot [S_1 \dot{-} (S_1 \dot{-} Y)] = 0$; $F(S_0X,Y) = [1 \dot{-} \alpha\{S_1 \dot{-} (S_1 \dot{-} Y)\}] \cdot \{S_1 \dot{-} [S_1 \dot{-} (S_0X + Y)]\} = \psi(Y)$.

Now $\psi(0) = S_1 \dot{-} (S_1 \dot{-} S_0X) = S_1 \dot{-} S_1 = 0$, $\psi(S_0Y) = 0$ as the last factor is 0, and $\psi(S_1Y) = 0$ as the first factor becomes $1 \dot{-} \alpha(S_1) = 0$. So $F(S_0X,Y) = 0$. At last, $F(S_1X,Y) = 0$ as then the first factor in (F11)' becomes $1 \dot{-} \alpha(S_1) = 0$.

To shorten the proofs for equations corresponding to axioms for disjunction, we introduce the function

$$(9)\quad A(X,Y) = [1 \dot{-} \alpha(1 \dot{-} X)] \cdot \{[1 \dot{-} \alpha(1 \dot{-} Y)] \cdot S_1 + [1 \dot{-} \alpha(S_1 \dot{-} Y)]\} + + [1 \dot{-} \alpha(S_1 \dot{-} X)] \cdot \{[1 \dot{-} \alpha(1 \dot{-} Y)] + [1 \dot{-} \alpha(S_1 \dot{-} Y)] \cdot S_1\}.$$

We note:

$$(10)\quad \begin{cases} A(0,Y) = 0; \\ A(S_0X,Y) = [1 \dot{-} \alpha(1 \dot{-} Y)] \cdot S + [1 \dot{-} \alpha(S_1 \dot{-} Y)] \; ; \\ A(S_1X,Y) = [1 \dot{-} \alpha(1 \dot{-} Y)] + [1 \dot{-} \alpha(S_1 \dot{-} Y)] \cdot S_1 \; ; \end{cases}$$

$$(11)\quad A(X,0) = 0 \, .$$

(F12) *CpApq*.

(F12)' $[1 \dot{-} \alpha(X)] \cdot A(X,Y) = 0$

$F(0,Y) = A(0,Y) = 0 \cdot F(S_\mu X,Y) = 0$ as the first factor is 0 for $\mu = 0,1$.

(F13) *CqApq*

(F13)' $[1 \dot{-} \alpha(Y)] \cdot A(X,Y) = 0$.

The easy proof by recursion in Y is omitted.

(F14) *CApqCCprCCqrr*.

(F14)' $\{1 \dot{-} \alpha[A(X,Y)]\} \cdot \{1 \dot{-} \alpha[(1 \dot{-} \alpha(X)) \cdot Z]\} \cdot \cdot \{1 \dot{-} \alpha[(1 \dot{-} \alpha(Y)) \cdot Z]\} \cdot Z = 0$.

or, using (5) and (6),

(F14)'' $\{1 \dot{-} \alpha[A(X,Y)]\} \cdot \{1 \dot{-} [1 \dot{-} \alpha(X)] \cdot \alpha(Z)\} \cdot \cdot \{1 \dot{-} [1 \dot{-} \alpha(Y)] \cdot \alpha(Z)\} \cdot Z = 0$.

First, we have $F(0,Y,Z) = \{1 \dot{-} \alpha(Z)\} \cdot \{1 \dot{-} [1 \dot{-} \alpha(Y)] \cdot \alpha(Z)\} \cdot Z = \psi(Y,Z)$. Now $\psi(Y,0) = 0$, as the last factor is 0, and $\psi(Y,S_\mu Z) = 0$ as $1 \dot{-} \alpha(S_\mu Z) = 0$. Therefore, $F(0,Y,Z) = 0$. Also, $F(S_0X,Y,Z) = \{1 \dot{-} \alpha[A(S_0X,Y)]\} \cdot \{1 \dot{-} [1 \dot{-} \alpha(Y)] \cdot \alpha(Z)\} \cdot Z = \psi_1(Y,Z)$. Now $\psi_1(0,Z) = \{1 \dot{-} \alpha[S_1]\} \cdot \{1 \dot{-} \alpha(Z)\} \cdot Z = 0$, $\psi_1(S_0Y,Z) = \{1 \dot{-} \alpha(S_1)\} \cdot Z = 0$ and $\psi_1(S_1Y,Z) = \{1 \dot{-} \alpha(1)\} \cdot Z = 0$. So

$F(S_0X, Y, Z) = 0$. At last $F(S_1X, Y, Z) = \{1 \dot{-} \alpha[A(S_1X, Y)]\} \cdot \{1 \dot{-} [1 \dot{-} \alpha(Y)] \cdot \alpha(Z)\} \cdot Z = \varphi(Y, Z)$. Further, $\phi(0, Z) = \{1 \dot{-} [1 \dot{-} 0] \cdot \alpha(Z)\} \cdot Z = (1 \dot{-} \alpha(Z)) \circ Z = 0$; $\phi(S_0Y, Z) = \{1 \dot{-} \alpha(1)\} \cdot \{. . .\} \cdot Z = 0$ and $\phi(S_1Y, Z) = \{1 \dot{-} \alpha(S_1)\} \cdot \{. . .\} \cdot Z = 0$. Therefore $F(S_1X, Y, Z) = 0$.

(F15) *CNApqCNpNq*.

(F15)' $\{1 \dot{-} \alpha[S_1 \dot{-} A(X, Y)]\} \cdot \{1 \dot{-} \alpha(S_1 \dot{-} X)\} \cdot (S_1 \dot{-} Y) = 0$.

$F(0, Y) = \{1 \dot{-} \alpha[S_1]\} \cdot \{1 \dot{-} \alpha(S_1)\} \cdot (S_1 \dot{-} Y) = 0$; $F(S_0X, Y) = \{1 \dot{-} \alpha[S_1 \dot{-} A(S_0X, Y)]\} \cdot \{1 \dot{-} \alpha(S_1 \dot{-} S_0X)\} \cdot (S_1 \dot{-} Y) = \{. . .\} \cdot \{1 \dot{-} \alpha(S_1)\} \cdot (. . .) = 0$; $F(S_1X, Y) = (1 \dot{-} \alpha[S_1 \dot{-} \{[1 \dot{-} \alpha(1 \dot{-} Y)] + [1 \dot{-} \alpha(S_1 \dot{-} Y)] \cdot S_1\}]) \cdot (S_1 \dot{-} Y) = \psi(Y)$. Then $\psi(0) = (1 \dot{-} \alpha[S_1]) \cdot S_1 = 0$, $\psi(S_0Y) = (1 \dot{-} \alpha[S_1 \dot{-} \{0\}]) \cdot S_1 = 0$ and $\psi(S_1Y) = (\ ) \cdot (S_1 \dot{-} S_1Y) = 0$. Therefore $F(S_1X, Y) = 0$ too.

(F16) *CNApqCNqNp*.

(F16)' $\{1 \dot{-} \alpha[S_1 \dot{-} A(X, Y)]\} \cdot [1 \dot{-} \alpha(S_1 \dot{-} Y)] \circ (S_1 \dot{-} X) = 0$.

The proof is similar to the proof of (F15)'.

(F17) *CNpCNqNApq*.

(F17)' $[1 \dot{-} \alpha(S_1 \dot{-} X)] \cdot [1 \dot{-} \alpha(S_1 \dot{-} Y)] \cdot [S_1 \dot{-} A(X, Y)] = 0$.

$F(0, Y) = [1 \dot{-} \alpha(S_1)] \cdot [. . .] \cdot [. . .] = 0$;
$F(S_0X, Y) = [1 \dot{-} \alpha(S_1)] \cdot [. . .] \cdot [. . .] = 0$.
$F(S_1X, Y) = [1 \dot{-} \alpha(S_1 \dot{-} Y)] \cdot [S_1 \dot{-} A(S_1X, Y)] = \psi[Y]$.

Now $\psi(0) = [1 \dot{-} \alpha(S_1)] \cdot [. . .] = 0$, $\psi(S_0Y) = [1 \dot{-} \alpha(S_1)] \cdot [. . .] = 0$ and at last $\psi(S_1Y) = S_1 \dot{-} A(S_1X, S_1Y) = S_1 \dot{-} S_1 = 0$. So $F(S_1X, Y) = 0$ too.

(F18) *CCpNpCCqNqCNNpCNNqNApq*.

Using (5) and (6) we can write the corresponding equation as

(F18)' $\{1 \dot{-} [1 \dot{-} \alpha(X)] \cdot \alpha(S_1 \dot{-} X)\} \cdot \{1 \dot{-} [1 \dot{-} \alpha(Y)] \cdot \alpha(S_1 \dot{-} Y)\} \cdot$
$\cdot \{1 \dot{-} \alpha[S_1 \dot{-} (S_1 \dot{-} X)]\} \cdot \{1 \dot{-} \alpha[S_1 \dot{-} (S_1 \dot{-} Y)]\} \cdot [S_1 \dot{-} A(X, Y)] = 0$.

First, $F(0, Y) = \{1 \dot{-} \alpha(S_1)\} \cdot \{. . .\} \cdot \{. . .\} \cdot \{. . .\} \cdot [. . .] = 0$. Secondly, $F(S_0X, Y) = \{1 \dot{-} [1 \dot{-} \alpha(Y)] \cdot \alpha(S_1 \dot{-} Y)\} \cdot \{1 \dot{-} \alpha[S_1 \dot{-} (S_1 \dot{-} Y)]\} \cdot [S_1 \dot{-} A(S_0X, Y)] = \psi(Y)$. Now $\psi(0) = \{1 \dot{-} \alpha(S_1)\} \cdot \{. . .\} \cdot [. . .] = 0$, $\psi(S_0Y) = S_1 \dot{-} A(S_0X, S_0Y) = S_1 \dot{-} S_1 = 0$ and $\psi(S_1Y) = \{1 \dot{-} 0\} \cdot \{1 \dot{-} \alpha[S_1]\} \cdot [. . .] = 0$. Therefore $F(S_0X, Y) = 0$. At last $F(S_1X, Y) = 0$ as the third factor in (F18)' becomes 0 in this case.

This brings to the end the proof that all axioms (F1)-(F18) become provable equations in the model. Therefore, every thesis of the system A is verified in the model.

LITERATURE

[1] B. Sobociński: On the Propositional System A of Vučković and its Extension. I. *Notre Dame Journal of Formal Logic* 5 (1964), 141-153.

[2] V. Vučković: Recursive Models of Some Non-Classical Propositional Calculi. *Filosofija* (Yugoslav J. of Philosophy), 4 (1960), 69-84. (Serbian; German summary).

[3] V. Vučković: Recursive Models for Three-Valued Propositional Calculi with Classical Implication. *Notre Dame Journal of Formal Logic,* v. VIII (1967), pp. 148-153.

[4] I. Thomas: Three-Valued Propositional Calculus with Classical Implication. *Notre Dame Journal of Formal Logic,* v. VIII (1967), pp. 145-147.

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