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THE MODAL STRUCTURE OF THE PRIOR-RESCHER FAMILY OF INFINITE PRODUCT SYSTEMS

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1. *Prior-Rescher Family of Product Systems*.* Let S be an arbitrary sentential system of m-valued truth-functional logic, $m \geq 2$. Following the notational conventions of Rescher ([6], p. 99), we mean by $\Pi_k(S)$ the truth-functional system that is the k-fold product of S with itself. That is, the truth values of $\Pi_k(S)$ are the k-tuples of the truth values of S, and the semantics of $\Pi_k(S)$ is based on the semantics of S in the following way. Let $\otimes$ be an n-ary connective. Then $\otimes(\langle\alpha_1^1, \ldots, \alpha_k^1\rangle, \ldots, \langle\alpha_1^n, \ldots, \alpha_k^n\rangle)$ is $\langle\otimes(\alpha_1^1, \ldots, \alpha_1^n), \ldots, \otimes(\alpha_k^1, \ldots, \alpha_k^n)\rangle$. Rescher observes that there are two plausible ways to treat truth-value designation in $\Pi_k(S)$. One might regard a truth value $\langle\alpha_1, \ldots, \alpha_k\rangle$ as designated in $\Pi_k(S)$ iff (a) each member of $\langle\alpha_1, \ldots, \alpha_k\rangle$ is designated in S, or iff (b) at least one member of $\langle\alpha_1, \ldots, \alpha_k\rangle$ is designated in S. Both alternatives lead to exactly the same theses for all the product systems discussed in this paper, so it is a matter of indifference which is chosen. For the sake of definiteness we adopt alternative (a). Again following Rescher's notation (*ibid.*), by $\Pi_{\aleph_0}(S)$ we mean the denumerable product of S with itself. That is, the truth values of $\Pi_{\aleph_0}(S)$ are the denumerable sequences $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ of the truth values of S, and the semantics of $\Pi_{\aleph_0}(S)$ is based on that of S in the same way that the semantics of $\Pi_k(S)$ is based on the semantics of S.

In [6], p. 195, Rescher considers the family of systems $\Pi_k(S)^+$ and $\Pi_{\aleph_0}(S)^+$, which we call the *Prior-Rescher family of product systems*. In all these systems the underlying truth-functional logic S has a "truest" designated value t and a "falsest" nondesignated value f. One obtains $\Pi_k(S)^+$ by supplementing $\Pi_k(S)$ with the singulary operator $\Box$ whose semantics is given as follows. The value of $\Box A$ is the k-tuple $\langle$t, . . . , t$\rangle$ if the value of A is that same k-tuple; otherwise, the value of $\Box A$ is the k-tuple $\langle$f, . . . , f$\rangle$. Similarly, one gets $\Pi_{\aleph_0}(S)^+$ by supplementing $\Pi_{\aleph_0}(S)$

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with $\Box$ evaluated as follows. The value of $\Box A$ is (t, t, t, . . .) if the value of A is that same infinite sequence; otherwise, the value of $\Box A$ is (f, f, f, . . .). (Prior has also discussed other product systems supplemented with modality, such as the Diodorean systesm whose modal structure is discussed in Bull [1].)

2. *Modal Structure of the Systems.* $\Pi_{\aleph_0}(S)^+$. In [5], pp. 21 ff., Prior introduced the systems $\Pi_k(C)^+$ and $\Pi_{\aleph_0}(C)^+$, where C is classical two-valued logic, and asserted that the theses of $\Pi_{\aleph_0}(C)^+$ are exactly the theses of the modal system S5. Prior's result about $\Pi_{\aleph_0}(C)^+$ has prompted Rescher ([6], p. 195) to raise but to leave unanswered the question of what is the modal structure of a product system $\Pi_{\aleph_0}(S)^+$, when $S \neq C$. There appear to be two ways to answer Rescher's query. The first way is to present an "interesting" axiomatization of S-with-modality and then to prove that the theorems of this axiomatic system are precisely the theses of $\Pi_{\aleph_0}(S)^+$. The second way, the way we shall adopt, consists in presenting a plausible modal semantics for S-with-modality, proving that the valid wffs of this system are precisely the theses of $\Pi_{\aleph_0}(S)^+$, and showing how the two semantics are related. To anticipate, we will show that the theses of $\Pi_{\aleph_0}(S)^+$ are the valid wffs of a Kripkean S5 modal system S^* having S as the underlying truth-functional logic.

It behooves us now to say what we mean by a Kripkean S5 modal system S^* based on a truth-functional system S. In addition to the connectives of S, S^* contains the singulary operator $\Box$. The semantics of S^* appeals to so-called S5 model structures $\langle G, K, R\rangle$, where K is a nonempty set (the set of possible worlds), $G \in K$, and R is an equivalence relation on K (the relation of accessibility). An *interpretation on* $\langle G, K, R\rangle$ *of a wff* is simply a function that assigns to each sentential variable of the wff some truth value or other at each possible world (member of K). The basic semantical notion is *the value of a wff at a world under an interpretation of it on a model structure.* For truth-functional connectives the semantical clauses of the inductive definition of the aforementioned notion are given in the usual way ([2], pp. 84 ff.). The basic Kripkean theme admits of some variation in the clause governing $\Box$ when the underlying truth-functional logic S is many-valued. Two natural alternatives present themselves. The first is to let the value of a wff A at a world W under an interpretation Σ on $\langle G, K, R\rangle$ be t (the "truest" value) if the value of A under Σ on $\langle G, K, R\rangle$ is t at every world accessible to W, and otherwise be f (the "falsest" value). The second is to let the value of A at W under Σ on $\langle G, K, R\rangle$ be the "least true" of the values that A has under Σ on $\langle G, K, R\rangle$ at worlds accessible to W, provided the value of A under Σ on $\langle G, K, R\rangle$ is designated at each of these worlds, and otherwise be f. Because the first alternative corresponds to the Prior-Rescher semantical rule for $\Box$ in product systems, we adopt it here. Henceforth, then, by S^* we shall mean the Kripkean S5 modal system that results when $\Box$ is added to the truth-functional system S, the semantics of $\Box$ being given by the first alternative just discussed. Notice that the second alternative is somewhat less economical than the first, presupposing not only a "truest" value and a "falsest" value but also a well-ordering of the designated values with respect to truthlikeness. The product-system counterpart to the second alternative is the following rule. For the value $\langle \alpha_1, \ldots, \alpha_k \rangle$ of A, the value of $\Box A$ is the k-tuple $\langle \alpha', \ldots, \alpha' \rangle$ provided that all the members of $\langle \alpha_1, \ldots, \alpha_k \rangle$ are designated and that α' is the least designated of them; otherwise, the value of $\Box A$ is the k-tuple $\langle \mathsf{f}, \ldots, \mathsf{f} \rangle$. Denumerable sequences of truth values are treated analogously.

3. *Formal Results.* When R is an equivalence relation, as in S5 model structures $\langle G, K, R \rangle$, the model structures become semantically superfluous. Rather than deal with interpretations on model structures as above, one may simply regard an S5 interpretation of A as an ordered pair $\langle G, K \rangle$, where K is a set of truth-value assignments to the sentential variables of A with $G \in K$. That is, one may treat the truth-value assignments of K as mutually accessible possible worlds. In a truth-tabular representation (see [3], pp. 593-595, and [4]), $\langle G, K \rangle$ corresponds to the value-assignment portion of a partial or full truth table for A in the following way. K is the set of value-assignment rows of the table, and G is a given one of these rows. Truth-functional connectives are handled in the usual truth-tabular way, and $\Box$ is treated thus: The value of $\Box A$ on a row of the table is t if the value of A is t on each row of the table; otherwise, the value of $\Box A$ on the given row is f. The equivalence of the Kripkean semantics to this truth-tabular representation may be put as follows. A wff A is valid in S^* iff the value of A is designated on every row of every finite truth table for A. Let n be the number of distinct sentential variables in A. It is readily verified that A is valid in S^* iff the value of A is designated on every row of every truth table for A that contains m^n or fewer rows, where m is the number of truth values in the system S. This gives us a decision procedure for validity for an arbitrary system S^*.

Theorem 1. *For any positive integer k, if A is not a thesis of $\Pi_k(S)^+$, then A is not a thesis* (valid wff) *of S^*.*

Theorem 1 is an immediate corollary of the following lemma.

Lemma. *Let $b_1, \ldots, b_n$ be a complete list of the distinct variables of A. Then A has in $\Pi_k(S)^+$ the value $\langle \beta_1, \ldots, \beta_k \rangle$ under the value assignment of $\langle \alpha_1^1, \ldots, \alpha_k^1 \rangle, \ldots, \langle \alpha_1^n, \ldots, \alpha_k^n \rangle$ to $b_1, \ldots, b_n$ respectively iff in the truth-tabular representation of S^* we have*

$b_1 \ldots b_n$	A
$\alpha_1^1 \ldots \alpha_1^n$	β_1
$\vdots \quad\quad \vdots$	$\vdots$
$\alpha_k^1 \quad\quad \alpha_k^n$	β_k

The lemma can be proved by a straightforward induction on the number of occurrences of connectives in A. The lemma shows that the product semantics of $\Pi_k(S)^+$ is merely a variant representation of the S5 truth tables containing exactly k rows. Thus the following theorem is also a corollary of the foregoing lemma.

Theorem 2. *If A is not a thesis of S*, then for some positive integer k, A is not a thesis of* $\Pi_k(S)^+$.

From Theorems 1 and 2 we have immediately:

Theorem 3. *A is a thesis of S* iff, for every positive integer k, A is a thesis of* $\Pi_k(S)^+$.

Next we show:

Theorem 4. *For any positive integer k, if A is not a thesis of* $\Pi_k(S)^+$, *then A is not a thesis of* $\Pi_{\aleph_0}(S)^+$.

To establish Theorem 4, one can prove by mathematical induction on the number of occurrences of connectives in A that if A has in $\Pi_k(S)^+$ the value $\langle\beta_1, \ldots, \beta_k\rangle$, for the value assignment of $\langle\alpha_1^1, \ldots, \alpha_k^1\rangle, \ldots, \langle\alpha_1^n, \ldots, \alpha_k^n\rangle$ to the variables $b_1, \ldots, b_n$ of A, then the value of A in $\Pi_{\aleph_0}(S)^+$ is the infinite sequence $(\beta_1, \ldots, \beta_k, \beta_1, \ldots, \beta_k, \ldots)$ for the value assignment of the infinite sequences $(\alpha_1^1, \ldots, \alpha_k^1, \alpha_1^1, \ldots, \alpha_k^1, \ldots), \ldots, (\alpha_1^n, \ldots, \alpha_k^n, \alpha_1^n, \ldots, \alpha_k^n, \ldots)$ to $b_1, \ldots, b_n$ respectively.

At this juncture Prior's result that $\Pi_{\aleph_0}(C)^+$ and C^* (i.e. S5) have the same theses can be derived from theorems 1, 2, 4 and verification of the fact that the axioms and rules of S5 are validated by the semantics of $\Pi_{\aleph_0}(C)^+$. This result is a special case of Theorem 6 below.

Theorem 5. *If A is not a thesis of* $\Pi_{\aleph_0}(S)^+$, *then for some positive integer k, A is not a thesis of* $\Pi_k(S)^+$.

To prove Theorem 5, let $b_1, \ldots, b_n$ be a complete list of the distinct variables of A, and let $(\beta_1, \beta_2, \beta_3, \ldots)$ be the value of A in $\Pi_{\aleph_0}(S)^+$ for the value assignment of $(\alpha_1^1, \alpha_2^1, \alpha_3^1, \ldots), \ldots, (\alpha_1^n, \alpha_2^n, \alpha_3^n, \ldots)$ to $b_1, \ldots, b_n$ respectively, and let $\langle\gamma_1^1, \ldots, \gamma_{n+1}^1\rangle, \ldots, \langle\gamma_1^s, \ldots, \gamma_{n+1}^s\rangle$ be a possibly redundant list of the distinct $(n+1)$-tuples in the infinite list $\langle\alpha_1^1, \ldots, \alpha_1^n, \beta_1\rangle$, $\langle\alpha_2^1, \ldots, \alpha_2^n, \beta_2\rangle, \ldots$. Then one can show by mathematical induction on the number of occurrences of connectives in A that the value of A in $\Pi_s(S)^+$ is $\langle\gamma_{n+1}^1, \ldots, \gamma_{n+1}^s\rangle$ for the value assignment of $\langle\gamma_1^1, \ldots, \gamma_1^s\rangle, \ldots, \langle\gamma_n^1, \ldots, \gamma_n^s\rangle$ to $b_1, \ldots, b_n$ respectively.

The next theorem, which follows immediately from Theorems 1, 2, 4 and 5, constitutes our answer to Rescher's query about the modal structure of $\Pi_{\aleph_0}(S)^+$.

Theorem 6. *S* and* $\Pi_{\aleph_0}(S)^+$ *have exactly the same theses.*

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