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FUNCTIONALS DEFINED BY RECURSION

LUIS ELPIDIO SANCHIS

Recursive functionals of finite type have been studied by several authors in recent years. The class of functionals that can be defined by primitive recursion of finite type is certainly more constructive than other more inclusive classes, as the general recursive functionals studied by Kleene. Moreover functionals defined by recursion are sufficient for the interpretation of formal systems of number theory in the manner described by Gödel in [3].

In this paper we study a formalization of the class of functionals which are closed under explicit definition and recursion. Combinators, first studied in combinatory logic, play a central role in this formalization. They are used first to obtain closure under explicit definition and second to formalize definitions by recursion. For this purpose new operators of a special kind must be introduced. But they behave in a manner quite similar to ordinary combinators, and we intend to use the same name for both kinds of operators. The system is constructed as an equation calculus in the usual way in combinatory logic. It is proved that the rules are complete in the sense that whenever an equation with variables is derivable, the corresponding equation (without variables) of higher type is also derivable without using variables. This generalizes the well known principle of extensionality in combinatory logic.[1] We also analyse a kind of reduction of terms by means of replacements. It is shown that every constant term of the type of natural numbers can be reduced in that way to a numeral. This can be generalized for constant terms of higher type and the result is applied to prove the consistency of the equation calculus. Results of the same sort, were obtained by Tait in [11].

We have used several ideas and methods that are current in combinatory logic, but the paper is self contained. In the work of Curry it has been customary to avoid the assignment of a definite type to the combinators. We shall depart from this procedure by requiring every entity of the system to have a definite type. We need in this way to assume an infinite number of combinators. Grzegorczyk has studied in [4] a very similar formalism. The standpoint there is mainly semantical. We plan to discuss in a forthcoming paper the possibility of formalizing the arguments of [4] in our

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system. We must mention also that the application of combinators to recursive functionals appears also in the work of Curry [1] and Lercher [8].

1. Types and terms. We shall define a set of entities called *types*, and a set of entities called *terms*. Each term will be of a definite type.

1.1 There is one primitive type, denoted with the letter N, which corresponds to the set of natural numbers. If α and β are types, then $(\mathbf{F}\alpha\beta)$ is also a type which corresponds to functionals with argument in α and value in β. We shall use the notation

$$\mathbf{F}_{n+1}\alpha_1 \ldots \alpha_{n+1}\beta \quad \text{for} \quad \mathbf{F}\alpha_1(\mathbf{F}_n\alpha_2 \ldots \alpha_{n+1}\beta)$$
$$\mathbf{F}_0\beta \quad \text{for} \quad \beta$$

It is clear that every type has a unique representation in the form $\mathbf{F}_n\alpha_1 \ldots \alpha_n N$ with $n \geqq 0$.

1.2 The terms are obtained from some given primitive atoms by a binary operation called *application*. The primitive atoms are *combinators*, *constants* and *variables*.

1.2.1 Combinators. If α, β and γ are types there are combinators

$\mathbf{I}_\alpha$	of type	$\mathbf{F}\alpha\alpha$
$\mathbf{K}_{\alpha\beta}$	of type	$\mathbf{F}_2\beta\alpha\beta$
$\mathbf{S}_{\alpha\beta\gamma}$	of type	$\mathbf{F}_3(\mathbf{F}_2\alpha\beta\gamma)(\mathbf{F}\alpha\beta)\alpha\gamma$
$\mathbf{R}_\alpha$	of type	$\mathbf{F}_3(\mathbf{F}_2 N\alpha\alpha)\alpha N\alpha$

1.2.2 Constants. There are only two constants:

O	of type	N
$\mathcal{J}$	of type	$\mathbf{F}NN$

1.2.3 Variables. For each type there are infinitely many variables of that type.

1.2.4 Application. If X is a term of type $\mathbf{F}\alpha\beta$ and Y is a term of type α, then (XY) is a term of type β.

Parentheses will be omitted with the usual conventions in combinatory logic. For instance in place of $(((XY)Z)U)$ we shall write $XYZU$. In this way every term has a unique representation in the form $X_1 \ldots X_n$, $n \geqq 1$, where X_1 is a primitive atom.

1.3 Letters U, V, W, X, Y and Z, with or without subscripts, are used for terms; letters x, y and z, with or without subscripts, are used for variables. Most of the time we shall not indicate the type of terms or variables we are talking about; we shall also omit the subscripts of the combinators. In that case it must be understood that the assertions about the terms and combinators hold for every reasonable assignment of types and subscripts.

The expression $X \equiv Y$ means that X and Y are exactly the same term. Terms of type N are called *numerical terms*. A *constant term* is a term which does not contain variables; the combinators are constant terms, but are not constants in the sense of **1.2.2**.

1.4 Numerals. O is a numeral. If X is a numeral then $\mathcal{J}X$ is also a numeral. We denote with O^n the numeral containing exactly n occurrences of $\mathcal{J}$.

1.4.1 Closed terms. A term of the form $X_1 \dots X_n$, $n \geqq 1$, where X_1 is either a constant or a variable, is called a *closed term.* The terms $X_2, \dots, X_n$ are called the *arguments*.

1.4.2 Substitution. Let $X, Y_1, \dots, Y_n$, $n \geqq 1$, be terms and $x_1, \dots, x_n$ be distinct variables such that Y_i and x_i are of the same type. The expression

$$[Y_1, \dots, Y_n/x_1, \dots, x_n]X$$

denotes the term obtained by simultaneous *substitution* of $Y_1, \dots, Y_n$ for $x_1, \dots, x_n$, respectively, in X.[2]

2. Conversion. A *redex* is a term of one of the forms in the list below. With each redex we associate a term of the same type which is called the *contractum* of the redex. Letters X, Y and Z in the list stand for arbitrary terms of the corresponding types. We recall the convention stated in the first paragraph of **1.3.**

REDEX	CONTRACTUM
$\mathbf{I}X$	X
$\mathbf{K}XY$	X
$\mathbf{S}XYZ$	$XZ(YZ)$
$\mathbf{R}XYO$	Y
$\mathbf{R}XYO^{k+1}$	$XO^k(\mathbf{R}XYO^k)$

Let X be a term containing disjoint redexes $U_1, \dots, U_n$, $n \geqq 0$. Here U_i denotes a definite occurrence in X. Let $V_1, \dots, V_n$ be the contracta of those redexes. If Y is the result of replacing each U_i by V_i in X, we say that Y is a *contraction* of X. If $n = 1$ we say that Y is a *simple contraction* of X.

We say that X *reduces* to Y with length k, and we write $X \ \mathbf{red}_k \ Y$ or $X \ \mathbf{red} \ Y$, if there is a finite sequence of terms $X_1, \dots, X_k$ such that $X \equiv X_1$, $Y \equiv Y_k$ and for every $i > 1$, X_i is a contraction of X_{i-1}.

The following properties of the reduction relation follow easily from the definition.

X **red** X
X **red** Y *and* Y **red** Z *then* X **red** Z
X **red** Y *then* $[X/z]\,Z$ **red** $[Y/x]\,Z$
X **red** Y *then* $[Z/x]X$ **red** $[Z/x]\,Y$

2.1 Abstraction. Let X be a term of type α and x a variable of type β. The term $[x]X$ of type $\mathbf{F}\beta\alpha$ is defined by the following rules:

(A1) If $X \equiv x$ *then* $[x]X \equiv \mathbf{I}$
(A2) *If* X *is atomic distinct from* x, *then* $[x]X \equiv \mathbf{K}X$
(A3) If $X \equiv YZ$, $U \equiv [x]\,Y$, $V \equiv [x]\,Z$, *then* $[x]X \equiv \mathbf{S}UV$

Reduction and abstraction are related by the following property that can be proved easily by induction on the structure of X:

$$([x]X)V \ \mathbf{red} \ [V/x]X$$

The preceding definition can be generalized for several variables:

$$[x_1, \ldots, x_{n+1}]X \equiv [x_1, \ldots, x_n][x_{n+1}]X$$

If $x_1, \ldots, x_n$ are distinct variables we have

$$([x_1, \ldots, x_n]X)V_1 \ldots V_n \ \mathbf{red}\ [V_1, \ldots, V_n/x_1, \ldots, x_n]X$$

2.2 Residuals. Let X be a term and $U_1, \ldots, U_n$, $n \geqq 0$, be disjoint redexes in X. Let Y be the contraction of X obtained by replacing each U_i by its contractum V_i. Suppose Z is some redex in X. We define the *residuals* of Z in Y by the following rules:

(R1) *If Z is one of the redexes $U_1, \ldots, U_n$ there is no residual of Z in Y.*
(R2) *If Z is disjoint from all redexes $U_1, \ldots, U_n$ then the residual of Z is the corresponding part of the same form which is not affected by the replacement.*
(R3) *If Z is a part of U_i there are in V_i no, one or two parts that correspond to Z. Since V_i is a part of Y those parts are also in Y and are the residuals of Z in Y.*
(R4) *If Z contains $U_{i_1}, \ldots, U_{i_k}$ and Z_1 is obtained by replacing those redexes in Z by their contracta, then Z_1 is a part of Y and is the residual of Z in Y.*

We note that every residual of Z is a redex which corresponds to the same combinator as Z; also two residuals of Z are disjoint in Y. Furthermore if Z_1 and Z_2 are disjoint redexes in X then the residuals of Z_1 and Z_2 in Y are also disjoint.

Now let X be a term, $U_1, \ldots, U_n$ disjoint redexes in X and $V_1, \ldots, V_m$ also disjoint redexes in X, $n \geqq 0$, $m \geqq 0$. Let Y_1 be the contraction of X corresponding to the redexes $U_1, \ldots, U_n$ and Y_2 the contraction of X corresponding to $V_1, \ldots V_m$. Then if we replace in Y_1 the residuals of $V_1, \ldots, V_m$ by their contracta, and we replace in Y_2 the residuals of $U_1, \ldots, U_n$ by their contracta, we get the same term Z. For suppose $n = 1$; if U_1 is a part of V_j and we consider all possible forms of V_j we shall see that it is the same to contract first U_1 and then the residual of V_j or to contract V_j and then the residuals of U_1. If U_1 contains $V_{j_1}, \ldots, V_{j_k}$ again considering the possible forms of U_1 we get the same result. The same analysis can be made if $m = 1$. Now if both $n > 1$ and $m > 1$ we can take the redexes in one group that are not contained in redexes of the other group and in this way the situation is reduced to cases of the form $n = 1$. From this property follows the following Lemma.

Lemma 1. *If Y_1 and Y_2 are contractions of X there is a term Z which is a contraction of Y_1 and also a contraction of Y_2.*

2.3 The result of Lemma 1 holds if we take reductions in place of contractions. This is called in the literature the Church-Rosser property.[3]

Lemma 2. *If X $\mathbf{red}_k$ Y and U is a contraction of X, then there is a Z which is a contraction of Y and U $\mathbf{red}_k$ Z.*

The proof is by induction on k. For $k = 1$ we take $U \equiv Z$. If. $k > 1$ there is a term V which is a contraction of X and V red_{k-1} Y. By Lemma 1 and the induction hypothesis the Lemma follows.

Lemma 3. *If* X **red** Y *and* X red_k Z *then there is a term* U *such that* Y **red** U *and* Z **red** U.

Proof by induction on k. For $k = 1$ we take $U \equiv Y$. For $k > 1$ there is a term V which is a contraction of X and V red_{k-1} Z. By Lemma 2 there is a term V_1 which is a contraction of Y and V **red** V_1. By the induction hypothesis there is a term U such that V_1 **red** U and Z **red** U. It follows that Y **red** U.

2.4 We say that a term X is *convertible* to a term Y, and we write X **conv** Y, if there is a term Z such that X **red** Z and Y **red** Z.

The following properties can be easily proved using the definition and the properties of reduction.

X **conv** X
X **conv** Y *then* Y **conv** X
X **red** Y *then* X **conv** Y
X **conv** Y *then* $[Z/x]X$ **conv** $[Z/x]Y$
X **conv** Y *then* $[X/x]Z$ **conv** $[Y/x]Z$

Lemma 4. *If* X **conv** Y *and* Y **conv** Z *then* X **conv** Z.

By hypothesis there are terms U and V such that

X **red** U	Y **red** V
Y **red** U	Z **red** V

By Lemma 3 there is a term U_1 such that

U **red** U_1	V **red** U_1

It follows that X **red** U_1, Z **red** U_1, hence X **conv** Z.

Theorem 1. X **conv** Y *if and only if there is a finite sequence of terms* $X_1, \ldots, X_k$, $k \geqq 1$, *such that* $X \equiv X_1$, $Y \equiv X_k$ *and for* $i > 1$, *either* X_i *is a contraction of* X_{i-1} *or* X_{i-1} *is a contraction of* X_i.

If X **conv** Y it is clear that the sequence exists. Conversely if the sequence exists we can prove by induction on k, using Lemma 4, that X **conv** Y.

We say that a term is *irreducible* if it does not contain redexes. From the definition it is clear that if X and Y and irreducible then X **conv** Y if and only if $X \equiv Y$. This entails the consistency of the conversion relation. Note that the result of Lemma 4, which is a completeness result, requires an analysis of the forms of the redexes and their contracta. This can be compared with the predicate calculus with Gentzen rules; in fact Lemma 4 is a kind of elimination theorem.

3. Equality. We have shown that conversion is an equality relation between terms. It is easy to give an example of terms representing (extensionally) the same functional, which are not convertible. For instance take

SK and **KI**, of type $\mathbf{F}(\mathbf{F}\alpha\beta)(\mathbf{F}\beta\beta)$. We introduce now a much stronger relation which is complete in the sense that whenever $Xx = Yx$ is derivable, where x does not occur in X or Y, then $X = Y$ is derivable.

3.1 The relation of *equality*, written as usual $X = Y$, is defined given several *axioms*, (E1)-(E11) and *rules of derivation* (I)-(IV). In the axioms and rules letters U,V,X,Y and Z stand for arbitrary terms, and letters x,y and z for distinct arbitrary variables, provided the types correspond in such a way that both terms in each equation take the same type.

(E1) $X = X$

(E2) $\mathbf{I}X = X$

(E3) $\mathbf{K}XY = X$

(E4) $\mathbf{S}XYZ = XZ(YZ)$

(E5) $[x,y]\mathbf{R}xyO = \mathbf{KI}$

(E6) $[x,y,z]\,\mathbf{R}xy(\mathcal{J}z) = [x,y,z]\,xz(\mathbf{R}xyz)$

(E7) $[x,y]\,\mathbf{S}(\mathbf{S}(\mathbf{KK})x)y = \mathbf{K}$

(E8) $[x,y,z]\,\mathbf{S}(\mathbf{S}(\mathbf{S}(\mathbf{KS})x)y)z = [x,y,z]\,\mathbf{S}(\mathbf{S}xz)(\mathbf{S}yz)$

(E9) $\mathbf{S}(\mathbf{KI}) = \mathbf{I}$

(E10) $[x]\,\mathbf{S}(\mathbf{K}x)\mathbf{I} = \mathbf{I}$

(E11) $[x,y]\,\mathbf{K}(xy) = [x,y]\,\mathbf{S}(\mathbf{K}x)(\mathbf{K}y)$

Note that the terms in axioms (E5)-(E11) do not contain variables. The variables appearing in the notation are used only to make explicit the reduction properties of the terms involved. Note also that each of the axioms is only a schema; proper axioms are obtained by an assignment of type to the variables and subscripts to the combinators. Given such an assignment the terms of the equations take some type. We list below the most general types the terms in axioms (E5)-(E11) can take

(E5); $\mathbf{F}_2(\mathbf{F}_2N\alpha\alpha)\alpha\alpha$

(E6): $\mathbf{F}'_3(\mathbf{F}_2N\alpha\alpha)\alpha N\alpha$

(E7): $\mathbf{F}_2(\mathbf{F}\gamma\alpha)(\mathbf{F}\gamma\beta)(\mathbf{F}\gamma\alpha)$

(E8): $\mathbf{F}_3(\mathbf{F}_3\delta\alpha\beta\gamma)(\mathbf{F}_2\delta\alpha\beta)(\mathbf{F}\delta\alpha)(\mathbf{F}\delta\gamma)$

(E9): $\mathbf{F}(\mathbf{F}\alpha\beta)(\mathbf{F}\alpha\beta)$

(E10): $\mathbf{F}(\mathbf{F}\alpha\beta)(\mathbf{F}\alpha\beta)$

(E11): $\mathbf{F}_3(\mathbf{F}\alpha\beta)\alpha\gamma\beta$

3.2 The rules are given as usual by inserting the premises above, and the conclusion below, a line.

Rule (I) $$\frac{X = Y}{Y = X}$$

Rule (II) $$\frac{X = Y \quad Y = Z}{X = Z}$$

Rule (III) $$\frac{X = Y \quad U = V}{XU = YV}$$

Rule (IV) $$\frac{\mathbf{S}(\mathbf{K}Y)\mathcal{J} = \mathbf{S}XY}{\mathbf{R}X(YO) = Y}$$

Rule (IV) is called the *induction rule*. Note that if Y is of type $\mathsf{F}N\alpha$ then X must be of type $\mathsf{F}_2 N\alpha\alpha$; we say in this case that α is the *type of the induction.*

A *derivation* of $X = Y$ is an arrangement of equations showing which are the axioms and how the rules are applied, terminating in $X = Y$. If in such derivation every occurrence of a variable x is replaced by a given term U, we obtain a derivation of $[U/x]X = [U/x]Y$. Hence if X and Y are constant terms there is a derivation of $X = Y$ in which the terms of every equation are constant terms.

3.3 Lemma 5. *If* $X = Y$, *then* $[X/x]U = [Y/x]U$.

The proof is by induction on the structure of U. The case U is x is trivial; if U is not atomic we apply the induction hypothesis and rule (III).

Lemma 6. *If* X **red** Y *then* $X = Y$ *can be derived without using rule* (IV).

First note that if U is a redex and V is its contractum, and the combinator of the redex is **I**,**K** or **S**, then $U = V$ is an axiom. Hence if in the reduction from X to Y only such redexes are contracted, from Lemma 5 it follows that $X = Y$. To complete the proof we need only to show that $U = V$ also in the case in which the combinator of the redex is **R**. This follows from axioms (E5) and (E6) noting that in the reductions corresponding to the abstraction operator only redexes of **I**,**K** and **S** are contracted.

Lemma 7. *If* X **conv** Y *then* $X = Y$ *can be derived without using rule* (IV).

Immediately from Lemma 6.

3.4 Now let $U = \mathbf{K}$ be a case of (E7); let X and Y be terms of the same type as the variables x and y in the axiom. Hence $UXY = \mathbf{K}XY$ and from this using reductions we get without using rule (IV)

(E7*) $$\mathbf{S}(\mathbf{S}(\mathbf{KK})X)Y = X$$

By the same procedure we get from (E8)-(E11)

(E8*) $$\mathbf{S}(\mathbf{S}(\mathbf{S}(\mathbf{KS})X)Y)Z = \mathbf{S}(\mathbf{S}XZ)(\mathbf{S}YZ)$$

(E9*) $$\mathbf{S}(\mathbf{KI})X = X$$

(E10*) $$\mathbf{S}(\mathbf{K}X)\mathbf{I} = X$$

(E11*) $$\mathbf{K}(XY) = \mathbf{S}(\mathbf{K}X)(\mathbf{K}Y)$$

3.5 Lemma 8. *If* X *does not contain the variable* x *then* $[x]X = \mathbf{K}X$ *and* $[x](Xx) = X$ *without using rule* (IV).

That $[x]X = \mathbf{K}X$ follows by induction on the structure of X using (E11*). Hence

$$[x](Xx) \equiv \mathbf{S}(\mathbf{K}X)\mathbf{I}$$
$$= X \qquad \text{(by (E10*))}$$

Theorem 2. *If* $U = V$ *is derivable without using rule* (IV) *then* $[x]U = [x]V$ *is derivable without using rule* (IV).

The proof is by induction on the derivation of $U = V$. The case of (E1) is trivial. Suppose $U = V$ is a case of (E2) say $\mathsf{I}X = X$. If $Y \equiv [x]X$ we must prove $\mathsf{S}(\mathsf{KI})Y = Y$ and this is (E9*). The cases of axioms (E3) and (E4) follow in the same way using (E7*) and (E8*). The other axioms do not contain variables, hence they follow by Lemma 8. For rules (I), (II) and (III) we use the induction hypothesis and again the same rule.

Theorem 3. *If $Xx = Yx$ is derivable without using rule* (IV) *then $X = Y$ is derivable without using rule* (IV), *provided x does not occur in X or Y.*

Using Theorem 2 and Lemma 8 we have

$$\begin{aligned} X &= [x](Xx) \\ &= [x](Yx) \\ &= Y \end{aligned}$$

3.6 We define now several special terms.

$$\mathsf{C}_{\alpha\beta\gamma} = [x]\,\mathsf{S}(\mathsf{K}(\mathsf{S}x))\mathsf{K} \quad \text{of type} \quad \mathsf{F}(\mathsf{F}_2\alpha\beta\gamma)(\mathsf{F}_2\beta\alpha\gamma)$$
$$\mathsf{D}_{\alpha\beta} = [x,y]\,\mathsf{S}(\mathsf{K}(xy)) \quad \text{of type} \quad \mathsf{F}_4(\mathsf{F}_2N\beta\alpha)N(\mathsf{F}\beta\beta)\beta\alpha$$

Lemma 9. $\mathsf{C}(\mathsf{C}X) = X$

We have $\mathsf{C}(\mathsf{C}X)xyz$ **conv** $Xxyz$, hence by Lemma 7 and Theorem 3 we get $\mathsf{C}(\mathsf{C}X) = X$.

Lemma 10. *If $\mathsf{C}X = \mathsf{C}Y$ then $X = Y$.*

From $\mathsf{C}X = \mathsf{C}Y$ we get $\mathsf{C}(\mathsf{C}X) = \mathsf{C}(\mathsf{C}Y)$, hence $X = Y$.

Theorem 4. *If $U = V$, then $[x]U = [x]V$.*

We complete the proof of Theorem 2 considering the case in which rule (IV) is used. Suppose we have

$$\frac{\mathsf{S}(\mathsf{K}Y)\mathcal{J} = \mathsf{S}XY}{\mathsf{R}X(YO) = Y}$$

We set:

$$\begin{aligned} X_1 &= [x]\mathsf{S}(\mathsf{K}Y)\mathcal{J} \\ Y_1 &= [x]\mathsf{S}XY \\ U_1 &= [x]\mathsf{R}X(YO) \\ V_1 &= [x]\,Y \end{aligned}$$

By the induction hypothesis we know that $X_1 = Y_1$; we want to prove $U_1 = V_1$.

First note that $\mathsf{C}U_1Ox$ **conv** $\mathsf{C}V_1Ox$, hence

$$(1)\quad \mathsf{C}U_1O = \mathsf{C}V_1O$$

Also $\mathsf{S}(\mathsf{K}(\mathsf{C}V_1))\mathcal{J}yx$ **conv** $\mathsf{C}X_1yx$, hence

$$(2)\quad \mathsf{S}(\mathsf{K}(\mathsf{C}V_1))\mathcal{J} = \mathsf{C}X_1 = \mathsf{C}Y_1$$

Furthermore $\mathsf{C}Y_1yx$ **conv** $\mathsf{S}(\mathsf{D}X)(\mathsf{C}V_1)yx$, hence

$$(3)\ \mathsf{S}(\mathsf{K}(\mathsf{C}V_1))\mathcal{J} = \mathsf{C}Y_1 = \mathsf{S}(\mathsf{D}X)(\mathsf{C}V_1)$$

and from (3), using rule (IV) we get

$$(4)\ \mathsf{R}(\mathsf{D}X)(\mathsf{C}V_1 O) = \mathsf{C}V_1$$

Furthermore $\mathsf{S}(\mathsf{K}(\mathsf{C}U_1))\mathcal{J}yx = \mathsf{S}(\mathsf{D}X)(\mathsf{C}U_1)yx$ without rule (IV), hence

$$(5)\ \mathsf{S}(\mathsf{K}(\mathsf{C}U_1))\mathcal{J} = \mathsf{S}(\mathsf{D}X)(\mathsf{C}U_1)$$

and from (5) using rule (IV) we get

$$(6)\ \mathsf{R}(\mathsf{D}X)(\mathsf{C}U_1 O) = \mathsf{C}U_1$$

From (1), (4) and (6) we obtain $\mathsf{C}U_1 = \mathsf{C}V_1$, hence by Lemma 10 $U_1 = V_1$ holds.

Theorem 5. *If $Xx = Yx$ where x does not occur in X or Y, then $X = Y$.*

Proof as in Theorem 3.

3.7 From the proof of Theorem 4 we see that if the given induction was of type α and the variable x is of type β, then we need new inductions of type $\mathsf{F}\beta\alpha$.

4. Regularity. Given a term we may try to get an irreducible form by means of repeated contractions. It is not clear now that the rules given in section 2 are sufficient for that purpose. We shall show that this is the case, hence that every term reduces to some irreducible term. Note that an irreducible constant numerical term must be a numeral.

4.1 Let X be a term. We define the *sucessors* of X by the following rules:

(S1) *If X is of type $\mathsf{F}\alpha\beta$ and x is a variable of type α not occurring in X, then Xx is a successor of X.*

(S2) *If X is a numerical term of the form $\mathsf{I}X_1 \dots X_n$, then $X_1 \dots X_n$ is a successor of X.*

(S3) *If X is a numerical term of the form $\mathsf{K}X_1 \dots X_n$, $n \geqq 2$ then X_2 and $X_1 X_3 \dots X_n$ are successors of X.*

(S4) *If X is a numerical term of the form $\mathsf{S}X_1 \dots X_n$, $n \geqq 3$, then $X_1 X_3 (X_2 X_3) X_4 \dots X_n$ is a successor of X.*

(S5) *If X is a numerical term of the form $\mathsf{R}X_1 X_2\, O X_3 \dots X_n$, $n \geqq 2$ then X_1 and $X_2 X_3 \dots X_n$ are successors of X.*

(S6) *If X is a numerical term of the form $\mathsf{R}X_1 X_2\, O^{k+1} X_3 \dots X_n$ $n \geqq 2$, then $X_1\, O^k (\mathsf{R}X_1 X_2\, O^k) X_3 \dots X_n$ is a successor of X.*

(S7) *If X is a numerical term of the form $\mathsf{R}X_1 X_2\, U X_3 \dots X_n$, $n \geqq 2$, and U is not a numeral then U and $\mathsf{R}X_1 X_2\, O^k X_3 \dots X_n$ for $k = 0,1, \dots$ are successors of X.*

(S8) *If X is a numerical term of the form $yX_1 \dots X_n$, $n \geqq 1$, then $X_1, \dots, X_n$ are successors of X.*

(S9) *If X is a numerical term of the form $\mathcal{J}Y$, then Y is a successor of X.*

4.2 A fundamental *sequence* of a term X, is a sequence $X_1, X_2, \dots$ where $X_1 = X$, and X_{i+1} is a successor of X. We say that X is *regular* if every fundamental sequence of X terminates. Hence a term X is regular if and only if all the successors of X are regular.

4.2.1 Numerical variables are regular since they have no successor. The same is true for O and for $\mathscr{J}$ by rules (S1) and (S9). We can prove also that any variable is regular, using induction on the type of the variable and rules (S1) and (S8). It is easy to show that numerals and terms containing only variables are regular.

4.2.2 The combinators **I**, **K** and **S** are regular. For instance to prove that **K** is regular is sufficient to prove that $\mathsf{K}x_1 \ldots x_n$ is regular for n depending on the type of **K** (by rule (S1)), and this is true by rule (S3) since x_2 and $x_1 x_3 \ldots x_n$ are regular.

4.2.3 The combinators **R** are also regular. First it can be proved that $\mathsf{R}xyO^k x_1 \ldots x_n$ is regular for every $k \geqq 0$, using induction on k. From this using rule (S7) it follows that $\mathsf{R}xyzx_1 \ldots x_n$ is regular, hence by rule (S1) that **R** is regular.

4.3 The preceding analysis shows that the atoms in our system are regular. We must prove now that regularity is preserved by application.

4.3.1 We note first that the following *induction principle* is available for regular terms. Let $\mathfrak{P}$ be a property such that: a) O and numerical variables have the property $\mathfrak{P}$; b) If every successor of a term X has the property $\mathfrak{P}$, then X has the property $\mathfrak{P}$. Then we can infer that every regular term has the property $\mathfrak{P}$. For if some term X does not have the property $\mathfrak{P}$ we can construct a non terminating fundamental sequence of X. We shall denote this kind of argument **R**-induction.[4]

4.4 We say that a term X is a *variant* of a term U if there are variables $x_1, \ldots, x_n, y_1, \ldots, y_n, n \geqq 0$, such that $X = [y_1, \ldots, y_n/x_1, \ldots, x_n]U$.

4.4.1 Lemma 11. *Let X be a regular term. Then given terms U and Y such that both X and Y are varients of U, Y is a regular term.*

The proof is by **R**-*induction*. For instance, suppose that X is a numerical term of the form $\mathsf{K}X_1 \ldots X_n, n \geqq 2$. Then $U \equiv \mathsf{K}U_1 \ldots U_n$ and $Y \equiv \mathsf{K}Y_1 \ldots Y_n$ where X_i and Y_i are variants of U_i, $i = 1, \ldots, n$. Since $X_1 X_3 \ldots X_n$ and X_2 are regular it follows that $Y_1 Y_3 \ldots Y_n$ and Y_2 are regular, hence that X is regular. If X is a term of type $\mathsf{F}\alpha\beta$, let y be a variable of type α not occurring in Y, and z a variable of the same type not occurring in X or U. Then Xz and Yy are both variants of Uz, hence Yy is regular. It follows that Y is regular.

4.4.2 Corollary. *Let X be a variant of U. Then X is regular if and only if U is regular.*

4.5 Theorem 6. *Let Y be a regular term of type β and x a variable of type β. Then for every regular term X, the term $Z \equiv [Y/x]X$ is regular.*

The proof is by induction on the structure of the type β; we assume the theorem is true for any Y and x of a type which is a proper part of β. Under this hypothesis and for given Y and x of type β, we prove by **R**-*induction* that given a regular term X, the term $Z \equiv [Y/x]X$ is regular.

Case 1. X is O or a numerical variable. It is clear that Z is regular.

Case 2. X is of type $\mathsf{F}\alpha\gamma$. Let z be a variable not occurring in X or Y, and we assume $[Y/x](Xz) \equiv Zz$ is regular. By **4.4.2**, Z is regular.

Case 3. X has successors under one of rules (S2)-(S6) or (S9). This case is trivial.

Case 4. X is of the form $\mathsf{R}X_1X_2UX_3\ldots X_n$ under rule (S7). Then $Z \equiv \mathsf{R}Z_1Z_2VZ_3\ldots Z_n$ and we assume V is regular and also that $\mathsf{R}Z_1Z_2O^kZ_3 \ldots Z_n$ are regular for $k \geqq 0$. Hence if V is a numeral Z is regular by assumption and if V is not a numeral then Z is regular by rule (S7)

Case 5. X is of the form $yX_1\ldots X_n$. If y is not x then $Z \equiv yY_1\ldots Y_n$ and our assumption is that $Y_1, \ldots, Y_n$ are regular; hence Z is regular. If y and x are the same variable, then $Z \equiv YY_1\ldots Y_n$, and our assumption is again that $Y_1, \ldots, Y_n$ are regular. Note that each Y_i is of a type which is a proper part of β, so we may use the induction hypothesis on the structure of β. Since Y is regular, $Yx_1\ldots x_n$ is also regular with $x_1, \ldots, x_n$ not occurring in Y. It follows that $[Y_1, \ldots, Y_n/x_1, \ldots, x_n](Yx_1\ldots x_n)$ is regular.

4.5.1 Theorem 7. *Every term is regular.*

We have shown that the theorem is true for the atoms. Now suppose $X \equiv UV$ where U and V are regular. Hence Ux is regular and by the preceding theorem $[V/x](Ux)$ is regular.

4.6 The importance of Theorem 7 is that we can use R-*induction* to prove properties of arbitrary terms. We shall give some applications in this direction.

4.6.1 A reduction sequence of a term X is a sequence $X_1, X_2, \ldots$ where $X_1 \equiv X$ and X_{i+1} is a simple contraction of X_i.

Theorem 8. *For each term X there is a number m such that every reduction sequence of X terminates in less than m steps.*

The proof is by R-*induction*. We show in one example the general methods for dealing with all the cases. Suppose X is of the form $\mathsf{R}X_1X_2UX_3\ldots X_n$ where U is not a numeral. Our assumption is that there is a number m_0 such that every reduction sequence of U terminates in less than m_0 steps. Also there are numbers $m_1, m_2, \ldots$ such that $\mathsf{R}X_1X_2O^kX_3\ldots X_n$ terminates in less than m_{k+1} steps. It follows that if U does not reduce to a numeral every reduction sequence of X terminates in less than $m_0 + m_1$ steps. If U reduces to a numeral O^k then every reduction sequence of X terminates in less than $m_0 + m_{k+1}$ steps.

4.6.2 As a consequence of Theorem 8 we get that every term reduces to a term which is irreducible. This reduction can always be performed by means of arbitrary contractions; no matter how the redex are chosen the procedure eventually terminates in an irreducible term. Moreover a numerical constant term which is irreducible is a numeral. Hence every numerical constant term reduces to a numeral. This numeral is unique by Lemma 3.

5. Congruence. We can obtain additional information about the equality relation of section **3** using the results on regularity. For that purpose we define a new binary relation between constant terms; this relation is called *congruence*. For constant numerical terms we say that X is congruent to Y, which is written X **cong** Y, exactly if X **conv** Y, i.e. if there is a numeral U such that X **red** U and Y **red** U. If X and Y are constant terms of type $\mathsf{F}\alpha\beta$ we say that X **cong** Y if and only if for arbitrary terms U and V of type α such that U **cong** V, we always have XU **cong** YV.[5]

5.1 Lemma 12. *If X **cong** Y then Y **cong** X.*

The proof by induction on the type of X is clear.

Lemma 13. *If X and Y are constant terms, X **red** X_1, Y **red** Y_1 and X_1 **cong** Y_1, then X **cong** Y.*

If X is of type N we have X **conv** Y. If X is of type $\mathsf{F}\alpha\beta$, and we suppose the lemma is true for terms of type β then for arbitrary U and V of type α such that U **cong** V, we have XU **red** $X_1 U$, YV **red** $Y_1 V$ and $X_1 U$ **cong** $Y_1 V$, hence XU **cong** YV. This means X **cong** Y.

5.1.1 Theorem 9. *Let X be a term with variables $x_1, \dots, x_k$ and let $U_1, \dots, U_k$, $V_1, \dots, V_k$ be terms of the corresponding types such that U_i **cong** V_i. Let $U \equiv [U_1, \dots, U_k/x_1, \dots, x_k]X$ and $V \equiv [V_1, \dots, V_k/x_1, \dots, x_k]X$, then U **cong** V.*

The proof is by R-*induction*. For instance let X be of the form $\mathsf{R}X_1 X_2 W X_3 \dots X_n$ where W is not a numeral. Then $U \equiv \mathsf{R}Y_1 Y_2 W_1 Y_3 \dots Y_n$ and $V \equiv \mathsf{R}Z_1 Z_2 W_2 Z_3 \dots Z_n$. By the induction hypothesis we know that

$$W_1 \ \mathbf{cong} \ W_2$$
$$\mathsf{R}Y_1 Y_2 O^m Y_3 \dots Y_n \ \mathbf{cong} \ \mathsf{R}Z_1 Z_2 O^m Z_3 \dots Z_n$$

for ever $m \geqq 0$. But W_1 **cong** W_2 means there is a numeral O^m such that W_1 **red** O^m and W_2 **red** O^m. Hence by Lemma 13 we have U **cong** V.

5.1.2 Corollary. *If X is a constant term then X **cong** X.*

5.2 Lemma 14. *If X **cong** Y and Y **cong** Z then X **cong** Z.*

Proof by induction on the type of X, using **5.1.2**.

Lemma 15. *Let X and Y be terms with variables $x_1, \dots, x_k$ such that X **conv** Y. Let $U_1, \dots, U_k, V_1, \dots, V_k$ be terms of the corresponding types such that U_i **cong** V_i. Then*

$$[U_1, \dots, U_k/x_1, \dots, x_k]X \ \mathbf{cong} \ [V_1, \dots, V_k/x_1, \dots, x_k]\, Y$$

For there is a Z such that X **red** Z and Y **red** Z. Using Theorem 9 for this Z and Lemma 13 we get the result.

5.2.1 Corollary. *Let X and Y be constant terms and $x_1, \dots, x_k$ distinct variables such that $Xx_1 \dots x_k$ **conv** $Yx_1 \dots x_k$, then X **cong** Y.*

5.3 Lemma 16. *Let X and Y be constant terms of type $\mathsf{F}N\alpha$. If for every k we have XO^k **cong** YO^k, then X **cong** Y.*

Given U and V of type N such that U **cong** V there is a numeral O^k such that U **red** O^k and V **red** O^k. Hence we use Lemma 13.

5.3.1 Theorem 10. *If* $X = Y$ *and* X *and* Y *are constant terms, then* X **cong** Y.

The proof is by induction on the derivation of $X = Y$. Every axiom, with the only exception of (E6) satisfies the condition of **5.2.1** for some $k \geqq 0$, hence we have X **cong** Y. For (E6) suppose U_1 **cong** V_1, U_2 **cong** V_2 and U_3 **cong** V_3 where U_1 is of type $\mathsf{F}_2 N\alpha\alpha$, U_2 of type α and U_3 of type N. Suppose $X = Y$ is the instance of (E7). Hence we need only to show that $XU_1 U_2 U_3$ **cong** $YV_1 V_2 V_3$. This is true because there is some O^k such that

$$XU_1 U_2 U_3 \text{ red } U_1 O^k (\mathbf{R} U_1 U_2 O^k)$$
$$YV_1 V_2 V_3 \text{ red } V_1 O^k (\mathbf{R} V_1 V_2 O^k)$$

so we can use Lemma 13.

We must show also that the rules preserve the property. This is clear for rules (I), (II) and (III). Suppose we have a case of rule (IV).

$$\frac{\mathbf{S}(\mathbf{K}Y)\mathcal{J} = \mathbf{S}XY}{\mathbf{R}X(YO) = Y}$$

Our induction hypothesis can be expressed in the form: $Y(\mathcal{J}O^k)$ **cong** $XO^k(YO^k)$ for every $k \geqq 0$. We must prove $\mathbf{R}X(YO)O^k$ **cong** YO^k for every $k \geqq 0$. For $k = 0$ this is trivial. Suppose it is true for some k, then

$$\begin{aligned} \mathbf{R}X(YO)O^{k+1} &\text{ cong } XO^k(\mathbf{R}X(YO)O^k) \\ &\text{ cong } XO^k(YO^k) \\ &\text{ cong } Y(\mathcal{J}O^k) \end{aligned}$$

NOTES

1. See [2], Chapter 6.
2. A formal definition is given in [2], p. 205.
3. Proved in [9] for a weaker system of combinatory logic.
4. We note that it would be possible to define the class of regular terms by an induction with inductive rules corresponding to the successor rules. In case of rule (S7) the inductive rule must have an infinite number of premises. We think that the justification for such induction is precisely that it is equivalent to assert that every fundamental sequence terminates.
5. The notion of congruence appears to be related with the property of being extensionally definite that Kreisel defines in [7], p. 124.

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Universidad de Buenos Aires
Buenos Aires, Argentina