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ON THE RECURSIVE UNSOLVABILITY OF THE PROVABILITY OF THE DEDUCTION THEOREM IN PARTIAL PROPOSITIONAL CALCULI

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1. *Introduction.** In a recent paper, Pogorzelski [2] proved the existence of a "weakest" partial propositional (implicational) calculus. Such a calculus is the weakest in the sense that its rules and axioms are among the derived rules and theorems respectively of any partial propositional calculus in which the deduction theorem holds. This result suggests the following question: Is it possible to algorithmically determine of an arbitrarily given partial propositional calculus whether or not the deduction theorem holds? In this paper we prove that the answer to this question is negative.

To accomplish our goal, we consider a special subclass of partial propositional calculi, namely those in which the rules of inference consist of modus ponens and substitution. It is known that the decision problem for the latter calculi is recursively unsolvable and we shall use Singletary's construction of such calculi in [3]. The proof that the problem of determining whether or not the deduction theorem holds for such a calculus arbitrarily given will parallel Yntema's proof of the recursive unsolvability of the completeness problem for a more restricted class of partial propositional calculi in [4]. Finally, using a well-known result of Boone [1], it will be an easy matter to show that for every recursively enumerable degree of unsolvability D, there exists a class of partial propositional calculi such that the problem of determining whether or not the deduction theorem holds for any member of the class is of degree D.

2. *Preliminaries.* For the purposes of this paper, we shall define a *generalized partial propositional calculus* to be a formal system whose

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symbols consist of (,), and $\supset$, and an infinite set of propositional variables $p_1, q_1, r_1, p_2, q_2, r_2, \ldots$. The well-formed formulas are taken to be (1) any propositional variable and (2) any expression of the form $(A \supset B)$ where A and B are any well-formed formulas. The set of axioms is simply any set of well-formed formulas and the rules of inference are any set of ordered tuples of well-formed formulas.

We define a *partial implicational calculus* to be a generalized partial propositional calculus in which the axioms are tautologies and the rules of inference are modus ponens and substitution. Clearly, the class of partial implicational calculi is a subclass of the class of generalized partial propositional calculi. Hence proving that there exists a partial implicational calculus such that the problem of determining whether or not the deduction theorem holds and that there exists such a calculus for every recursively enumerable degree of unsolvability automatically gives the corresponding results for generalized partial propositional calculi.

As previously mentioned, we shall use Singletary's construction of a partial implicational calculus with recursively unsolvable decision problem. We begin by outlining this construction and supplying the necessary definitions.

A *semi-Thue system* T consists of a finite alphabet $\mathbf{Z}(T)$ and a finite set of pairs of words on $\mathbf{Z}(T)$, called the defining relations of T. We shall denote that $(U, \overline{U})$ is a defining relation by writing $U \underset{T}{\rightarrow} \overline{U}$. If U and V are words of $\mathbf{Z}(T)$ then $U \vdash_T V$ means that there exists a finite sequence of assertions $U_1 \vdash_T V_1, U_2 \vdash_T V_2, \ldots, U_n \vdash_T V_n$, where U_n is U, V_n is V and for $i = 1, 2, \ldots, n$, one of the following holds:

1. There exists a word Y on $\mathbf{Z}(T)$ and some j, $1 \leq j < i$ such that U_i is $U_j Y$ and V_i is $V_j Y$.
2. There exists a word Y on $\mathbf{Z}(T)$ and some j, $1 \leq j < i$ such that U_i is YU_j and V_i is YV_j.
3. V_i is U_i.
4. U_i is U and V_i is $\overline{U}$ where $U \underset{T}{\rightarrow} \overline{U}$ is a defining relation of T.
5. There exist j and k where $1 \leq j < i$ and $1 \leq k < i$ such that U_i is U_j, V_j is U_k, and V_i is V_k.

A semi-Thue system T is called *standard* if $\mathbf{Z}(T) = \{1, b\}$ and no word in any defining relation of T is empty. For any nonempty word on $\{1, b\}$ we define W^* as follows:

$$\begin{aligned} 1^* &= p_2 \supset (p_2 \supset p_2) \\ b^* &= p_2 \supset (p_2 \supset (p_2 \supset p_2)) \\ (X1)^* &= X^* \vee 1^* \\ (Xb)^* &= X^* \vee b^* \end{aligned}$$

where X is any nonempty word on $\{1, b\}$ and where $(A \vee B)$ is an abbreviation of the well-formed formula $((A \supset B) \supset B)$. For any nonempty word W on $\{1, b\}$, let $f(W)$ denote $W^* \vee h$, where h is $(p_2 \supset (p_2 \supset (p_2 \supset (p_2 \supset p_2))))$.

For any standard semi-Thue system T with defining relations $U_i \rightarrow \overline{U}_i$, $i = 1, 2, \ldots, m$, let $\mathbf{P}(T)$ be the partial implicational calculus whose axioms are as follows:

(1) $(((p_1 \vee q_1) \vee r_1) \vee h) \supset ((p_1 \vee (q_1 \vee r_1)) \vee h)$
(2) $((p_1 \vee (q_1 \vee r_1)) \vee h) \supset (((p_1 \vee q_1) \vee r_1) \vee h)$
(3) $((p_1 \vee h) \supset (q_1 \vee h)) \supset (((p_1 \vee r_1) \vee h) \supset ((q_1 \vee r_1) \vee h))$
(4) $((p_1 \vee h) \supset (q_1 \vee h)) \supset (((r_1 \vee p_1) \vee h) \supset ((r_1 \vee q_1) \vee h))$
(5) $(p_1 \vee h) \supset (p_1 \vee h)$
(6) $f(U_i) \supset f(\overline{U}_i)$, $i = 1, 2, \ldots, m$
(7) $((p_1 \vee h) \supset (q_1 \vee h)) \supset (((q_1 \vee h) \supset (r_1 \vee h)) \supset ((p_1 \vee h) \supset (r_1 \vee h)))$

Singletary proves that for any nonempty words W_1 and W_2 on $\{1, b\}$, $W_1 \vdash_T W_2$ if and only if $\vdash f(W_1) \supset f(W_2)$ in $\mathbf{P}(T)$. Thus, since there exists a standard semi-Thue system with recursively unsolvable decision problem, it follows that there exists a partial implicational calculus with recursively unsolvable decision problem.

A well-formed formula A of $\mathbf{P}(T)$ is *semi-regular* if A is 1^* or b^* or is of the form $A_1 \vee A_2$ where A_1 and A_2 are semi-regular. A well-formed formula A of $\mathbf{P}(T)$ is *regular* if and only if it is of the form $B \vee h$, where B is semi-regular.

For any well-formed formula B of $\mathbf{P}(T)$ whose only propositional variable is p_2 and for any well-formed formula A of $\mathbf{P}(T)$, let $\mathbf{B}(A)$ denote the result of replacing each occurrence of p_2 in B by A. A well-formed formula B of $\mathbf{P}(T)$ is *S-regular* if and only if there is a regular well-formed formula C such that B is $\mathbf{C}(A)$ for some well-formed formula A.

A well-formed formula B of $\mathbf{P}(T)$ is *S-valid* if and only if B is of the form $B_1 \supset B_2$ but not of the form $A_1 \vee A_2$ and either (1) there are regular well-formed formulas C_1 and C_2 and a well-formed formula A such that B_1 is $\mathbf{C}_1(A)$, B_2 is $\mathbf{C}_2(A)$, and $C_1 \supset C_2$ in $\mathbf{P}(T)$, or (2) B_1 is not S-regular, B_2 is not S-regular and if B_1 is S-valid then B_2 is S-valid.

3. *Recursive unsolvability.*

Lemma 1 (Singletary). *Every theorem of* $\mathbf{P}(T)$ *has one of the following forms, where* H *is a substitution instance of* h.

Form a. $((A_1 \vee H) \supset (A_2 \vee H))$
Form b. $((A_1 \vee H) \supset (A_2 \vee H)) \supset ((A_3 \vee H) \supset (A_4 \vee H))$
Form c. $((A_1 \vee H) \supset (A_2 \vee H)) \supset (((A_2 \vee H) \supset (A_3 \vee H)) \supset ((A_1 \vee H) \supset (A_3 \vee H)))$

Lemma 2 (Singletary). *All theorems of* $\mathbf{P}(T)$ *are S-valid.*

Lemma 3. *If* V_1, V_2 *and* A *are well-formed formulas of* $\mathbf{P}(T)$ *such that* V_1 *and* V_2 *are regular then* $\vdash V_1(A) \supset V_2(A)$ *in* $\mathbf{P}(T)$ *if and only if* $\vdash V_1 \supset V_2$ *in* $\mathbf{P}(T)$.

Proof. If $\vdash V_1 \supset V_2$ in $\mathbf{P}(T)$ then also $\vdash V_1(A) \supset V_2(A)$ by substitution. Conversely if $\vdash V_1(A) \supset V_2(A)$ then by Lemma 2, $V_1(A) \supset V_2(A)$ is S-valid and hence $\vdash V_1 \supset V_2$ in $\mathbf{P}(T)$ by the definition of S-validity.

For any partial propositional calculus P, and any theorem A of P, we shall say that A *depends* on an axiom B of P if and only if B is a step of every proof of A.

The classical proof of the deduction theorem for the complete propositional calculus **L** is a result of the fact that $p_1 \supset (q_1 \supset p_1)$ and $(p_1 \supset (q_1 \supset r_1)) \supset ((p_1 \supset q_1) \supset (p_1 \supset r_1))$ are theorems of **L**. Hence it is also true that the deduction theorem holds for any partial implicational calculus in which they are theorems. Conversely, if P is a partial implicational calculus in which the deduction theorem holds we have $p_1, q_1 \vdash p_1$ which implies that $\vdash p_1 \supset (q_1 \supset p_1)$. Furthermore, $p_1 \supset (q_1 \supset r_1), p_1 \supset q_1, p_1 \vdash q_1$ and $p_1 \supset (q_1 \supset r_1), p_1 \supset q_1, p_1 \vdash q_1 \supset r_1$ which imply $p_1 \supset (q_1 \supset r_1), p_1 \supset q_1, p_1 \vdash r_1$ and so $\vdash (p_1 \supset (q_1 \supset r_1)) \supset ((p_1 \supset q_1) \supset (p_1 \supset r_1))$. We use this fact to prove:

Lemma 4. *Let W_1 and W_2 be any two regular well-formed formulas of* $\mathbf{P}(T)$ *and let* $\mathbf{R}[T]\,(W_1, W_2)$ *be obtained from* $\mathbf{P}(T)$ *by adding two axioms*

(8) $(W_1 \supset W_2) \supset (p_1 \supset (q_1 \supset p_1))$

(9) $(W_1 \supset W_2) \supset ((p_1 \supset (q_1 \supset r_1)) \supset ((p_1 \supset q_1) \supset (p_1 \supset r_1)))$.

Then the deduction theorem holds in $\mathbf{R}[T]\,(W_1, W_2)$ *if and only if* $\vdash W_1 \supset W_2$ *in* $\mathbf{P}(T)$.

Proof. If $\vdash W_1 \supset W_2$ in $\mathbf{P}(T)$ then $\vdash W_1 \supset W_2$ in $\mathbf{R}[T]\,(W_1, W_2)$ and so $\vdash p_1 \supset (q_1 \supset p_1)$ and $\vdash (p_1 \supset (q_1 \supset r_1)) \supset ((p_1 \supset q_1) \supset (p_1 \supset r_1))$ in $\mathbf{R}[T]\,(W_1, W_2)$ and so the deduction theorem holds.

Conversely, suppose the deduction theorem holds in $\mathbf{R}[T]\,(W_1, W_2)$. Then $\vdash p_1 \supset (q_1 \supset p_1)$ in $\mathbf{R}[T]\,(W_1, W_2)$. Now $p_1 \supset (q_1 \supset p_1)$ is not a theorem of $\mathbf{P}(T)$ since it does not have *Form a*, *b*, or *c*. Hence it depends on either (8) or (9). Furthermore, $p_1 \supset (q_1 \supset p_1)$ is not a substitution instance of any axiom of $\mathbf{R}[T]\,(W_1, W_2)$ and so every proof of it contains at least one application of modus ponens.

Now consider any proof in $\mathbf{R}[T]\,(W_1, W_2)$ of $p_1 \supset (q_1 \supset p_1)$. At least one application of modus ponens must be used in conjunction with (8) or (9) for otherwise $p_1 \supset (q_1 \supset p_1)$ would not depend on (8) or (9). Consider the first such application. Then there is a well-formed formula which follows, by virtue of modus ponens, form two well-formed formulas of the form $(B \supset C) \supset D$ and E, where D is a substitution instance of $p_1 \supset (q_1 \supset p_1)$ or of $(p_1 \supset (q_1 \supset r_1)) \supset ((p_1 \supset q_1) \supset (p_1 \supset r_1))$, B is W_1 and C is W_2 or B is $W_1(A)$ and C is $W_2(A)$ for some well-formed formula A, and E is a substitution instance of axioms of $\mathbf{R}[T]\,(W_1, W_2)$ or is a theorem of $\mathbf{P}(T)$.

Now D cannot be of the form $F \vee G$ since if $P \supset (Q \supset P)$ were of the form $(F \supset G) \supset G$ then P would coincide with $F \supset G$ and $Q \supset P$ would coincide with G, which is impossible. Also, $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$ cannot be of the form $(F \supset G) \supset G$ since if it were, $Q \supset R$ and $(P \supset Q) \supset (P \supset R)$ would both coincide with G, which is impossible.

Since D cannot be of the form $F \vee G$, D cannot be regular. Hence $(B \supset C) \supset D$ cannot be the antecedent of axioms (8) or (9). Furthermore, $(B \supset C) \supset D$ cannot be the antecedent of any theorem of $\mathbf{P}(T)$ of *Forms b* or *c*, since otherwise D would be of the form $A_2 \vee H$.

Finally, $(B \supset C) \supset D$ is not of the form $A_1 \vee H$, i.e. of the form $(A_1 \supset H) \supset H$, where H is a substitution instance of h, since if it were, D would be C and hence D would be S-regular, a contradiction. Hence $(B \supset C) \supset D$ is not an antecedent of any theorem of $\mathbf{P}(T)$ of *Form a*.

Thus, E is an antecedent of $(B \supset C) \supset C$. That is, E is of the form $W_1(A) \supset W_2(A)$ or $W_1 \supset W_2$. However, E is a substitution instance of one of the axioms of $\mathbf{R}[T]$ (W_1, W_2) or is a theorem of $\mathbf{P}(T)$. But since no substitution instance of $p_1 \supset (q_1 \supset p_1)$ or $(p_1 \supset (q_1 \supset r_1)) \supset ((p_1 \supset q_1) \supset (p_1 \supset r_1))$ is S-regular, E is not a substitution instance of either axiom (8) or (9). Hence $\vdash E$ in $\mathbf{P}(T)$. Thus $\vdash W_1 \supset W_2$ in $\mathbf{P}(T)$ if E is $W_1 \supset W_2$ and $\vdash W_1(A) \supset W_2(A)$ in $\mathbf{P}(T)$ if E is $W_1(A) \supset W_2(A)$. Hence by Lemma 3, we have in either case, $\vdash W_1 \supset W_2$ in $\mathbf{P}(T)$.

Taking T to be a standard semi-Thue system with recursively unsolvable decision problem yields:

Theorem 1: *The problem of determining whether or not the deduction theorem holds for an arbitrarily given partial implicational calculus (and hence for an arbitrarily given generalized partial propositional calculus) is recursively unsolvable.*

Boone has shown that for any recursively enumerable degree of unsolvability D, there exists a standard semi-Thue system, $\mathbf{T}(D)$, such that the word problem for $\mathbf{T}(D)$ has degree D. Now for any standard semi-Thue system $\mathbf{T}(D)$ of degree D, let $\mathbf{C}(D)$ be the class of all partial implicational calculi of the form $\mathbf{R}[\mathbf{T}(D)]$ $(f(W_1), f(W_2))$ where W_1 and W_2 are nonempty words on $\{1, b\}$. Then $\mathbf{R}[\mathbf{T}(D)]$ defines a one-one mapping from the set of pairs of nonempty words on $\{1, b\}$ onto $\mathbf{C}(D)$ such that $W_1 \vdash W_2$ in $\mathbf{T}(D)$ if and only if the deduction theorem holds for $\mathbf{R}[\mathbf{T}(D)]$ $(f(W_1), f(W_2))$. Hence we have:

Theorem 2: *For every recursively enumerable degree of unsolvability D there exists a class of partial implicational calculi $\mathbf{C}(D)$ (and hence a class of generalized partial propositional calculi $\mathbf{C}(D)$) such that the problem of determining whether or not the deduction theorem holds for an arbitrary member of $\mathbf{C}(D)$ has degree D.*

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