Notre Dame Journal of Formal Logic
Volume VIII, Numbers 1 and 2, April 1967

RECURSIVE MODELS FOR THREE-VALUED PROPOSITIONAL CALCULI WITH CLASSICAL IMPLICATION

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1. *Introduction.* The aim of this paper is to complete the author's paper [1], exhibiting various systems of propositional calculi which have models inside the recursive arithmetic of words. We limit our exposition to three-valued case; nevertheless, the method can be applied to the calculi with more than 3 truth-values.

In the elaboration of this paper we considered first four such systems, which raised naturally in an attempt to eliminate an error in our paper [1], which was remarked by B. Sobociński in [2] and [3], and we gave the proofs of their completeness along the lines of the well-known Kalmar proof for the completeness of the classical propositional calculus. Later discussions with I. Thomas ([6]) contributed to look for models of general three-valued propositional fragments with classical implication. As now the paper [6] provides the proof of completeness we restrict ourself to the construction of models only.

2. *Recursive arithmetic of words.* Recursive arithmetic of words (short: **RAW**) is an equation calculus over the words of an alphabet

(2.1) $$\mathcal{J}_n = \{S_0, S_1, \ldots, S_{n-1}\}$$

with more than one letter, which is built up as follows.

Denote the empty word by 0.

Introduce $n + 2$ initial functions

(2.2) $$Z(X) = 0,$$
(2.3) $$I(X) = X$$

and

(2.4) $$S_i(X) = S_iX, \quad i = 0, 1, \ldots, (n-1)$$

where S_iX is the word obtained from the word X by writing the letter S_i on its beginning. All variables X, Y, Z (with possible indices) run over the set $\Omega(\mathcal{J}_n)$ of all words written by letters of $\mathcal{J}_n$ (and also over the empty word, which is supposed to be a member of $\Omega(\mathcal{J}_n)$). Formation rules are the

Received March 11, 1965.

substitution of functions and words for variables and the definitions by primitive recursion.

A function $f(X_1, \dots, X_m, Y)$ is defined by simple primitive recursion by the following $(n+1)$ equations:

$$(2.5)\qquad f(X_1,..,X_m, 0) = a(X_1,..,X_m)$$
$$f(X_1,..,X_m,S_iY) = b_i(X_1,..,X_m,Y, f(X_1,...,X_m, Y)),\ i = 0,..,n-1$$

where a and all b_i are or initial functions or previously defined by the scheme (2.5). A function $f(X,Y)$ is defined by double primitive recursion by the following $n^2 + n + 1$ equations:

$$(2.6)\qquad \begin{aligned} f(X, 0) &= a(X), \\ f(0, S_iY) &= b_i(Y),\ i = 0,..,n-1, \\ f(S_iX, S_jY) &= c_{ij}(X, Y, f(X, Y)),\ i,j = 0,..,n-1 \end{aligned}$$

where a, all b_i and all c_{ij} are or initial function or previously defined by (2.5) or (2.6). A function is primitive recursive if it is an initial function, or if it is defined by primitive recursion (of both types), or if it is obtained from such a function by substitution with such functions. We note that (2.6) can be reduced to (2.5) (see f.i. [4]). We introduce (2.6) in order to simplify the exposition.

The only expressions which form **RAW** are equations between primitive recursive functions. We admit only proved equations. An equation $f = \phi$ between two word-functions is proved, if and only if f and ϕ satisfy the same defining equations (2.5) or (2.6), or if f and ϕ are obtained from such functions by the same substitutions. It can be proved that **RAW** is non-contradictory in the following sense: if the equation

$$f(X_1,..,X_m) = g(X_1,..,X_m)$$

is proved and if $A_1,..,A_m$ are any words in $\Omega(\mathcal{J}_n)$, then $f(A_1,..,A_m)$ and $g(A_1,.., A_m)$ are one and the same word. A complete exposition of **RAW** is given in [5]. Here we present a very minor part of it, which is sufficient for our purposes. We need first n additive operations Xo_iY, which are defined by $(i = 0,..,n-1)$

$$(2.7)\qquad \begin{aligned} Xo_i0 &= X \\ Xo_iS_j Y &= S_{i+j}(Xo_iY),\quad j = o,..,n-1. \end{aligned}$$

The addition $i + j$ of indices is modulo n.

Especially, the operation o_0 is called addition and denoted by $+$. We repeat its definition:

$$(2.8)\qquad \begin{aligned} X + 0 &= X \\ X + S_j Y &= S_j(X + Y),\quad j = o,..,n-1. \end{aligned}$$

$X + Y$ is the concatenation YX. Oo_iX, written simply as o_iX, is obtained from X by augmenting the indices of all letters of X for i, modulo n. We note a few proved equations; on the right side we refer to the corresponding equation of [5].

$$(2.9)\qquad Xo_iY = X + o_iY.$$

(2.10) $$0 + X = X.$$
(2.11) $$X + (Y + Z) = (X + Y) + Z.$$

The multiplication $X \cdot Y$ is defined by

(2.12) $$\begin{gathered} X \cdot 0 = 0 \\ X \cdot S_j Y = (X \cdot Y) + o_j X, \quad j = o, \dots, n\text{-}1. \end{gathered}$$

Note that

(2.13) $$S_0 \cdot X = X \cdot S_0 = X,$$

which suggests consideration of S_0 as the unit for multiplication. Therefore, we write sometimes 1 for S_0.

The difference $X \dot{-} Y$ is defined by double primitive recursion:

(2.14) $$\begin{gathered} X \dot{-} 0 = X, \\ 0 \dot{-} S_j Y = 0, \; j = o, \dots, n\text{-}1, \\ S_i X \dot{-} S_j Y = \begin{cases} X \dot{-} Y, & \text{if } i = j, \\ S_i(X \dot{-} Y) & \text{if } i \neq j \end{cases} \quad i, j = o, \dots, n\text{-}1. \end{gathered}$$

Some elementary properties of the difference are

(2.15) $$0 \dot{-} X = 0.$$ (5.10)
(2.16) $$Y \dot{-} (X + Y) = 0.$$ (5.11)
(2.17) $$(Y + X) \dot{-} X = Y.$$ (5.12)
(2.18) $$X \dot{-} X = 0.$$ (5.14)

Note that $1 \dot{-} S_i$ is 0 if and only if $i = 0$. In all other cases $1 \dot{-} S_i = 1$.

The last function to be introduced is $\alpha(X)$:

(2.19) $$\begin{gathered} \alpha(0) = 0, \\ \alpha(S_i X) = 1, \; i = o, \dots, n\text{-}1. \end{gathered}$$

We quote:

(2.20) $$(1 \dot{-} \alpha(X)) \cdot X = 0.$$ (6.16)

If we define the absolute difference $|X, Y|$ by

(2.21) $$|X, Y| = (X \dot{-} Y) + (Y \dot{-} X),$$

it can be proved that $X = Y$ is equivalent with $|X, Y| = 0$ ([5], (7.3)). Therefore: every equation in **RAW** can be put in the form $f = 0$.

Finally, note the validity of the proof-schema:

(2.22) $$\frac{\begin{gathered} X = 0 \\ (1 \dot{-} \alpha(X)) \cdot Y = 0 \end{gathered}}{Y = 0.}$$

whose meaning is: if the first two rows are provable, then the third row is provable.

3. *Fundamental equations*: Here we present that part of **RAW** which is needed for the construction of models, limiting ourselves to a **RAW** over the alphabet $\mathcal{J}_2 = \{S_0, S_1\}$ with two letters.

Introduce two functions

(3.1) $N_i(X) = \alpha(S_i \dot{-} X), \quad i = 0,1.$

Remark that

(3.2) $N_i(X) = 0$ if and only if $X = S_i Y$; in other cases $N_i(X) = 1$.

The following set of equations is easily provable. There, i and j take the values 0 and 1.

(3.3) $\{1 \dot{-} \alpha[(1 \dot{-} \alpha[(1 \dot{-} \alpha(X))\cdot Y])\cdot Z\}\cdot\{1 \dot{-} \alpha[(1 \dot{-} \alpha(Z))\cdot X]\}\cdot[1 \dot{-} \alpha(V)]\cdot X = 0;$

(3.4.i) $[1 \dot{-} \alpha(X)]\cdot N_0(N_i(X)) = 0;$

(3.5.i,j) $\{1 \dot{-} \alpha[N_i(X)]\}\cdot N_0(N_j(X)) = 0, \quad i \neq j;$

(3.6.i) $\{1 \dot{-} \alpha\{(N_i(X)\}\cdot[1 \dot{-} \alpha(X)]\cdot Y = 0;$

(3.7.i) $[1 \dot{-} \alpha(X)]\cdot\{1 \dot{-} \alpha[N_i(Y)]\}\cdot N_i\{[1 \dot{-} \alpha(X)]\cdot Y\} = 0;$

(3.8) $\{1 \dot{-} \alpha[\{1 \dot{-} \alpha[N_1(X)]\}\cdot X]\}$
$\cdot\{1 \dot{-} \alpha[\{1 \dot{-} \alpha[N_0(X)]\}\cdot X]\}\cdot X = 0.$

F.i. to prove (3.3) denote its left side by $f(X,Y,Z,V)$. Then $f(0,Y,Z,V) = 0$ and $f(S_k X,Y,Z,V) = [1 \dot{-} \alpha(Z)]\cdot\{1 \dot{-}[1 \dot{-} \alpha(Z)]\}\cdot[1 \dot{-} \alpha(V)] = 0$, as easily seen by recursion in Z. To prove (3.4.i) it suffices to show that the left side is 0 for $X = 0$. As $N_i(0) = 1$ and $N_0(1) = 0$ (by (3.2)), the result follows. Remaining equations are provable in a similar way. To shorten the exposition we write $N_2(X)$ for X and by $X = S_2 Z$ we mean $X = 0$.

Call a word function $\tilde{f}$, whose range is in $\{0, S_0, S_1\}$ regular if from

(3.9) $\tilde{f}(S_{i_1}, S_{i_2}, \ldots, S_{i_n}) = S_{i,n-1},$

where every i_k is or o, or 1 or 2 (in the last case S_2 means 0), follows

(3.10) $\tilde{f}(S_{i_1} Z_1, S_{i_2} Z_2, \ldots, S_{i_n} Z_n) = S_{i_{n-1}},$

for any $Z_1, Z_2, \ldots, Z_n \in \Omega(S_2)$.

Every regular function can be defined in the following way. First, by "truth tables" we define a mapping f of the set $\{0, S_0, S_1\}$ into $\{0, S_0, S_1\}$. The truth table has 3^n rows and $n+1$ columns: (we write italics for variables running only over letters and the empty word)

(3.11)

x_1	x_2	$\ldots$	x_n	$f(x_1, x_2, \ldots, x_n)$
0	0		0	$f(0,0,\ldots,0)$
0	0		S_0	$f(0,0,\ldots,S_0)$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
S_1	S_1		S_1	$f(S_1,S_1,\ldots,S_1)$.

Then define $\tilde{f}$ by

(3.12) $\tilde{f}(S_{i_1} Z_1, S_{i_2} Z_2, \ldots, S_{i_n} Z_n) = f(S_{i_1}, S_{i_2}, \ldots, S_{i_n}),$

for any $Z_1, \ldots, Z_n$.

$\tilde{f}$ is defined by 3^n conditions, so it is primitive recursive. Let $\tilde{f}$ be a regular function. To every row of the truth table for the corresponding f, say to j-th row, we assign the function

(3.13) $\psi_j(X_1,\dots,X_n) = [1 \dot{-} \alpha(X_1')]\dots\dots[1 \dot{-} \alpha(X_n')] \cdot \tilde{f}'(X_1,\dots,X_n)$,

where

$$X_i' = \begin{cases} X_i & \text{, if in the } i\text{-th column of } j\text{-th row stands } S_2 \text{ (i.e. } 0) \\ N_0(X_i) & \text{if in the } i\text{-th column of } j\text{-th row stands } S_0 \\ N_1(X_i) & \text{if in the } i\text{-th column of } j\text{-th row stands } S_1 \end{cases}$$

and where

$$\tilde{f}' = \begin{cases} \tilde{f} & \text{, if in the } (n+1)\text{-th column of } j\text{-th row stands } S_2 \\ N_0(\tilde{f}) & \text{, if in the } (n+1)\text{-th column of } j\text{-th row stands } S_0 \\ N_1(\tilde{f}) & \text{, if in the } (n+1)\text{-th column of } j\text{-th row stands } S_1 \text{ .} \end{cases}$$

We prove: for every $j = 1, 2, \dots, 3^n$

(3.14) $$\psi_j(X_1,\dots,X_n) = 0.$$

Remark that

$$\psi_j = \left\{\prod_{i=1}^{n} [1 \dot{-} \alpha(N_{j_i}(X_i)\right\} \cdot N_{j_{n-1}}(\tilde{f}(X_1,\dots,X_n)),$$

where $\prod_1^n \alpha_i = \alpha_1, \alpha_2 \dots \alpha_n$.

The expression in { } is $\neq 0$ if and only if $N_{j_i}(X_i) = 0$, $i = 1,\dots,n$. By the definition of functions N_k, $k = 0,1,2$

$N_{j_i}(X_i) = 0$ if and only if $X_i = S_{j_i} Z_i$.

As then

$$\tilde{f}(X_1,\dots,X_n) = \tilde{f}(S_{j_1}Z_1,\dots,S_{j_n}Z_n) = S_{j_{n+1}}$$

we have

$$N_{j_{n+1}}(\tilde{f}(X_1,\dots,X_n)) = N_{j_{n+1}}(S_{j_{n+1}}) = 0.$$

This proves (3.14). We make the convention that (3.14) stands for all 3^n such equations.

4. Construction of models. To construct models for the propositional fragments of [6] interpret

(4.1) Cpq as $[1 \dot{-} \alpha(X)] \cdot Y$,

(4.2) $N_1 p$ as $N_0(X)$

and

(4.3) $N_2 p$ as $N_1(X)$.

Every proposition involving C, N_1 and N_2 is interpreted in **RAW** as an equation, with 0 on the right side and with the corresponding interpretation of its symbols by means of (4.1)-(4.3) on the left side. F.1. CpN_1N_jp becomes the equation

$$[1 \dot{-} \alpha(X)] \cdot N_0(N_j(X)) = 0, j = 0,1.$$

If $\phi(x_1,\dots,x_n)$ is any n-argument functor, as his representant we introduce the regular function $f(X_1,\dots,X_n)$ defined as follows:

To the values 0,1,2 of arguments x_i and of $\phi(x_1, \ldots, x_n)$ for an assignment of those values in the truth table of ϕ, we correspond the words $0, S_0, S_1$ respectively. In this way, we define first a mapping f with domain $\{0, S_0, S_1\}$ and with the range in the same set. For $\tilde{f}$ we take then the regular extension of f, as defined by (3.12). With this, the first 6 rows of axioms in [6] become equations (3.3)-(3.8) of section 3 of this paper, and the 3^n axioms in the row 7 of the axiom list of [6] becomes 3^n equations (3.14). (2.22) becomes the detachment rule

$$\frac{\begin{array}{l}\vdash \alpha \\ \vdash C\alpha\beta\end{array}}{\vdash \beta}$$

and as a substitution rule, corresponding to the substitution rule of the propositional calculus, is valid in **RAW** we conclude: if any proposition is provable in the propositional fragment of [6], its corresponding equation in **RAW** is provable too.

Remark. To construct corresponding models for n-valued calculi we have to use an **RAW** over the alphabet with n-1 letters.

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