Notre Dame Journal of Formal Logic
Volume XIII, Number 2, April 1972
NDJFAM

INDUCTION ON FIELDS OF BINARY RELATIONS

W. RUSSELL BELDING

In [1] the following principle of induction, introduced by Montague in [2],

A. *If* $\varphi(x)$ *is a formula not containing the variable* y *and* R *is a well-founded relation, then*

$$(y)\,(y \in \mathsf{Fld}R \wedge (x)\,(xRy \rightarrow \varphi(x)) \rightarrow \varphi(y)) \rightarrow (y)\,(y \in \mathsf{Fld}R \rightarrow \varphi(y))$$

is proved in the field of G.B. set theory. It is shown below that the restriction that R be well-founded can be removed and the induction will still hold provided a restriction is placed on the formula $\varphi(x)$. The relationship between the various induction principles of [1] and the induction principle proved in this paper (Theorem 1) is discussed. The notation and definitions used in this paper are explained and defined in [1]. The relations considered in this paper are always binary relations.

The following theorem gives a new sufficient condition for induction of binary relations.

Theorem 1. (Induction Principle **E**). *For every* R *and every* $\varphi(x)$, *if* R *is a binary relation,* $\varphi(x)$ *a formula not containing the variable* y *and* $\varphi(x)$ *has the property that for every sequence of sets* $\{a_n\}_{n<\omega}$ *such that* $a_{n+1}Ra_n$ *for every* $n \geqslant 0$, *there is at least one integer* $m \geqslant 0$ *such that* $\varphi(a_m)$ *holds, then*

$$(y)\,(y \in \mathsf{Fld}R \wedge (x)\,(xRy \rightarrow \varphi(x)) \rightarrow \varphi(y)) \rightarrow (y)\,(y \in \mathsf{Fld}R \rightarrow \varphi(y)).$$

Proof: An indirect proof is used. Assume the hypothesis and suppose the induction fails. That is,

(1) $(y)\,(y \in \mathsf{Fld}R \wedge (x)\,(xRy \rightarrow \varphi(x)) \rightarrow \varphi(y))$
(2) $(\exists y)\,(y \in \mathsf{Fld}R \wedge \sim \varphi(y))$

Suppose a_0 is such that $a_0 \in \mathsf{Fld}R$ and $\sim\varphi(a_0)$. First suppose that a_0 has no predecessor or that φ holds for every predecessor of a_0. Then clearly

(3) $a_0 \in \mathsf{Fld}R \wedge (x)\,(xRa_0 \rightarrow \varphi(x))$

By (3) and (1) it follows that $\varphi(a_0)$ holds, contrary to (2). Thus,

Received August 4, 1970

(4) $(\exists y)(yRa_0 \wedge \sim\varphi(y))$

Let a_1 be such that a_1Ra_0 and $\sim\varphi(a_1)$; clearly $a_1 \in \mathrm{Fld}R$. Now apply the same argument to a_1 as was applied to a_0 to obtain

(5) $(\exists y)(yRa_1 \wedge \sim\varphi(y))$

Continuing in this manner a sequence of sets $\{a_n\}_{n<\omega}$ is obtained such that $a_{n+1}Ra_n$ and $\sim\varphi(a_n)$ holds for every $n \geq 0$. This contradicts the hypothesis. Hence (2) is false, and the theorem is proved.

In the proof of Theorem 1, only the variables x and y are used. And, it is also ensured that the formula $\varphi(x)$ does not contain the variable y. It is possible that a formula $\varphi(x)$ may contain variables other than x or y and these variables may be bound or free.

To prove that Montague's induction principle **A** is a special case of induction principle **E** it is sufficient to prove the following theorem.

Theorem 2. *For every R and every $\varphi(x)$, if R is a binary relation and $\varphi(x)$ a formula not containing the variable y then, if R is well-founded then for every sequence of sets $\{a_n\}_{n<\omega}$ such that $a_{n+1}Ra_n$ for every $n \geq 0$, there is an integer $m \geq 0$ such that $\varphi(a_m)$ holds.*

Proof: Let R be a binary relation, $\varphi(x)$ a formula not containing the variable y and R be well-founded. Since R is well-founded there is no sequence of sets $\{a_n\}_{n<\omega}$ in $\mathrm{Fld}R$ such that $a_{n+1}Ra_n$ for every $n \geq 0$. Therefore if $\{a_n\}_{n<\omega}$ is a sequence of sets such that $a_{n+1}Ra_n$ for every $n \geq 0$ there is an integer $m \geq 0$ such that $\varphi(a_m)$ holds. Theorem 2 is proved.

Corollary 3. *Induction principle* **A** *is a special case of induction principle* **E**.

Proof: If the hypotheses of induction principle **A** are satisfied by an R and $\varphi(x)$ then by Theorem 2 the hypotheses of induction principle **E** are satisfied. Since **A** and **E** have the same conclusions the corollary is proved.

Besides induction principle **A**, stated above, and induction principle **E**, proved above, there are induction principles **B**, **C**, and **D** which are proved in [1].

B. (Tarski). *If $\varphi(x)$ is a formula not containing the variable y then*

$$(y)(y \in V \wedge (x)(x \in y \rightarrow \varphi(x)) \rightarrow \varphi(y)) \rightarrow (y)(y \in V \rightarrow \varphi(y))$$

C. (Poss). *If A is a transitive class and $\varphi(x)$ a formula not containing the variable y then*

$$(y)(y \in A \wedge (x)(x \in y \rightarrow \varphi(x)) \rightarrow \varphi(y)) \rightarrow (y)(y \in A \rightarrow \varphi(y))$$

D. (Belding). *If A is a supertransitive class, $\varphi(x)$ a formula not containing the variable y and having the property that for every sequence $\{a_n\}_{n<\omega}$ such that $a_{n+1} \subset a_n$ and $a_n \in A$ for every $n \geq 0$, there is at least one integer $m \geq 0$ such that $\varphi(a_m)$ holds then*

$$(y)(y \in A \wedge (x)(x \subset y \rightarrow \varphi(x)) \rightarrow \varphi(y)) \rightarrow (y)(y \in A \rightarrow \varphi(y))$$

Theorem 4. *Induction principle* **D** *is a special case of induction principle* **E**.

Proof: Let A be a supertransitive class, $\varphi(x)$ a formula not containing the variable y and having the property that for every sequence of sets $\{a_n\}_{n<\omega}$ in A such that $a_{n+1} \subset a_n$ for every $n \geq 0$, there is an integer $m \geq 0$ such that $\varphi(a_m)$ holds. Define a binary relation R as follows:

$$(u)(v)(uRv \equiv u \subset v \wedge v \in A)$$

Since A is supertransitive it is clear that

(1) $\text{Fld}R \subseteq A$

If there is an x such that $x \in A - \text{Fld}R$ then by definition of R, x has no proper subset. Thus $x = \phi$. However, if $\text{card}(A) \geq 2$ then there is some y in A such that $y \neq \phi$. In this case $\phi \subset y$ and $\phi \in \text{Fld}R$. So the only case in which $A \neq \text{Fld}R$ is the case $A = \{\phi\}$. Induction on this set is trivial and the induction formula of **D** holds on this set. Having eliminated this case, now assume that

(2) $A = \text{Fld}R$

Suppose that $\{a_n\}_{n<\omega}$ is a sequence of sets in A such that $a_{n+1} \subset a_n$ for every $n \geq 0$. Thus $a_{n+1}Ra_n$ for every $n \geq 0$. By hypothesis, there is an integer $m \geq 0$ such that $\varphi(a_m)$ holds. Now induction principle **E**, may be used to obtain

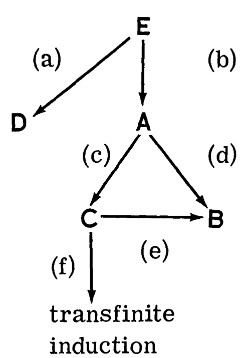
(3) $(y)(y \in \text{Fld}R \wedge (x)(xRy \rightarrow \varphi(x)) \rightarrow \varphi(y)) \rightarrow (y)(y \in \text{Fld}R \rightarrow \varphi(y))$

By (2), definition of R and (3),

(4) $(y)(y \in A \wedge (x)(x \subset y \rightarrow \varphi(x)) \rightarrow \varphi(y)) \rightarrow (y)(y \in A \rightarrow \varphi(y))$

By the hypothesis and (4), induction principle **D** has been obtained from induction principle **E**, as required.

Theorem 4 yields the result that whenever induction principle **D** can be applied, induction principle **E** can be applied with the same results. The results of [1] and this paper enable the following diagram to be drawn. In this diagram an arrow from **A** to **B**, for example, shall mean that induction principle **B** is a special case of induction principle **A**.



(a) Theorem 4
(b) Corollary 3
(c) Lemma 10 of [1]
(d) Lemma 3 of [1]
(e) Theorem 11 of [1]
(f) Theorem 12 of [1].

Thus the induction principles **A**, **B**, **C**, **D** and transfinite induction are all special cases of induction principle **E**. Since each of these induction principles are theorems of G. B. set theory, they are logically equivalent. Having shown that some of these induction principles are special cases of other induction principles it should be remarked that it may be possible to reverse some of the arrows in the diagram above. That is, for example, it may be possible to show, by an appropriate choice of several relations or by some other method, that induction principle **E** is a special case of induction principle **A**. Such questions will not be considered here.

Remark: Let R be a binary relation and $\varphi(x)$ a formula not containing the variable y. In effect, induction principle **E** gives a sufficient condition for the induction formula

$$(\alpha) \qquad (y)(y \in \mathsf{Fld} R \wedge (x)(xRy \rightarrow \varphi(x)) \rightarrow \varphi(y)) \rightarrow (y)(y \in \mathsf{Fld} R \rightarrow \varphi(y))$$

to hold. Regarding necessary conditions, two results were obtained in [1], namely, Theorems 8 and 21. In both cases the relation R was explicitly stated and an allowable formula $\varphi(x)$ was defined and substituted in (α) to obtain the necessary condition.

I wish to thank Professor B. Sobociński for his helpful advice.

BIBLIOGRAPHY

[1] Belding, W. R., R. L. Poss, and P. J. Welsh, Jr., "Transitivity, Supertransitivity and Induction," *The Notre Dame Journal of Formal Logic*, vol. XIII (1972), pp. 177-190.

[2] Montague, R., "Well-founded relations; generalizations of principles of induction and recursion," *Bulletin of the American Mathematical Society*, vol. 61 (1955), p. 442.

Seminar in Symbolic Logic
University of Notre Dame
Notre Dame, Indiana