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CERTAIN SETS OF POSTULATES FOR DISTRIBUTIVE LATTICES WITH THE CONSTANT ELEMENTS

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The single aim of this note is to establish such axiomatizations of distributive lattice with the constant elements, i.e. either with I and O, or with I only or with O only, that each of the equational axiom-systems presented here will contain one and only one axiom in which no constant element occurs. Since the constructions of such axiomatizations are related to certain results previously obtained and published by some other authors, the involved investigations will be referred to briefly in section **1**.

1 G. D. Birkhoff and G. Birkhoff have established, *cf.* [1], [2], pp. 135-137, and [3], pp. 34-35, that *any algebraic system*

$$\mathfrak{D} = \langle A, \cap, \cup, I \rangle$$

with two binary operations $\cap$ *and* $\cup$, *and with one constant element* $I \epsilon A$ *which satisfies the following seven postulates*

$K1$ $[a] : a \epsilon A \, . \supset . \, I = a \cup I$
$K2$ $[a] : a \epsilon A \, . \supset . \, I = I \cup a$
$K3$ $[a] : a \epsilon A \, . \supset . \, a = a \cap I$
$K4$ $[a] : a \epsilon A \, . \supset . \, a = I \cap a$
$K5$ $[a] : a \epsilon A \, . \supset . \, a = a \cap a$
$K6$ $[abc] : a, b, c \epsilon A \, . \supset . \, a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$
$K7$ $[abc] : a, b, c \epsilon A \, . \supset . \, (b \cup c) \cap a = (b \cap a) \cup (c \cap a)$

is a distributive lattice with I.

In [4], pp. 26-27, Croisot has shown that these axioms are mutually independent, *cf.* [2], p. 139, problem 65, and, moreover, he has proved that the axioms $K1$-$K7$ are inferentially equivalent to the axioms $K1$, $K3$, $K5$ and

$L1$ $[abc] : a, b, c \epsilon A \, . \supset . \, a \cap (b \cup c) = (c \cap a) \cup (b \cap a)$

2 Theorem 1. *Any algebraic system*

$$\mathfrak{A} = \langle A, \cap, \cup, I, O \rangle$$

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with two binary operations $\cap$ *and* $\cup$, *and with two constants* $I \epsilon A$ *and* $O \epsilon A$ *which satisfies the following four postulates*

$A1$ $[a] : a \epsilon A \,.\supset.\, I = a \cup I$

$A2$ $[a] : a \epsilon A \,.\supset.\, a = a \cap I$

$A3$ $[a] : a \epsilon A \,.\supset.\, a = a \cup O$

$A4$ $[abc] : a, b, c \epsilon A \,.\supset.\, a \cap ((b \cap b) \cup c) = (c \cap a) \cup (b \cap a)$

is a distributive lattice with I *and* O.

Proof: Let us assume the formulas $A1$-$A4$. Then:

$A5$ $[ab] : a, b \epsilon A \,.\supset.\, a = (I \cap a) \cup (b \cap a)$

PR $[ab] : a, b \epsilon A \,.\supset.$

$a = a \cap I = a \cap ((b \cap b) \cup I) = (I \cap a) \cup (b \cap a)$ [$A2$; $A1$; $A4$]

$A6$ $[abc] : a, b, c \epsilon A \,.\supset.\, (b \cap c) \cup (a \cap c) = c \cap ((b \cup a) \cup (b \cup a))$

PR $[abc] : a, b, c \epsilon A \,.\supset.$

$(b \cap c) \cup (a \cap c) = c \cap ((a \cap a) \cup b)$ [$A4$]

$= c \cap ((I \cap ((a \cap a) \cup b)) \cup (I \cap ((a \cap a) \cup b)))$ [$A5$]

$= c \cap (((b \cap I) \cup (a \cap I)) \cup ((b \cap I) \cup (a \cap I)))$ [$A4$]

$= c \cap ((b \cup a) \cup (b \cup a))$ [$A2$]

$A7$ $[a] : a \epsilon A \,.\supset.\, I \cap (a \cup a) = a$

PR $[a] : a \epsilon A \,.\supset.$

$I \cap (a \cup a) = I \cap (((I \cap a) \cup (I \cap a)) \cup ((I \cap a) \cup (I \cap a)))$ [$A5$]

$= ((I \cap a) \cap I) \cup ((I \cap a) \cap I)$ [$A6$]

$= (I \cap a) \cup (I \cap a) = a$ [$A2$; $A5$]

$A8$ $[a] : a \epsilon A \,.\supset.\, I \cup a = I$

PR $[a] : a \epsilon A \,.\supset.$

$I \cup a = (I \cap I) \cup (a \cap I) = I \cap ((a \cap a) \cup I) = I \cap I = I$ [$A2$; $A4$; $A1$]

$A9$ $[ab] : a, b \epsilon A \,.\supset.\, a = (b \cap a) \cup (I \cap a)$

PR $[ab] : a, b \epsilon A \,.\supset.$

$a = a \cap I = a \cap (I \cup b) = a \cap ((I \cap I) \cup b) = (b \cap a) \cup (I \cap a)$

[$A2$, $A8$; $A2$; $A4$]

$A10$ $[ab] : a, b \epsilon A \,.\supset.\, (I \cap a) \cup (b \cap a) = (b \cap a) \cup (I \cap a)$ [$A5$; $A9$]

$A11$ $[ab] : a, b \epsilon A \,.\supset.\, a \cup (b \cap (a \cup a)) = (b \cap (a \cup a)) \cup a$

PR $[ab] : a, b \epsilon A \,.\supset.$

$a \cup (b \cap (a \cup a)) = (I \cap (a \cup a)) \cup (b \cap (a \cup a))$ [$A7$]

$= (b \cap (a \cup a)) \cup (I \cap (a \cup a))$ [$A10$]

$= (b \cap (a \cup a)) \cup a$ [$A7$]

$A12$ $I \cap O = O$ [$A3$; $A7$]

$A13$ $[a] : a \epsilon A \,.\supset.\, O = a \cap O$

PR $[a] : a \epsilon A \,.\supset.$

$O = (a \cap O) \cup (I \cap O) = (a \cap O) \cup O = a \cap O$ [$A9$; $A12$; $A3$]

$A14$ $[a] : a \epsilon A \,.\supset.\, O = O \cap a$

PR $[a] : a \epsilon A \,.\supset.$

$O = I \cap O = I \cap (a \cap O) = I \cap (a \cap (O \cup O)) = I \cap (a \cap ((O \cap O) \cup O))$

[$A12$; $A13$; $A3$; $A13$]

$= I \cap ((O \cap a) \cup (O \cap a)) = O \cap a$ [$A4$; $A7$]

$A15$ $[a] : a \epsilon A \,.\supset.\, a = O \cup a$

PR $[a] : a \epsilon A \;.\supset .$
$a = a \cup O = a \cup (O \cap (a \cup a)) = (O \cap (a \cup a)) \cup a = O \cup a$ [$A3; A14; A11; A14$]

A16 $[ab] : a, b \epsilon A \;.\supset .\; a \cap (b \cap b) = b \cap a$

PR $[ab] : a, b \epsilon A \;.\supset .$
$a \cap (b \cap b) = a \cap ((b \cap b) \cup O) = (O \cap a) \cup (b \cap a)$ [$A3; A4$]
$= O \cup (b \cap a) = b \cap a$ [$A14; A15$]

A17 $[a] : a \epsilon A \;.\supset .\; I \cap a = a$

PR $[a] : a \epsilon A \;.\supset .$
$I \cap a = a \cap (I \cap I) = a \cap I = a$ [$A16; A2; A2$]

A18 $[a] : a \epsilon A \;.\supset .\; a = a \cap a$

PR $[a] : a \epsilon A \;.\supset .$
$a = a \cap I = I \cap (a \cap a) = a \cap a$ [$A2; A16; A17$]

A19 $[abc] : a, b, c \epsilon A \;.\supset .\; a \cap (b \cup c) = (c \cap a) \cup (b \cap a)$ [$A18; A4$]

Thus, since the axioms $A1; A2; A3$ and $A4$ imply $A18$ and $A19$, it has been proved that $\{A1; A2; A3; A4\} \rightleftarrows \{K1; K3; K5; L1\}$. Therefore, the proof of Theorem 1 is complete. It should be remarked that in the axiom-system discussed above the postulate $A4$ can be substituted by

*A4** $[abc] : a, b, c \epsilon A \;.\supset .\; a \cap (b \cup (c \cap c)) = (c \cap a) \cup (b \cap a)$

The proof that $\{A1; A2; A3; A4^*\} \rightleftarrows \{K1; K3; K5; L1\}$ requires the use of deductions entirely analogous to that which are given above.

3 For distributive lattices with I or O we have similar theorems. Namely:

Theorem 2. *Any algebraic system*

$$\mathfrak{B} = \langle A, \cap, \cup, I\rangle$$

with two binary operations $\cap$ *and* $\cup$, *and with one constant element* $I \epsilon A$ *which satisfies the postulates A1, A2, A17 and A4* (see section **2** above) *is a distributive lattice with I.*

and

Theorem 3. *Any algebraic system*

$$\mathfrak{C} = \langle A, \cap, \cup, O\rangle$$

with two binary operations $\cap$ *and* $\cup$, *and one constant element* $O \epsilon A$ *which satisfies the postulates*

C1 $[a] : a \epsilon A \;.\supset .\; a = a \cup O$
C2 $[a] : a \epsilon A \;.\supset .\; a = O \cup a$
C3 $[a] : a \epsilon A \;.\supset .\; O = a \cap O$
C4 $[abc] : a, b, c \epsilon A \;.\supset .\; a \cup ((b \cup b) \cap c) = (c \cup a) \cap (b \cup a)$

is a distributive lattice with O.

Proof: We can prove Theorem 2 more easily than Theorem 1. Namely, let us assume $A1, A2, A17$ and $A4$. Then:

B1 $[ab] : a, b \epsilon A . \supset . a = a \cup (b \cap a)$

PR $[ab] : a, b \epsilon A . \supset .$

$a = a \cap I = a \cap ((b \cap b) \cup I) = (I \cap a) \cup (b \cap a) = a \cup (b \cap a)$ [*A2*; *A1*; *A4*; *A17*]

B2 $[a] : a \epsilon A . \supset . a = a \cup a$ [*B1*; *A17*]

B3 $[ab] : a, b \epsilon A . \supset . (a \cap a) \cup b = b \cup a$

PR $[ab] : a, b \epsilon A . \supset .$

$(a \cap a) \cup b = I \cap ((a \cap a) \cup b) = (b \cap I) \cup (a \cup I)$ [*A17*; *A4*]

$= b \cup a$ [*A2*]

B4 $[a] : a \epsilon A . \supset . (a \cap a) = a$

PR $[a] : a \epsilon A . \supset .$

$(a \cap a) = (a \cap a) \cup (a \cap a) = (a \cap a) \cup a$ [*B2*; *B3*]

$= I \cap ((a \cup a) \cap a) = (a \cap I) \cup (a \cap I) = a \cup a = a$ [*A17*; *A4*; *A2*; *B2*]

B5 $[abc] : a, b, c \epsilon A . \supset . a \cap (b \cup c) = (c \cap a) \cup (b \cap a)$ [*A4*; *B4*]

Since *B4* and *B5* are the consequences of *A1*, *A2*, *A17* and *A4*, we have $\{A1; A2; A17; A4\} \rightleftarrows \{K1; K3; K5; L1\}$. Therefore, the proof is complete. Similarly to the previous Theorem in the present axiomatization we can substitute *A4* by *A4**.

A proof of Theorem 3 is omitted here, since it is self-evident that it is a dual of the deductions which were used in order to obtain Theorem 2, Croisot's theorem, and finally the theorem of Birkhoffs.

4 The mutual independence of the axioms *A1*, *A2*, *A3* and *A4*. It is obvious that *A3* does not follow from *A1*, *A2* and *A4*, since in the field of distributive lattice with I it is impossible to define the constant element O by the means of $\cap$, $\cup$ and I alone. On the other hand, the following matrices

$\mathfrak{M}1$

$\cap$	α	I	O
α	α	α	O
I	α	I	O
O	O	O	O

$\cup$	α	I	O
α	α	α	α
I	α	α	I
O	α	I	O

$\mathfrak{M}2$

$\cap$	α	I	O
α	I	I	I
I	I	I	I
O	I	I	I

$\cup$	α	I	O
α	I	I	α
I	I	I	I
O	I	I	O

$\mathfrak{M}3$

$\cap$	α	I	O
α	α	α	O
I	I	I	O
O	O	O	O

$\cup$	α	I	O
α	I	I	α
I	I	I	I
O	α	I	O

which are the suitable modifications of Croisot's examples E'_{2a}, E'_3 and E'_4 respectively, *cf.* [4], p. 27, are such that

(a) Matrix $\mathfrak{M}1$ verifies *A2*, *A3* and *A4*, but it falsifies *A1*.

(b) Matrix $\mathfrak{M}2$ verifies $A1$, $A3$ and $A4$, but it falsifies $A2$.
(c) Matrix $\mathfrak{M}3$ verifies $A1$, $A2$ and $A3$, but falsifies $A4$ for a/α, b/I and c/I.

Thus, the axioms $A1, A2, A3, A4$ are mutually independent. Since the axiom-systems given in section **3** above are almost banal and not especially interesting, the mutual independence of the axioms belonging to them is not discussed here.

5 Final remark. In the axiom-systems $\{A1; A2; A3; A4\}$ and $\{A1; A2; A17; A4\}$ $A4$ cannot be substituted by $L1$, since the following matrix

$$\mathfrak{M}4 \qquad \begin{array}{c|cccc} \cap & \alpha & \beta & I & O \\ \hline \alpha & \alpha & \alpha & \alpha & O \\ \beta & \alpha & \alpha & \beta & O \\ I & \alpha & \beta & I & O \\ O & O & O & O & O \end{array} \qquad \begin{array}{c|cccc} \cup & \alpha & \beta & I & O \\ \hline \alpha & \alpha & \beta & I & \alpha \\ \beta & \beta & \beta & I & \beta \\ I & I & I & I & I \\ O & \alpha & \beta & I & O \end{array}$$

which is a modification of Croisot's example E_1, *cf.* [4], p. 26, verifies $A1$, $A2$, $A3$, $A17$ and $L1$, but falsifies $A4$ for a/I, b/β and c/α. Similarly, we can prove that in $\{C1; C2; C3; C4\}$ $C4$ cannot be substituted by the dual of $L1$.

N.B. After this paper was composed, the author unexpectedly obtained a stronger result which makes the deductions presented here obsolete. Namely, it has been proved that the axioms $A1$, $A2$, and $A4$ imply formula $A17$. For this reason this paper should be compared with [5].

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