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AXIOMATIC INSCRIPTIONAL SYNTAX
PART I: GENERAL SYNTAX

V. FREDERICK RICKEY

Inscriptional syntax is that study of syntax wherein the linguistic entities are studied as inscriptions, *i.e.*, as physical objects and not as abstract entities. In this paper* we shall axiomatize the syntax which is common to all languages, *i.e.*, General Syntax. In *Chapter I* of this paper we elucidate the notion of an inscription, expose some pre-logical assumptions, describe the three primitive terms of inscriptional syntax, and discuss our logical basis (*viz.*, Leśniewski's Ontology). In *Chapter II* we present the axioms for the syntactical system **M**, define the usual notions of general syntax, and prove some typical theorems of general syntax. Our aim is not to obtain new syntactical results, but rather to put the theory of syntax on a secure foundation. Accordingly, we shall only develop system **M** to the point where most syntactical investigations begin. In particular, concatenation is defined in our system, whereas it is usually taken as primitive.

The initial task of syntax is to formulate precise statements of the formative and deductive rules of a particular formal language. After these rules have been stated it is of interest to develop their consequences by proving derived rules and to investigate the interconnections between primitive and derived rules. All of these tasks can be accomplished using system **M**. To support this claim we shall formulate the rule of Protothetic in the second part of this paper.

Introduction. When Frege axiomatized the propositional calculus in his *Begriffsschrift* he realized that the deductive rules could not be expressed in the system itself[1], and so expressed them in ordinary language with the

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help of schematic diagrams. Since then many methods of presenting and discussing the syntax of a particular language or language in general have been developed.

The first attempt at axiomatization of the meta-language was by Tarski in 1930 in his famous paper on the concept of truth [43]. As primitive terms he took several logical constants and the notion of concatenation of expressions, from which he built up longer expressions, formulas, etc. A particularly fruitful approach to metatheory was initiated by Gödel [9] who arithmeticized the syntax so that concatenation became a certain arithmetical operation.

In both of these approaches the logical basis is considerable; the first uses set theory, the second arithmetic. Martin [31], [30] has urged, and his views have been supported by Carnap [5], that *the meta-language be taken as weak as possible*. Toward this end we shall construct a theory of syntax wherein variables of only one type are used.[2]

To say what these variables designate it is necessary to recall Peirce's distinction between *types* and *tokens*. If asked how many letters occur in the word

Frege

one would ordinarily say five. Another person—understanding the word "letter" in a distinct but perfectly correct sense—would reply that there are only four, *viz.*,

F, r, e, g.

Peirce would say that there are five letter tokens and four letter types in this word; moreover, that two of the letter tokens are instances of the same type. Hence he distinguishes between the general pattern or type of a letter and a particular instance or token of that letter.

This distinction gives rise to two views of syntax depending on whether the variables designate types or tokens. The classical view of syntax, initiated by Tarski and Gödel uses variables which designate types. When the variables designate tokens we obtain *inscriptional syntax*, so called because it refers only to the inscriptions of the language. Leśniewski [18] was the first to take this view of syntax. Since then it has been developed by Goodman and Quine [11], Goodman [10], and Martin [27], [28], [29], [30].

When the inscriptionalistic view is taken, the distinction between types and tokens becomes superfluous. There is however a need to eliminate expressions like "tokens of the same type," which is a relation between inscriptions established via comparison with the abstract type. In inscriptional syntax this notion is replaced by a relation of *equiformity* between inscriptions, which makes no use of abstract entities.

In this paper we take the inscriptionalistic (or nominalistic) point of view. The technical advantages of this method are that we can use a weaker logical basis and that the notion of equiformity is laid bare for closer scrutiny. There are ample philosophical reasons for preferring the inscriptionalistic viewpoint (*cf.* Goodman [10], p. 360ff. and Martin [30], Ch. XIII). These reasons are so strong that sometimes syntax is defined to be inscriptional (Curry [8], p. 36). Finally, since syntax is the study of linguistic objects, it seems preferable to study them as inscriptions rather than as abstractions of inscriptions.

The investigation of formation and deductive rules requires a specific language or family of related languages. In this paper we shall only investigate the syntactical properties common to all languages. This is the study of *general syntax*.

CHAPTER I

1 *Syntax and its Pre-logical Assumptions*. *Inscriptional Syntax*, which is our approach to syntax, deals with the objects of a (formalized) language as physical objects and as such admits reference only to the form and arrangement of these inscriptions.[3] The linguistic objects studied are *inscriptions*, and we deal with them *qua* inscriptions, not as abstract entities.

Inscriptions are physical objects consisting of a finite number of individual marks called words. They may be spatio-temporally scattered or discontinuous and need not consist of an uninterrupted string of symbols.[4] For example, each of the following objects are inscriptions:[5]

$$=,\ s\,0\,t,\ f(p),\ x = y.$$

The object consisting of the first, third and last words of the inscription

$$(*\ m\ n\quad 0 = +$$

is also an inscription. We only consider inscriptions which actually exist at some particular point in space-time. *Thus there is no such thing as the empty inscription.*[6]

By a *word* we mean any mark which, in any given linguistic context, is considered only as a whole. As examples of words we have:

$$a,\ \&,\ p,\ (,\ R,\ \phi,\ \llcorner,\ \urcorner.$$

The following are inscriptions which are not words but consist of two, four, and six words respectively:

$$p\ q,\ g(t),\ a/ * (C).$$

Syntax is not concerned with parts of words and indeed contains no apparatus for discussing them, it is concerned only with typographical characters in their entirety.[7]

In *Chapter II* we will see that words are definable in terms of inscriptions, our primitive notion. It is quite natural then to require that inscriptions be uniquely decomposable into words. The following two pre-logical assumptions guarantee this[8]:

1. Words are connected symbols.
2. Distinct words do not touch.

If the first of these conditions is disregarded we would be able to decompose some inscriptions in several ways. For example, if our language contained the words

$$a,\ a',\ 'a,$$

then the inscription ≪$a'a$≫[9] could be interpreted both as "a" followed by "$'a$" and as "a'" followed by "a".

In practice we naturally relax this condition so that indices of various sorts and also common words like

$$=,\ \bar{z},\ p',\ j\ ,\ !\ ,\ i,$$

can be used. But keep in mind that this is only an informal convention.[10]

The proscription against words touching prevents ≪ w ≫ from being variously interpreted as "w" followed by "v", or "v" followed by "w", or even as three consecutive "v" 's.[11]

In most presentations of a formal system certain fonts of letters are used for specific purposes. For example, in the first order predicate calculus, the words

$$x,\ y,\ z,\ \ldots$$

are used as individual variables and

$$P,\ Q,\ R,\ \ldots$$

are used as predicates. In general syntax no such restrictions are necessary, although for special theories such restrictions are sometimes used. We never make any restrictions about the shape of symbols or preassign symbols to specific categories, this being a matter of indifference.

The availability of symbols, and in particular of non-equiform symbols (*i.e.*, symbols of different shapes, *cf. Ch. I,* 2), is a subject of some debate. Leśniewski, among others[12], presumed that there is always the "possibility" of creating new symbols non-equiform with any in existence. Considering this to be a question for the metaphysician or empirical scientist and recalling the difficulties of Goodman and Quine [11] on this topic, we shall remain neutral. The number of existing or possible inscriptions will not affect our work.

2 *The Primitive Terms for General Syntax.* Since syntax is the study of the *form* and *arrangement* of symbols we will need several primitive metalogical terms to discuss these concepts.

In classical syntax the form of symbols is avoided; two symbols are of the same form if they are tokens of the same type. In inscriptional syntax the notion "of the same type" is replaced by a relation between inscriptions which we call *equiformity*.

Intuitively, and of course pre-axiomatically, we say two words are equiform when they have the same shape, *i.e.*, when they can be printed from the same piece of type.[13] For example the words ≪A≫ and ≪ A≫ are equiform, but no two of the following words are:

$$a,\ \mathrm{A},\ \mathsf{a},\ \mathsf{A},\ \lessdot,\ \forall\ .$$

Equiformity is a relation between two inscriptions, not just words. Intuitively two inscriptions are equiform when they have the same number of words and corresponding words are equiform. This is expressed

$$A \ \varepsilon\ \mathbf{cnf}(B)$$

and is read "A is equiform (conformal) to B". This is one of our primitive syntactical terms.

The inscriptions $\ll(p)\gg$ and $\ll(\ p)\gg$ are equiform (so spacing is irrelevant), but neither of these is equiform to $\ll((p)\gg$ or to any inscription not consisting of exactly three words. In the inscription[14]

$$A\colon\ \underbrace{\overbrace{(\,p\ q}^{C}\ )\ /\ s\ q}_{B}\ )\ *$$

the inscription B consisting of the 1st, 2nd, 7th, and 8th words of A[15] is equiform to the inscription C consisting of the first four words of A.[16] It is very important to realize that *two equiform words in two different places are never the same word*.[17] There is no ideal letter "A" of which all concrete letters "A" are instances.

Discussion of the arrangement of inscriptions requires relations of precedence and containment. To insure that our language be linear[18] we need a total ordering on the words. We shall use

$$A \ \varepsilon\ \mathbf{pr}(B),$$

which is read "*A precedes B*", or more precisely "The word A is preceding the word B". Without the restriction to words we would not have trichotomy, *i.e.*,

$$A\,\varepsilon\,\mathbf{pr}(B)\ .\lor.\ A = B\ .\lor.\ B\,\varepsilon\,\mathbf{pr}(A),$$

for consider the inscription

$$C\colon\ \underbrace{a\ \overbrace{b\ c\ d}^{B}\ e}_{A}$$

where A is the inscription consisting of the 1st, 3rd, and last words of C, and B consists of the 2nd and 4th words of C. Neither of these inscriptions precedes the other. Later (*cf. Ch. II*) we will define an ordering on certain non-overlapping non-meshing inscriptions.

It suffices to take a very weak inclusion relationship, *viz.*, that a word is part of an inscription. We write this[19]

$$A \ \varepsilon\ \mathbf{vrb}(B)$$

and read "A is a word in the inscription B". For example in the inscription

$$A\colon\ \llcorner \underset{\substack{\uparrow\\ C}}{p} \lrcorner\ \ulcorner \overbrace{\phi(\,\underset{\substack{\uparrow\\ B}}{p}\ p\ )\urcorner}^{D}$$

C and B are words in A, but C is not a word in D, nor is D a word in A. Hence "is equiform to a word in" is not synonymous with "is a word in".

Leśniewski [18] used as primitive the term "inscription A is part of inscription B," where by this we mean that every word in A is a word in B. This term is stronger, both in the sense of Leśniewski (*cf.* Sobociński [41]) and Goodman [10], so we have chosen "**vrb**(B)" as primitive.

Notice that "word" is definable in terms of "**vrb**(B)":

$$[A]{:}A\,\varepsilon\,\mathbf{vrb}\,.\equiv.\,[\exists B]\,.\,A\,\varepsilon\,\mathbf{vrb}(B)$$

and so need not be taken as a primitive term.

Since we will assume that inscriptions contain only a finite number of words (the name) **vrb**(B) is well ordered by **pr**, and hence it is meaningful to speak of the 1st, 2nd, . . . , last word of A. (*cf. Ch. II, D3, D4, D5*).

Our choice of primitive syntactical terms has been guided by the theoretical and aesthetic requirements on deductive theories outlined by Sobociński [41]. Naturally our terms have been chosen so that they are adequate to define the syntactical terms needed to state the rules of deductive theories; PART II of the paper will bear this out. The terms are independent in the sense that no two can define the third, and they are simpler than any other known set.[20] Admittedly we could give a single term which would suffice to define our three primitive terms but it would not be perspicuous.

3 *The Logical Basis for Syntax*: *Ontology*. Ontology is the most general theory of the connections between names. Its sole primitive term is the "ε" in the individual proposition

$$A\,\varepsilon\,a.$$

We read this "*A is a*" and interpret it as meaning

(1) A is an unempty name.
(2) A is unique.
(3) Anything which is A is also a.

Both A and a are names and so belong to the *same* semantical category or logical type. Consequently it is quite meaningful to write

$$A\,\varepsilon\,A.$$

This means simply that A is a name which denotes precisely one object, *i.e.*, A is an individual name. This interpretation, when formalized, gives the following axiom for Ontology:

$$AxO\quad [Aa]\,.\!\cdot\!.\,A\,\varepsilon\,a\,.\equiv:[\exists B]\,.\,B\,\varepsilon\,A:\;[BC]{:}\,B\,\varepsilon\,A\,.\,C\,\varepsilon\,A\,.\supset.\,B\,\varepsilon\,C:\;[B]{:}\,B\,\varepsilon\,A\,.\supset.\,B\,\varepsilon\,a$$

From this axiom one can deduce the following four theses which characterize Ontology.

Ont. 1 $[ABa]{:}\,A\,\varepsilon\,B\,.\,B\,\varepsilon\,a\,.\supset.\,B\,\varepsilon\,A$
Ont. 2 $[Aa]{:}\,A\,\varepsilon\,a\,.\supset.\,A\,\varepsilon\,A$

Ont. 3 $[Aa]: A \varepsilon a . \supset . [\exists B] . B \varepsilon A$
Ont. 4 $[ABa]: A \varepsilon B . B \varepsilon a . \supset . A \varepsilon a$

It is impossible to give here more than the definitions and less known theses of Ontology which we shall use. For a formulation of the rule of Ontology see Leśniewski [19], and for a discussion of its axiomatization see Sobociński [38]. For an introduction to Ontology see Sobociński [39], Lejewski [15], Słupecki [37], and Küng [13]. Leśniewski's views on names can be found in Lejewski [16] and Sinisi [33], [34], [35]. The motivation for the semantical categories can be found in Hiż [12] and Bocheński [2]; the formal properties in Ajdukiewicz [1] and Machover [26].

The following definitions are well known from the literature of Ontology.

DO1 $[a]: !\{a\} . \equiv . [\exists A] . A \varepsilon a$

a is *non empty*.

DO2 $[a] \therefore \rightharpoonup \{a\} . \equiv : [BC]: B \varepsilon a . C \varepsilon a . \supset . B \varepsilon C$

a is *unique*.

DO3 $[A]: A \varepsilon \vee . \equiv . A \varepsilon A$

A is *an individual*.

DO4 $[A]: A \varepsilon \wedge . \equiv . A \varepsilon A . \sim (A \varepsilon A)$

A is the *empty object*. The name $\wedge$ does not denote (or denotes nothing, if you will).

DO5 $[AB]: A = B . \equiv . A \varepsilon B . B \varepsilon A$

A is *identical* with individual B.

DO6 $[Ab]: A \neq B . \equiv . A \varepsilon A . B \varepsilon B . \sim (A = B)$

Individuals A and B are *different*.

DO7 $[ab] \therefore a \subset b . \equiv : [A]: A \varepsilon a . \supset . A \varepsilon b$

a is *contained in* b. This is Boolean inclusion.

DO8 $[ab] \therefore a \circ b . \equiv : [A]: A \varepsilon a . \equiv . A \varepsilon b$

a equals b. Equality holds between names, either empty, individual, or general whereas identity (=) holds only between individual names.

DO9 $[ab]: a \Subset b . \equiv . a \subset b . \sim (a \circ b)$

a is *properly included into* b.

DO10 $[Aab] \therefore A \varepsilon a \cup b . \equiv : A \varepsilon A : A \varepsilon a . \vee . A \varepsilon b$

A is *a or b*.

DO11 $[Aab]: A \varepsilon a \cap b . \equiv . A \varepsilon a . A \varepsilon b$

A is *a and b*.

DO12 $[Aa]: A \varepsilon \sim (a) . \equiv . A \varepsilon A . \sim (A \varepsilon a).$
A is *non* a.

DO13 $[\phi] .\because . \rightleftarrows \{\phi\} . \equiv : [abc] : \phi\{ab\} . \phi\{ac\} . \supset . b \circ c : [abc] : \phi\{ab\} . \phi\{cb\} . \supset . a \circ c$
ϕ is a *1-1 binary connection.*

DO14 $[ab] :: a \infty b . \equiv :. [\exists \phi] .\because . \rightleftarrows \{\phi\} : [A] : A \varepsilon a . \equiv . [\exists B] . \phi\{AB\} . B \varepsilon b : [A] :$
$A \varepsilon b . \equiv . [\exists B] . \phi\{BA\} . B \varepsilon a$
a is *equinumerous* with b.

DO15 $[ab] .\because . a < b . \equiv : [\exists c] . c \subset b . c \infty a : \sim (a \infty b)$
a is *less-numerous* than b.

DO16 $[ab] .\because . a \leq b . \equiv : a < b . \vee . a \infty b$
a is *equi-or less-numerous* than b.

DO17 $[\phi a] .\because . \mathrm{Irr} < \phi > \{a\} . \equiv : \phi\{a\} : [b] : b \subset a . \phi\{b\} . \supset . a \circ b$
a is *minimal* with respect to ϕ.

DO18 $[a] :: \mathrm{Fin} \{a\} . \equiv .\because . [\phi b] .\because . \phi\{b\} : [d] : \phi\{d\} . \supset . d \subset a : \supset . [\exists c] . \mathrm{Irr} < \phi > \{c\}$
a is *finite*. This definition is due to Tarski [42]. We will need the following theorems of Ontology:

Ont. 5 $[AB\phi] : A = B . \phi\{A\} . \supset . \phi\{B\}$
Ont. 6 $[ab\phi] : a \circ b . \phi\{a\} . \supset . \phi\{b\}$

These theses of extensionality will be used frequently. Actually *Ont. 5* is a special case of *Ont. 6*.

Ont. 7 $[Aa] : A \varepsilon a . \supset . A = A$
Ont. 8 $[A] : A \varepsilon A . \equiv . \rightharpoonup\{A\} . ! \{A\}$
Ont. 9 $[Ab] : A \varepsilon A . \supset . \rightharpoonup\{A \cap b\}$
Ont. 10 $[ab] : \rightharpoonup\{a\} . b < a . \supset . a \varepsilon \vee$
Ont. 11 $[Ab] : A \cap b \varepsilon \vee . A \varepsilon A . \supset . A \varepsilon b$
Ont. 12 $[Aab] : a < b . A \varepsilon \sim (b) . \supset . a \cup A < b \cup A$
Ont. 13 $[Aab] : A \varepsilon A . a < b \cup A . \supset . a \leq b$
Ont. 14 $[a] . a \infty a$
Ont. 15 $[ab] : a \infty b . \supset . b \infty a$
Ont. 16 $[abc] : a \infty b . b \infty c . \supset . a \infty c$
Ont. 17 $[ab] : \mathrm{Fin} \{a\} . \supset . \mathrm{Fin} \{a \cap b\}$
Ont. 18 $[ab] : a \mathbin{\subset\!\!\!\cdot} b . \mathrm{Fin} \{b\} . \supset . \sim (a \infty b)$
Ont. 19 $[ab] : a \infty b . \rightharpoonup\{a\} . \supset . \rightharpoonup\{b\}$
Ont. 20 $[ab] : a \leq b . b \leq a . \supset . a \infty b$
Ont. 21 $[ABC] : A \varepsilon B \cup C . B \varepsilon B . C \varepsilon C . \supset : A = B . \vee . A = C$

The above has been a sketch of the Ontology which we will use in the sequel. In the following chapters all of the theses quantify only over the semantical category of names. It is however the case that some of the theses of Ontology which we use, but which are not part of our syntactical systems, require variables of higher types either for their statement or proof. Most notable among such theses is the law of extensionality for the semantical category of names (*Ont. 6*). The definitions of equinumerosity and finite require variables of higher types, but we do not use these higher variables in the sequel. In order to have a short name for the part of Ontology which we have described above we call it "first order" Ontology.

CHAPTER II

1 *The Axioms of System* **M**. Our theory of general syntax is based on Leśniewski's Ontology; not the full system, but just the "first order" fragment. This means that we shall only quantify variables from the semantical category of names. Moreover, since syntax is concerned only with inscriptions, we shall further restrict the range of our variables to names denoting inscriptions.[21]

As explained in *Chapter I,* **2**, system **M** has three primitive terms

$A \,\varepsilon\, \mathbf{vrb}(B)$	A is a word in B
$A \,\varepsilon\, \mathbf{pr}(B)$	A is a word preceding B
$A \,\varepsilon\, \mathbf{cnf}(B)$	A is equiform to B

The arguments of these terms are names, but the resulting sentences can only be true when the arguments are individual names, *i.e.*, names denoting exactly one inscription. It is only in the last axiom (*A8* below) that the names are general names[22], *i.e.*, names denoting more than one inscription. As will be seen later this axiom is extremely powerful. The following are the axioms for the syntactical system **M**:

Group I.

A1 $[AB] \therefore A = B \,.\!\equiv\!:\, [\exists C] \,.\, C \varepsilon\, \mathbf{vrb}(A) : [C] : C \varepsilon\, \mathbf{vrb}(A) \,.\!\equiv\!.\, C \varepsilon\, \mathbf{vrb}(B)$

A2 $[AB] \therefore A \varepsilon\, \mathbf{vrb}(B) \,.\!\equiv\!:\, A \varepsilon A : [C] : C \varepsilon\, \mathbf{vrb}(A) \,.\supset.\, C = A :$
$\quad [\exists C] \,.\, C \varepsilon\, \mathbf{vrb}(B) \,.\! \sim (A \varepsilon\, \mathbf{pr}(C)) \,.\! \sim (C \varepsilon\, \mathbf{pr}(A))$

Group II.

A3 $[AB] : A \varepsilon\, \mathbf{pr}(B) \,.\supset.\, A \varepsilon\, \mathbf{vrb}(A)$

A4 $[AB] : A \varepsilon\, \mathbf{pr}(B) \,.\supset.\, B \varepsilon\, \mathbf{vrb}(B)$

A5 $[ABC] : A \varepsilon\, \mathbf{pr}(B) \,.\, B \varepsilon\, \mathbf{pr}(C) \,.\supset.\, A \varepsilon\, \mathbf{pr}(C)$

Group III.

A6 $[AB] \therefore A \varepsilon\, \mathbf{cnf}(B) \,.\!\equiv\!:\, A \varepsilon A : [C] : B \varepsilon\, \mathbf{cnf}(C) \,.\!\equiv\!.\, A \varepsilon\, \mathbf{cnf}(C) :$
$\quad \mathbf{vrb}(A) \propto \mathbf{vrb}(B) : [CD] : \sim (A = B) \,.\, C \varepsilon\, \mathbf{vrb}(A) \,.\, D \varepsilon\, \mathbf{vrb}(B) .$
$\quad (\mathbf{vrb}(A) \cap \mathbf{pr}(C)) \propto (\mathbf{vrb}(B) \cap \mathbf{pr}(D)) \,.\supset.\, C \varepsilon\, \mathbf{cnf}(D)$

Group IV.

A7 $[A] : A \varepsilon A \,.\supset.\, \mathrm{Fin}\, \{\mathbf{vrb}(A)\}$

A8 $[Aa] : A \varepsilon a \,.\supset:\, [\exists B] : B \varepsilon B : [C] : C \varepsilon\, \mathbf{vrb}(B) \,.\!\equiv\!.\, [\exists D] \,.\, D \varepsilon a \,.\, C \varepsilon\, \mathbf{vrb}(D)$

To simplify the statement of *A8* we introduce the following definition:

D1 $[Aa] \therefore A \varepsilon\, \mathbf{Kl}\,(a) \,.\!\equiv\!:\, A \varepsilon A : [B] : B \varepsilon\, \mathbf{vrb}(A) \,.\!\equiv\!.\, [\exists C] \,.\, C \varepsilon a \,.\, B \varepsilon\, \mathbf{vrb}(C)$
A is the *Klass* of a.

This notion, which is akin to Leśniewski's Mereological notion of class[33], will be discussed in **3** below. Using this definition we have at once

T2.1.1 $[Aa] : A \varepsilon a \,.\supset.\, [\exists B] \,.\, B \varepsilon\, \mathbf{Kl}\,(a)$ [*A8*, *D1*]

The rules for system **M** are the same as those for Ontology. It must be remembered that our quantifiers are restricted to names of inscriptions.

Tarski was the first to give axioms for meta-theory [43], p. 173, but he did not construct a deductive theory. His logical basis, including recursive definitions, is considerably stronger than ours. Martin [30], p. 229ff., attempts to axiomatize the inscriptional theory of Goodman and Quine [11]. Chomsky [6] also gives some axioms. But none of these proceed in a formal way. The axioms are merely stated, then used intuitively. We do not use recursive definitions. Admittedly using Frege's device we could always eliminate recursive definitions, but it is not clear that they can always be eliminated without using variables of higher order.

2 *Elementary Consequences and Comments on the Axioms.*
Comments on Group I. Axiom *A1* can be split into three parts:

T2.2.1 $[A]:A\varepsilon A\,.\supset.[\exists B].B\,\varepsilon\,\mathbf{vrb}(A)$ [*A1*, *B*/*A*, *DO5*]
T2.2.2 $[ABC].\therefore C\varepsilon\,\mathbf{vrb}(A):[C]:C\varepsilon\,\mathbf{vrb}(A).\equiv.C\varepsilon\,\mathbf{vrb}(B):\supset.A=B$ [*A1*]
T2.2.3 $[AB].\therefore A=B.\supset:[C]:C\varepsilon\,\mathbf{vrb}(A).\equiv.C\varepsilon\,\mathbf{vrb}(B)$ [*A1*]

In the full Ontology *T2.2.3* follows from extensionality [*Ont. 5*], and so *A1* is externally dependent.[23] Thus we have

$$\{A1\}\rightleftarrows\{T2.2.1,\ T2.2.2\}$$

T2.2.1 is important for it shows that every individual, *i.e.*, every inscription, contains at least one word. Later we will show that inscriptions consist entirely of words [*T2.3.2.*] and that they are uniquely decomposable into words [*T2.3.8*]. Two inscriptions are identical precisely when they contain the same words:

T2.2.4 $[AB].\therefore A\varepsilon A:[C]:C\varepsilon\,\mathbf{vrb}(A).\equiv.C\varepsilon\,\mathbf{vrb}(B):\supset.A=B$
[*T2.2.1*, *T2.2.2*]

Consequently if an inscription has a word it is an individual:

T2.2.5 $[AB]:A\varepsilon\,\mathbf{vrb}(B).\supset.B\varepsilon B$ [*T2.2.2*, *DO5*]
T2.2.6 $[A]:A\varepsilon A.\equiv.[\exists B].B\varepsilon\,\mathbf{vrb}(A)$ [*T2.2.1*, *T2.2.5*]

For another important consequence of *A1* see *T2.3.1* below. From *A2* we obtain immediately:

T2.2.7 $[ABC]:A\varepsilon\,\mathbf{vrb}(B).B\varepsilon\,\mathbf{vrb}(C).\supset.A=B$ [*A2*]
T2.2.8 $[AB]:A\varepsilon\,\mathbf{vrb}(B).\supset.A\varepsilon\,\mathbf{vrb}(A)$
Hyp(1).$\supset$.
$[\exists C]$.
(2) $C\varepsilon\,\mathbf{vrb}(A)$. [1, *Ont. 2*, *T2.2.1*]
(3) $C=A$. [*T2.2.7*, 1, 2]
$A\varepsilon\,\mathbf{vrb}(A)$ [2, 3]

This theorem[24], motivates the following definition.

D2 $[A]:A\varepsilon\,\mathbf{vrb}.\equiv.A\varepsilon\,\mathbf{vrb}(A)$

This is to be read "*A* is a *word*".[25] As an equivalent definition we could take:

$$[A]:A\,\varepsilon\,\mathbf{vrb}\,.\equiv.\,[\exists B]\,.\,A\,\varepsilon\,\mathbf{vrb}(B)\,.$$

Using *D2* we can prove that all words in inscriptions are words:

T2.2.9 $[AB]:A\,\varepsilon\,\mathbf{vrb}(B)\,.\supset.\,A\,\varepsilon\,\mathbf{vrb}$ [*T2.2.8*, *D2*]

We stated in *Chapter I*, **2** that words are atoms, *i.e.*, the smallest inscriptions of our investigation. This is contained in the following frequently used theorem:

T2.2.10 $[AB]:A\,\varepsilon\,\mathbf{vrb}(B)\,.\,B\,\varepsilon\,\mathbf{vrb}\,.\supset.\,A=B$ [*D2*, *T2.2.7*]
T2.2.11 $[ABC]:A\,\varepsilon\,\mathbf{vrb}(B)\,.\,B\,\varepsilon\,\mathbf{vrb}(C)\,.\supset.\,A\,\varepsilon\,\mathbf{vrb}(C)$ [*T2.1.7*]

We now prove that anti-symmetry for **pr** follows from Axioms *A2* and *A3*.

T2.2.12 $[A]:A\,\varepsilon\,\mathbf{vrb}(A)\,.\supset.\sim(A\,\varepsilon\,\mathbf{pr}(A))$
Hyp(1) $.\supset.$
$[\exists C]\,.$
(2) $C\,\varepsilon\,\mathbf{vrb}(A)\,.$
(3) $\sim(A\,\varepsilon\,\mathbf{pr}(C))\,.$ [1, *A2*]
(4) $C=A\,.$ [*T2.2.7*, 2, 1]
$\sim(A\,\varepsilon\,\mathbf{pr}(A))$ [3, 4]
T2.2.13 $[A]\,.\sim(A\,\varepsilon\,\mathbf{pr}(A))$
Dem: $[A]$:
(1) $A\,\varepsilon\,\mathbf{vrb}(A)\,.\lor.\sim(A\,\varepsilon\,\mathbf{vrb}(A)):$ [*B*]
$\sim(A\,\varepsilon\,\mathbf{pr}(A))$ [1, *T2.2.12*, *A3*]
T2.2.14 $[AB]:A\,\varepsilon\,\mathbf{pr}(B)\,.\supset.\,A\neq B$
Hyp(1) $.\supset:$
(2) $B\,\varepsilon\,B:$ [*A4*, 1]
(3) $A=B\,.\lor.\,A\neq B$ [1, 2, *DO6*]
$A\neq B$ [3, *T2.2.13*, 1]

These theorems[26] bring out the strength of the innocent looking third conjunct of *A2*. At first it seems unnecessary since

$$A\,\varepsilon\,\mathbf{vrb}(B)\,.\supset.\,[\exists C]\,.\,C\,\varepsilon\,\mathbf{vrb}(B)$$

follows by quantification theory with $A = C$. Then with $A = C$ and using *T2.2.13* we could drop

$$\sim(A\,\varepsilon\,\mathbf{pr}(C))\,.\sim(C\,\varepsilon\,\mathbf{pr}(A))$$

and hence the whole third conjunct of *A2*. (Of course we would then have to take *T2.2.13* as an axiom.) Doing this would so weaken *A2* that we could prove

$$A\,\varepsilon\,\mathbf{vrb}\,.\supset.\,A\,\varepsilon\,\mathbf{vrb}(B)$$

Thus what the third conjunct of *A2* guarantees is that *A* is actually a word in *B*. This is done by incorporating the law of trichotomy for words in *A2*.

T2.2.15 $[AB]\,.\!\cdot.\,A\,\varepsilon\,\mathbf{vrb}\,.\,B\,\varepsilon\,\mathbf{vrb}\,.\,A\neq B\,.\supset:A\,\varepsilon\,\mathbf{pr}(B)\,.\lor.\,B\,\varepsilon\,\mathbf{pr}(A)$
Hyp(3) $.\supset.\!\cdot.$

(4) $[C]: C \varepsilon \mathbf{vrb}(A) . \supset . C = A :$ [$T2.2.10$, 1]
(5) $\sim(B \varepsilon \mathbf{vrb}(A)) :$ [3, 4]
(6) $[C]: C \varepsilon \mathbf{vrb}(B) . \supset . C = B .\because$ [$T2.2.10$, 2]
(7) $[C] .\therefore C \varepsilon \mathbf{vrb}(A) . \supset : B \varepsilon \mathbf{pr}(C) . \vee . C \varepsilon \mathbf{pr}(B) .\because$ [$A2$, 5, 2, 6]
$A \varepsilon \mathbf{pr}(B) . \vee . B \varepsilon \mathbf{pr}(A)$ [7, 1, $D2$]

T2.2.16 $[AB] .\because A \varepsilon \mathbf{vrb} . B \varepsilon \mathbf{vrb} . \supset : A = B . \vee . B \varepsilon \mathbf{pr}(A) . \vee . A \varepsilon \mathbf{pr}(B)$ [$T2.2.15$]

Comments on Group II. $A3$ and $A4$ show that **pr** is a relation restricted to words. $A5$ is transitivity for **pr** and we have already shown trichotomy [$T2.2.16$] and antisymmetry [$T2.2.13$]; to obtain a total ordering we need only irreflexivity:

T2.2.17 $[AB]: A \varepsilon \mathbf{pr}(B) . \supset . \sim(B \varepsilon \mathbf{pr}(A)).$ [$A5$, $T2.2.13$]

If one dislikes getting antisymmetry and trichotomy from the *Group I* axioms and prefers a single axiom giving all the properties of precedence he could use:

$$[AB] :: A \varepsilon \mathbf{pr}(B) . \equiv .\because A \varepsilon \mathbf{vrb}(A) . B \varepsilon \mathbf{vrb}(B) . \sim(B \varepsilon \mathbf{pr}(A)) . \sim(A = B) .\because A \varepsilon \mathbf{pr}(B) . \supset : [C] : B \varepsilon \mathbf{pr}(C) . \supset . A \varepsilon \mathbf{pr}(C)$$

This is equivalent to {$A3$, $A4$, $A5$, $T2.2.17$, $T2.2.14$, $T2.2.16$}. It is not organic[27], so there is some hope of simplifying it. Its main interest is that it can be recast into an axiom for totally ordered sets:

$$[xy] :: x < y . \equiv .\because \sim(y < x) . \sim(x = y) .\because x < y . \supset : [z] : y < z . \supset . x < z$$

For more work along this line see Clay [7].

Comments on Group III.

T2.2.18 $[A]: A \varepsilon A . \supset . A \varepsilon \mathbf{cnf}(A)$ [$A6$, B/A, *Ont. 14*, *Ont. 7*]
T2.2.19 $[AB]: A \varepsilon \mathbf{cnf}(B) . \supset . [C] : B \varepsilon \mathbf{cnf}(C) . \equiv . A \varepsilon \mathbf{cnf}(C)$ [$A6$]
T2.2.20 $[AB]: A \varepsilon \mathbf{cnf}(B) . \supset . B \varepsilon \mathbf{cnf}(A)$ [$T2.2.19$, C/A, $T2.2.18$]
T2.2.21 $[ABC]: A \varepsilon \mathbf{cnf}(B) . B \varepsilon \mathbf{cnf}(C) . \supset . A \varepsilon \mathbf{cnf}(C)$ [$T2.2.19$]

Hence equiformity is a "weak" equivalence relation, "weak" in the sense that reflexivity is proved under some hypothesis.[28] It is clear that equiform inscriptions have the same length:

T2.2.22 $[AB]: A \varepsilon \mathbf{cnf}(B) . \supset . \mathbf{vrb}(A) \infty \mathbf{vrb}(B)$ [$A6$]

and now we show they have the same spelling:

T2.2.23 $[ACD]: C \varepsilon \mathbf{vrb}(A) . C \varepsilon \mathbf{pr}(D) . (\mathbf{vrb}(A) \cap \mathbf{pr}(C)) \infty (\mathbf{vrb}(A) \cap \mathbf{pr}(D)) . \supset . C = D$

Hyp(3) . $\supset$.
(4) $C \varepsilon \mathbf{vrb}(A) \cap \mathbf{pr}(D) .$ [1,2, $DO11$]
(5) $C \varepsilon N(\mathbf{vrb}(A) \cap \mathbf{pr}(C)) .$ [$T2.2.13$, $DO12$, 1]
(6) $\mathbf{pr}(C) \subset \mathbf{pr}(D) .$ [$A5$, 2, $DO7$]
(7) $\mathbf{vrb}(A) \cap \mathbf{pr}(C) \subset \mathbf{vrb}(A) \cap \mathbf{pr}(D) .$ [6]
(8) $\mathbf{vrb}(A) \cap \mathbf{pr}(C) \Subset \mathbf{vrb}(A) \cap \mathbf{pr}(D) .$ [$DO9$, 7, 4, 5]
(9) $\mathrm{Fin}\,\{\mathbf{vrb}(A) \cap \mathbf{pr}(D)\} .$ [1, $T2.2.5$, $A7$, *Ont.17*]
$C = D$ [8, 9, *Ont.18*, 3]

T2.2.24 $[ACD]$: $C \,\varepsilon\, \mathbf{vrb}(A) \,.\, D \,\varepsilon\, \mathbf{vrb}(A) \,.\, (\mathbf{vrb}(A) \cap \mathbf{pr}(C)) \infty (\mathbf{vrb}(A) \cap \mathbf{pr}(D)) \,.$
$\supset . \, C = D$ [*T2.2.23, T2.2.16, T2.2.9*]

This lemma[29] establishes the spelling theorem:

T2.2.25 $[ABCD]$: $A \,\varepsilon\, \mathbf{cnf}(B) \,.\, C \,\varepsilon\, \mathbf{vrb}(A) \,.\, D \,\varepsilon\, \mathbf{vrb}(B) \,.$
$(\mathbf{vrb}(A) \cap \mathbf{pr}(C)) \infty (\mathbf{vrb}(B) \cap \mathbf{pr}(D)) \,.\supset .\, C \,\varepsilon\, \mathbf{cnf}(D)$
[*T2.2.24, T2.2.18, A6*]

Inscriptions can satisfy the spelling theorem without being equiform, for consider the inscriptions:

$$pqr, \qquad pqrs.$$

The first, second, and third words of both are equiform, but the first inscription has no fourth word and so these inscriptions satisfy the spelling theorem. They are not equiform as they are of different lengths.

The spelling theorem required the use of *A7, i.e.,* the finiteness of inscriptions. If we were dealing with inscriptions of infinite length we could still obtain *T2.2.25* if the fourth conjunct of *A6* were replaced by the following where the added hypothesis is necessary to obtain weak reflexivity.[30]

$A \,\varepsilon\, \mathbf{cnf}(A) \,.\supset : [CD] : C \,\varepsilon\, \mathbf{vrb}(A) \,.\, D \,\varepsilon\, \mathbf{vrb}(B) \,.\, (\mathbf{vrb}(A) \cap \mathbf{pr}(C)) \infty (\mathbf{vrb}(B) \cap \mathbf{pr}(D)) \,.$
$\supset .\, C \,\varepsilon\, \mathbf{cnf}(D)$

Comments on Group IV. A7, that inscriptions contain only a finite number of words, has already been used in the proof of the spelling theorem [*T2.2.25*]. We mentioned ways of eliminating its use there so that usage is minor. The main advantage of finite inscriptions is that we can speak of the first, second, . . . , last word of an inscription. These concepts are easily defined, but their existence requires the finiteness of our inscriptions.

D3 $[AB] .\!\cdot\!. A \,\varepsilon\, \mathbf{1vrb}(B) \,.\equiv .: A \,\varepsilon\, \mathbf{vrb}(B) : [C] : C \,\varepsilon\, \mathbf{vrb}(B) \,.\supset .\sim (C \,\varepsilon\, \mathbf{pr}(A))$

A is the *first word* in *B*.

D4 $[AB] :: A \,\varepsilon\, \mathbf{2vrb}(B) \,.\equiv .\!\cdot\!. A \,\varepsilon\, \mathbf{vrb}(B) .\!\cdot\!. [C] .\!\cdot\!. C \,\varepsilon\, \mathbf{vrb}(B) \,.\supset :$
$C \,\varepsilon\, \mathbf{pr}(A) \,.\equiv .\, C \,\varepsilon\, \mathbf{1vrb}(B)$

A is the *second word* in *B*.

D5 $[AB] .\!\cdot\!. A \,\varepsilon\, \mathbf{Uvrb}(B) \,.\equiv : A \,\varepsilon\, \mathbf{vrb}(B) : [C] : C \,\varepsilon\, \mathbf{vrb}(B) \,.\supset .\sim (A \,\varepsilon\, \mathbf{pr}(C))$

A is the *last word* in *B*.

In a similar fashion we can define $\mathbf{3vrb}(B)$, $\mathbf{4vrb}(B)$, . . . , but we cannot define n-$\mathbf{vrb}(B)$ without using variables of higher type. Since our aim has been to quantify over only one type of variable we shall just presume the definition of n-$\mathbf{vrb}$ for whatever fixed values of n it is used. We now proceed to show that every inscription has a first word.[31]

D6.1 $[AB] .\!\cdot\!. A \,\varepsilon\, \mathbf{w\text{-}pr}(B) \,.\equiv : A \,\varepsilon\, \mathbf{pr}(B) \,.\vee .\, A = B : A \,\varepsilon\, A$

A is *weakly-preceding B*.

T2.2.26.1 $[B]$: $B \,\varepsilon\, B \,.\supset .\, B \,\varepsilon\, \mathbf{w\text{-}pr}(B)$ [*D6.1*]

T2.2.26.2 $[AB]:A\varepsilon\,\mathbf{pr}(B).\supset.\mathbf{w\text{-}pr}(A)\subset\mathbf{w\text{-}pr}(B)$ [*D6.1*, *A5*]

T2.2.26.3 $[ABC]:\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(C)\circ\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(B).$
$B\varepsilon\,\mathbf{vrb}(A).\ C\varepsilon\,\mathbf{pr}(B).\supset.C=B$

Hyp(3) $.\supset.$

(4) $B\varepsilon\,\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(B).$ [2, *T2.2.26.1*, *DO11*]

(5) $B\neq C.$ [3, *T2.2.14*]

(6) $\sim(B\varepsilon\,\mathbf{pr}(C)).$ [3, *T2.2.17*]

(7) $\sim(B\varepsilon\,\mathbf{w\text{-}pr}(C)).$ [*D6.1*, 5, 6]

(8) $\sim(B\varepsilon\,\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(C)).$ [7]

$C=B$ [1, 4, 8, *DO8*]

T2.2.26.4 $[ABC]:\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(C)\circ\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(B).C\varepsilon\,\mathbf{vrb}(A).$
$B\varepsilon\,\mathbf{vrb}(A).\supset.C=B$ [*T2.2.16*, *T2.2.9*, *T2.2.26.3*]

D6.2 $[AB]:*\!\prec\! A\!\succ\!\{b\}.\equiv.[\exists B].b\circ\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(B).B\varepsilon\,\mathbf{vrb}(A)$

T2.2.26.5 $[A]:A\varepsilon A.\supset.[\exists b].*\!\prec\! A\!\succ\!\{b\}$ [*D6.2*, *T2.2.1*]

T2.2.26.6 $[Ab]:*\!\prec\! A\!\succ\!\{b\}.\supset.b\subset\mathbf{vrb}(A)$ [*D6.2*, *DO7*]

T2.2.26.7 $[A]::A\varepsilon A.\supset\therefore[\exists B]\therefore B\varepsilon\,\mathbf{vrb}(A):[bC]:b\subset\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(B).$
$b\circ\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(C).C\varepsilon\,\mathbf{vrb}(A).\supset.b\circ\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(B)$

Hyp(1) $.\supset\therefore$

(2) $\mathrm{Fin}\{\mathbf{vrb}(A)\}\therefore$ [1, *A7*]

(3) $[\phi b]\therefore\phi\{b\}:[d]:\phi\{d\}.\supset.d\subset\mathbf{vrb}(A):\supset.[\exists e].\mathrm{Irr}<\phi>\{e\}\therefore$ [2, *DO18*]

$[\exists e]:$

(4) $\mathrm{Irr}<*\!\prec\! A\!\succ\!>\{e\}.$ [3, *T2.2.26.5*, 1, *T2.2.26.6*]

(5) $*\!\prec\! A\!\succ\!\{e\}:$

(6) $[b]:b\subset e.*\!\prec\! A\!\succ\!\{b\}.\supset.b\circ e:$ [4, *DO17*]

$[\exists B]:$

(7) $e\circ\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(B).$

(8) $B\varepsilon\,\mathbf{vrb}(A):$ [5, *D6.2*]

(9) $[bC]:b\subset e.b\circ\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(C).C\varepsilon\,\mathbf{vrb}(A).\supset.e\circ b\therefore$ [6, *D6.2*]

$[\exists B]\therefore B\varepsilon\,\mathbf{vrb}(A):[bC]:b\subset\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(B).b\circ\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(C).$
$C\varepsilon\,\mathbf{vrb}(A).\supset.b\circ\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(B)$ [8, 9, 7]

T2.2.26.8 $[XAB]\therefore X\varepsilon\,\mathbf{vrb}(A):[bC]:b\subset\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(B).$
$b\circ\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(C).C\varepsilon\,\mathbf{vrb}(A).\supset.b\circ\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(B):$
$B\varepsilon\,\mathbf{vrb}(A).X\varepsilon\,\mathbf{pr}(B).\supset.\sim(X\varepsilon\,\mathbf{pr}(B))$

Hyp(4) $.\supset.$

(5) $\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(X)\subset\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(B).$ [*T2.2.26.2*, 4]

(6) $\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(X)\circ\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(B).$ [2, 5, 1]

(7) $X=B.$ [*T2.2.26.4*, 6, 1, 3]

$\sim(X\varepsilon\,\mathbf{pr}(B))$ [7, *T2.2.14*]

T2.2.27 $[A]:A\varepsilon A.\supset.[\exists B]:B\varepsilon 1\mathbf{vrb}(A)$

Hyp(1) $.\supset\therefore$

$[\exists B]\therefore$

(2) $B\varepsilon\,\mathbf{vrb}(A):$

(3) $[bC]:b\subset\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(B).b\circ\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(C).$ [1, *T2.2.26.7*]
$C\varepsilon\,\mathbf{vrb}(A).\supset.b\circ\mathbf{vrb}(A)\cap\mathbf{w\text{-}pr}(B):$

(4) $[X]:X\varepsilon\,\mathbf{vrb}(A).\supset.\sim(X\varepsilon\,\mathbf{pr}(B))\therefore$ [*T2.2.26.8*, 3, 2]

$[\exists B].B\varepsilon 1\mathbf{vrb}(A)$ [*D3*, 2, 4]

In a completely similar fashion we can prove[32]:

T2.2.28 $[A]:A\,\varepsilon\,A\,.\supset.[\exists B]\,.\,B\,\varepsilon\,\mathbf{Uvrb}(A)$ [*Sim.*, *T2.2.27*]
T2.2.29 $[A]:A\,\varepsilon\,\mathcal{N}(\mathbf{vrb})\,.\supset.[\exists B]\,.\,B\,\varepsilon\,\mathbf{2vrb}(A)$ [*Sim.*, *T2.2.27*]

The uniqueness of first words can be established in an elementary fashion:

T2.2.30 $[ABC]:B\,\varepsilon\,\mathbf{1vrb}(A)\,.\,C\,\varepsilon\,\mathbf{1vrb}(A)\,.\supset.\,B=C$
Hyp(2) $.\supset:$
(3) $B\,\varepsilon\,\mathbf{vrb}(A):$
(4) $[D]:D\,\varepsilon\,\mathbf{vrb}(A)\,.\supset.\sim(D\,\varepsilon\,\mathbf{pr}(B)):$ [1, *D3*]
(5) $C\,\varepsilon\,\mathbf{vrb}(A):$
(6) $[D]:D\,\varepsilon\,\mathbf{vrb}(A)\,.\supset.\sim(D\,\varepsilon\,\mathbf{pr}(C)):$ [2, *D3*]
(7) $\sim(C\,\varepsilon\,\mathbf{pr}(B)).$ [4, 5]
(8) $\sim(B\,\varepsilon\,\mathbf{pr}(C)).$ [3, 6]
$B=C$ [*T2.2.16*, *D2*, *T2.2.8*, 3, 5, 7, 8]
T2.2.31 $[ABC]:B\,\varepsilon\,\mathbf{Uvrb}(A)\,.\,C\,\varepsilon\,\mathbf{Uvrb}(A)\,.\supset.\,B=C$ [*Sim.*, *T2.2.30*]

We conclude this section with several lemmas.

T2.2.32 $[ADE]:D\,\varepsilon\,\mathbf{pr}(E)\,.\,D\,\varepsilon\,\mathbf{vrb}(A)\,.\,E\,\varepsilon\,\mathbf{1vrb}(A)\,.\supset.\sim(E\,\varepsilon\,\mathbf{1vrb}(A))$
Hyp(3) $.\supset:$
(4) $[C]:C\,\varepsilon\,\mathbf{vrb}(A)\,.\supset.\sim(C\,\varepsilon\,\mathbf{pr}(E)):$ [3, *D3*]
(5) $\sim(D\,\varepsilon\,\mathbf{pr}(E)).$ [4, 2]
$\sim(E\,\varepsilon\,\mathbf{1vrb}(A))$ [1, 5]
T2.2.33 $[ADE]:D\,\varepsilon\,\mathbf{pr}(E)\,.\,D\,\varepsilon\,\mathbf{vrb}(A)\,.\supset.\,E\,\varepsilon\,\mathcal{N}(\mathbf{1vrb}(A))$ [*T2.2.32*]
T2.2.34 $[AEF]:E\,\varepsilon\,\mathbf{pr}(F)\,.\,F\,\varepsilon\,\mathbf{vrb}(A)\,.\supset.\,E\,\varepsilon\,\mathcal{N}(\mathbf{Uvrb}(A))$ [*Sim.*, *T2.2.33*]
T2.2.35 $[AB]\therefore B\,\varepsilon\,\mathbf{vrb}(A)\,.\supset:\mathbf{1vrb}(A)\,\varepsilon\,\mathbf{pr}(B)\,.\vee.\,B=\mathbf{1vrb}(A)$
Hyp(1) $.\supset:$
$[\exists D]:$
(2) $D\,\varepsilon\,\mathbf{1vrb}(A)\,.$ [1, *T2.2.5*, *T2.2.27*]
(3) $\sim(B\,\varepsilon\,\mathbf{pr}(D)):$ [*D3*, 2, 1]
(4) $D\,\varepsilon\,\mathbf{pr}(B)\,.\vee.\,D=B:$ [*T2.2.16*, 1, *T2.2.9*, 2, *D3*, *T2.2.9*, 3]
$\mathbf{1vrb}(A)\,\varepsilon\,\mathbf{pr}(B)\,.\vee.\,B=\mathbf{1vrb}(A)$ [4, 2, *T2.2.30*]
T2.2.36 $[AB]\therefore B\,\varepsilon\,\mathbf{vrb}(A)\,.\supset:\mathbf{Uvrb}(A)=B\,.\vee.\,B\,\varepsilon\,\mathbf{pr}(\mathbf{Uvrb}(A))$
[*Sim.*, *T2.2.35*]
T2.2.37 $[ABC]:B\,\varepsilon\,\mathbf{1vrb}(A)\,.\,C\,\varepsilon\,\mathbf{vrb}(A)\,.\,C\neq B\,.\supset.\,B\,\varepsilon\,\mathbf{pr}(C)$
Hyp(3) $.\supset:$
(4) $\mathbf{1vrb}(A)\,\varepsilon\,\mathbf{pr}(C)\,.\vee.\,C\,\varepsilon\,\mathbf{1vrb}(A):$ [*T2.2.35*, 2]
$B\,\varepsilon\,\mathbf{pr}(C)$ [4, 1, *T2.2.30*, 3]
T2.2.38 $[ABC]:A\,\varepsilon\,\mathbf{1vrb}(C)\,.\,B\,\varepsilon\,\mathbf{2vrb}(C)\,.\supset.\,A\,\varepsilon\,\mathbf{pr}(B)$
Hyp(2) $.\supset\therefore$
(3) $A\,\varepsilon\,\mathbf{vrb}(C)\therefore$ [1, *D3*]
(4) $[D]\therefore D\,\varepsilon\,\mathbf{vrb}(C)\,.\supset:D\,\varepsilon\,\mathbf{pr}(B)\,.\equiv.\,D\,\varepsilon\,\mathbf{1vrb}(C)\therefore$ [2, *D4*]
(5) $A\,\varepsilon\,\mathbf{pr}(B)\,.\equiv.\,A\,\varepsilon\,\mathbf{1vrb}(C)\therefore$ [3, 4]
$A\,\varepsilon\,\mathbf{pr}(B)$ [1, 5]
T2.2.39 $[ABC]:A\,\varepsilon\,\mathbf{1vrb}(C)\,.\,B\,\varepsilon\,\mathbf{2vrb}(C)\,.\supset.\,A\neq B\,.$ [*T2.2.38*, *T2.2.14*]
T2.2.40 $[ABC]:A\,\varepsilon\,\mathbf{2vrb}(C)\,.\,B\,\varepsilon\,\mathbf{2vrb}(C)\,.\supset.\,A=B$

Hyp(2) . $\supset$.˙.

(3) $A \varepsilon \mathbf{vrb}(C)$.˙.

(4) $[D]$.˙. $D \varepsilon \mathbf{vrb}(C) . \supset : D \varepsilon \mathbf{pr}(A) . \equiv . D \varepsilon \mathbf{1vrb}(C)$.˙. $\Big\}$ [1, *D4*]

(5) $B \varepsilon \mathbf{vrb}(C)$.˙.

(6) $[D]$.˙. $D \varepsilon \mathbf{vrb}(C) . \supset : D \varepsilon \mathbf{pr}(B) . \equiv . D \varepsilon \mathbf{1vrb}(C)$.˙. $\Big\}$ [2, *D4*]

(7) $A \varepsilon \mathbf{pr}(B) . \equiv . A \varepsilon \mathbf{1vrb}(C) :$ [3, 6]

(8) $B \varepsilon \mathbf{pr}(A) . \equiv . B \varepsilon \mathbf{1vrb}(C) :$ [4, 5]

(9) $\sim (A \varepsilon \mathbf{pr}(B))$. [*T2.2.39*, 1, 7]

(10) $\sim (B \varepsilon \mathbf{pr}(A))$.˙. [*T2.2.39*, 2, 8]

$A = B$ [*T2.2.16*, 3, 5, *T2.2.9*, 9, 10]

T2.2.41 $[AB] : \mathbf{2vrb}(A) \varepsilon \mathbf{vrb}(B) . \mathbf{1vrb}(A) \varepsilon \mathcal{N}(\mathbf{vrb}(B)) . B \varepsilon \mathbf{vrb}(A) . \supset .$
$\mathbf{2vrb}(A) = \mathbf{1vrb}(B)$ [*Sim.*, *T2.2.35*]

3 *Klasses.* We have been dealing with inscriptions and their parts but have been hampered by the weakness of our tools. For example, we have not been able to prove that inscriptions consist entirely of words. The reason is easy to come by—we have no way of referring to such a "set" of words. Our primitive and defined terms can be true only with individuals as arguments, never with general names. This is precisely the difficulty. To cope with general names we need a method of reducing them to individual names. In the language of inscriptions this means that we must be able to consider several inscriptions as one. What we need then is an individual inscription which consists precisely of those words which are words in one of the several inscriptions denoted by the general name. The notion of Klass[33] defined in *Group IV* has the desired property:

D1 $[Aa]$.˙. $A \varepsilon \mathbf{Kl}(a) . \equiv : A \varepsilon A : [B] : B \varepsilon \mathbf{vrb}(A) . \equiv . [\exists C] . C \varepsilon a . B \varepsilon \mathbf{vrb}(C)$

This definition is analogous to the definition of class in Mereology:[34]

$[Aa]$.˙. $A \varepsilon \mathsf{Kl_M}(a) . \equiv : A \varepsilon A : [C] : C \varepsilon a . \supset . C \varepsilon \mathsf{el}(A) :$
$[B] : B \varepsilon \mathsf{el}(A) . \supset . [\exists CD] . C \varepsilon a . D \varepsilon \mathsf{el}(C) . D \varepsilon \mathsf{el}(B)$

To interpret this in system **M** notice that the a's are not necessarily words so the el (ingredient) of the second conjunct of the definiens should be understood as $\mathbf{vrb}(C) \subset \mathbf{vrb}(A)$ rather than **vrb**. Then the second conjunct becomes:

(1) $[BC] : C \varepsilon a . B \varepsilon \mathbf{vrb}(C) . \supset . B \varepsilon \mathbf{vrb}(A)$

In the third conjunct it suffices to consider only those B's which are words. Hence we have:

(2) $[B] : B \varepsilon \mathbf{vrb}(A) . \supset . [\exists CD] . C \varepsilon a . D \varepsilon \mathbf{vrb}(C) . D \varepsilon \mathbf{vrb}(B)$

But since B is a word we must have $D = B$ by *T2.2.7*, and so we can simplify (2) to:

(3) $[B] : B \varepsilon \mathbf{vrb}(A) . \supset . [\exists C] . C \varepsilon a . B \varepsilon \mathbf{vrb}(C)$

Combining (1) and (3) we obtain:

(4) $[B] : B \varepsilon \mathbf{vrb}(A) . \equiv . [\exists C] . C \varepsilon a . B \varepsilon \mathbf{vrb}(C)$

i.e., the definiens of *D1*. This shows our definition of Klass is quite natural.[35] As a final remark, for those familiar with Mereology, let us say that *A8*, the existence of Klasses, is quite a reasonable assumption. The uniqueness of Klasses is also desired. Axiom *A1* was designed for this purpose.[36]

T2.3.1 $[ABa]: A \varepsilon \mathbf{Kl}(a) . B \varepsilon \mathbf{Kl}(a) . \supset . A = B$
Hyp(2) . $\supset$:
(3) $[C]: C \varepsilon \mathbf{vrb}(A) . \equiv . [\exists D] . D \varepsilon a . C \varepsilon \mathbf{vrb}(D):$ [*D1*, 1]
(4) $[C]: C \varepsilon \mathbf{vrb}(B) . \equiv . [\exists D] . D \varepsilon a . C \varepsilon \mathbf{vrb}(D):$ [*D1*, 2]
(5) $[C]: C \varepsilon \mathbf{vrb}(A) . \equiv . C \varepsilon \mathbf{vrb}(B):$ [3, 4]
$A = B$ [*T2.2.4*, 1, 5]
T2.3.2 $[A]: A \varepsilon A . \supset . A \varepsilon \mathbf{Kl}(\mathbf{vrb}(A))$
Hyp(1) . $\supset$:
(2) $[B]: B \varepsilon \mathbf{vrb}(A) . \supset . [\exists C] . C \varepsilon \mathbf{vrb}(A) . B \varepsilon \mathbf{vrb}(C):$ [*T2.2.8*]
(3) $[BC]: B \varepsilon \mathbf{vrb}(C) . C \varepsilon \mathbf{vrb}(A) . \supset . B \varepsilon \mathbf{vrb}(A):$ [*T2.2.11*]
$A \varepsilon \mathbf{Kl}(\mathbf{vrb}(A))$ [*D1*, 1, 2, 3]
T2.3.3 $[A]: A \varepsilon A . \supset . [\exists a] . A \varepsilon \mathbf{Kl}(a) . a \subset \mathbf{vrb}$ [*T2.3.2*, *T2.2.9*]

This shows that every inscription consists entirely of words.

T2.3.4 $[Aa]: A \varepsilon \mathbf{Kl}(a) . \supset . [\exists B] . B \varepsilon a$
Hyp(1) . $\supset$.
(2) $[\exists C] . C \varepsilon \mathbf{vrb}(A) .$ [1, *T2.2.1*]
$[\exists B] . B \varepsilon a$ [*D1*, 1, 2]
T2.3.5 $[A]: A \varepsilon A . \supset . [\exists B] . B \varepsilon \mathbf{vrb}(A)$ [*i.e. T2.2.1*]
Hyp(1) . $\supset$.
(2) $A \varepsilon \mathbf{Kl}(\mathbf{vrb}(A)) .$ [1, *T2.3.2*]
$[\exists B] . B \varepsilon \mathbf{vrb}(A)$ [2, *T2.3.4*]

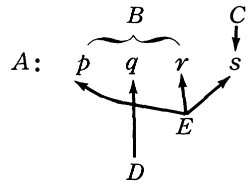
This shows that *T2.3.4* could replace *T2.2.1* as part of Axiom *A1.* In fact we could use:

T2.3.6 $[a]: [\exists A] . A \varepsilon a . \equiv . [\exists B] . B \varepsilon \mathbf{Kl}(a)$ [*A8*, *T2.3.4*]

as an axiom instead of *A8* and half of *A1*.

T2.3.7 $[AXab]: A \varepsilon \mathbf{Kl}(a) . A \varepsilon \mathbf{Kl}(b) . a \subset \mathbf{vrb} . b \subset \mathbf{vrb} . X \varepsilon a . \supset . X \varepsilon b$
Hyp(5) . $\supset$.
(6) $X \varepsilon \mathbf{vrb}(X) .$ [3, 5, *D2*]
(7) $X \varepsilon \mathbf{vrb}(A) .$ [*D1*, 1, 5, 6]
$[\exists Y] .$
(8) $Y \varepsilon b .$
(9) $X \varepsilon \mathbf{vrb}(Y) .$ [2, 7, *D1*]
(10) $Y \varepsilon \mathbf{vrb} .$ [4, 8]
(11) $X = Y .$ [9, 10, *T2.2.10*]
$X \varepsilon b$ [8, 11]
T2.3.8 $[Aab]: A \varepsilon \mathbf{Kl}(a) . A \varepsilon \mathbf{Kl}(b) . a \subset \mathbf{vrb} . b \subset \mathbf{vrb} . \supset . a \circ b$ [*T2.3.7*, *DO8*]

This is the much desired theorem that inscriptions be uniquely decomposable into words.[37] The hypotheses that the *a*'s and *b*'s be words is essential, for consider the inscription:



If $a \circ B \cup C$, $b \circ D \cup E$, then we have $A \varepsilon \mathbf{Kl}(a) . A \varepsilon \mathbf{Kl}(b)$, but not $a \circ b$. The following theorems will be used frequently.

$T2.3.9$ $[Ba]: B\varepsilon a \,.\, B \varepsilon \mathbf{vrb} . \supset . B \varepsilon \mathbf{vrb}(\mathbf{Kl}(a))$
Hyp(2) . ⊃ .
$[\exists A]$.
(3) $A \varepsilon \mathbf{Kl}(a)$. [1, $A8$]
(4) $B \varepsilon \mathbf{vrb}(A)$. [3, $D1$, 1, 2]
$B \varepsilon \mathbf{vrb}(\mathbf{Kl}(a))$ [3, 4, $T2.3.1$]
$T2.3.10$ $[ABa]: A \varepsilon \mathbf{Kl}(a) . a \subset \mathbf{vrb} . B\varepsilon \mathbf{vrb}(A) . \supset . B \varepsilon a$ [$D1$, $T2.2.10$]
$T2.3.11$ $[A]: A \varepsilon \vee . \supset . [\exists B] . B \varepsilon \mathbf{Kl}(\vee)$ [$T2.1.1$, $DO3$]

Thus if there are any inscriptions at all the Klass of all of them exists. By $A8$ this Klass is finite. This shows the strength of our nominalistic commitment.

4 *Expressions.* We have remarked numerous times that inscriptions can be "scattered" individuals, *i.e.*, the words making up an inscription need not form a consecutive string, but could be interrupted by symbols from other inscriptions. Those inscriptions which are uninterrupted strings of words are important enough to deserve a special name:

$D7$ $[A] .\therefore A \varepsilon \mathbf{expr} . \equiv : A \varepsilon A :$
$[BCD] : B \varepsilon \mathbf{vrb}(A) . D \varepsilon \mathbf{vrb}(A) . B \varepsilon \mathbf{pr}(C) . C \varepsilon \mathbf{pr}(D) . \supset . C \varepsilon \mathbf{vrb}(A)$
A is an *expression.*

Hence an inscription A is an expression if and only if every word between two words in A is in A. Tarski [43] uses **expr** as one of his primitive terms.

$T2.4.1$ $[ACDE] : A \varepsilon \mathbf{vrb} . E \varepsilon \mathbf{vrb}(A) . D\varepsilon \mathbf{vrb}(A) . E\varepsilon \mathbf{pr}(C) . C \varepsilon \mathbf{pr}(D) . \supset .$
$C \varepsilon \mathbf{vrb}(A)$
Hyp(5) . ⊃ .
(6) $E = A$. [$T2.2.10$, 1, 2]
(7) $D = A$. [$T2.2.10$, 1, 3]
(8) $A \varepsilon \mathbf{pr}(C)$. [4, 6]
(9) $C \varepsilon \mathbf{pr}(A)$. [5, 7]
$C \varepsilon \mathbf{vrb}(A)$ [8, 9, $T2.2.17$]

This lemma establishes the following important theorem.

$T2.4.2$ $[A] : A \varepsilon \mathbf{vrb} . \supset . A \varepsilon \mathbf{expr}$ [$D7$, $T2.4.1$]

This theorem is a formal statement of a remark of Leśniewski about one of his undefined terms, *viz.*, "Jedes Wort ist ein Ausdruck."[38]

T2.4.3 $[A]: A \varepsilon \mathbf{expr} . \supset . \text{Fin}\{\mathbf{vrb}(A)\}$ [*A7*]

"Ich würde keine solche Zusammenfassung von Wörtern einen Ausdruck nennen, welche aus unendlish vielen Wortern bestände."[38] Notice that the hypothesis is stronger than it need be. We attempt to translate Leśniewski's remarks precisely since these theorems will be used later in PART II of this paper to show that everything he does can be done in system **M**.

T2.4.4 $[A]: A \varepsilon \mathbf{expr} . \supset . A \varepsilon \mathbf{Kl}(\mathbf{vrb}(A))$ [*T2.3.2*]

"Jeder Ausdruck besteht aus Wörtern."[38] *T2.3.2* which is used in the proof formalizes "Gegenstände [sind] die Zussamenfassung von Wörtern."[38]

In all that follows the inscriptions considered are almost always expressions. Even though this is the case there are several reasons for considering inscriptions as the basic objects rather than expressions. If dealing only with formalized languages expressions would possibly[39] suffice, but in natural languages there are many proper inscriptions, for example, "if ... then" and "either ... or" in English and separable verbs in German. A technical reason why inscriptions are preferable is the question of existence of Klasses. The Klass of 1st, 3rd, and 5th words of an expression does not exist as an expression, but only as an inscription. We now restrict our terms **pr** and **vrb**(B) so that their arguments are expressions.

D8. $[AB] \therefore A \varepsilon \mathbf{prcd}(B) . \equiv : A \varepsilon \mathbf{expr} . B \varepsilon \mathbf{expr} :$
$[CD]: C \varepsilon \mathbf{vrb}(A) . D \varepsilon \mathbf{vrb}(B) . \supset . C \varepsilon \mathbf{pr}(D)$

A is an *expression preceding expression* B.

D9 $[AB]: A \varepsilon \mathbf{scd}(B) . \equiv . B \varepsilon \mathbf{prcd}(A) . A \varepsilon A$
A is *following* (secundum) B.[40]

T2.4.5 $[AB]: A \varepsilon \mathbf{prcd}(B) . \supset . \sim (B \varepsilon \mathbf{prcd}(A))$ [*D8*, *T2.2.17*]

D10 $[AB] \therefore A \varepsilon \mathbf{ingr}(B) . \equiv : A \varepsilon \mathbf{expr} . B \varepsilon \mathbf{expr} :$
$[C]: C \varepsilon \mathbf{vrb}(A) . \supset . C \varepsilon \mathbf{vrb}(B)$
A is an *ingredient* of B.

The term **ingr** is analogous to the primitive term of Mereology, *viz.*, element, *cf.* Sobociński [40]. The difference is that our theory is atomic (words are atoms) so we are not using the full strength of the Mereological notion. Our term **ingr** is *not* the same as used by Leśniewski in [18]. He does not require that A and B be expressions. We now bring out the interconnections between the notions just defined.

T2.4.6. $[A]: A \varepsilon \mathbf{expr} . \supset . A \varepsilon \mathbf{ingr}(A)$ [*D10*]

"Der Ausdrücke vom Typus 'ingr(A)' bediene ich mich auf eine Weise, dir mir gestattet, von einem beliebigen Ausdruche A zu behaupten, dass er ein ingr(A) ist."[38]

T2.4.7 $[AB]: A \varepsilon \mathbf{ingr}(B) . \supset . A \varepsilon \mathbf{expr}$ [*D10*]
T2.4.8 $[AB]: A \varepsilon \mathbf{ingr}(B) . \supset . B \varepsilon \mathbf{expr}$ [*D10*]

These two theorems are extremely important. They will be used repeatedly.

T2.4.9 $[AB]: A \,\varepsilon\, \mathbf{vrb}(B) . B \,\varepsilon\, \mathbf{expr} . \supset . A \,\varepsilon\, \mathbf{ingr}(B)$
Hyp(2) $. \supset :$
(3) $A \,\varepsilon\, \mathbf{expr} :$ [1, *T2.2.9*, *T2.4.2*]
(4) $[C] : C \,\varepsilon\, \mathbf{vrb}(A) . \supset . C \,\varepsilon\, \mathbf{vrb}(B) :$ [*T2.2.11*, 1]
$A \,\varepsilon\, \mathbf{ingr}(B) .$ [*D10*, 3, 2, 4]
T2.4.10 $[ABC] : A \,\varepsilon\, \mathbf{vrb}(B) . B \,\varepsilon\, \mathbf{ingr}(C) . \supset . A \,\varepsilon\, \mathbf{vrb}(C)$ [*D10*]
T2.4.11 $[AB] : A \,\varepsilon\, \mathbf{vrb} . A \,\varepsilon\, \mathbf{ingr}(B) . \equiv . A \,\varepsilon\, \mathbf{vrb}(B) . B \,\varepsilon\, \mathbf{expr}$
[*T2.4.9, T2.2.9, T2.4.10, D2, D10*]
T2.4.12 $[A] : A \,\varepsilon\, \mathbf{expr} . \supset . [\exists B] . B \,\varepsilon\, \mathbf{ingr}(A) . B \,\varepsilon\, \mathbf{vrb}$
Hyp(1) $. \supset .$
(2) $[\exists B] . B \,\varepsilon\, \mathbf{vrb}(A) .$ [1, *T2.2.1*]
(3) $B \,\varepsilon\, \mathbf{ingr}(A) .$ [*T2.4.9*, 1, 2]
$[\exists B] . B \,\varepsilon\, \mathbf{ingr}(A) . B \,\varepsilon\, \mathbf{vrb}$ [2, 3, *T2.2.9*]
T2.4.13 $[A] : A \,\varepsilon\, \mathbf{expr} . \equiv . [\exists B] . B \,\varepsilon\, \mathbf{ingr}(A)$ [*T2.4.12, T2.4.8*]

Thus expressions are just those inscriptions which have ingredients. *Cf. T2.2.6.*

T2.4.14 $[A] : A \,\varepsilon\, \mathbf{expr} . \supset . A \,\varepsilon\, \mathbf{cnf}(A)$ [*T2.2.18*]

"Jeder Ausdruck ist ein mit sich selbst gleichgestalteter Ausdruck."[38]

T2.4.15 $[AB] : A \,\varepsilon\, \mathbf{cnf}(B) . B \,\varepsilon\, \mathbf{prcd}(A) . \supset . B \neq A$ [*T2.2.1, D8, T2.2.13*]

"Zwei miteinander gleichgestaltete Ausdrücke, an zwei verschiedenen Stellen geschrieben, sind niemals *derselbe* Ausdruck."[38]

T2.4.17 $[ABC] : A \,\varepsilon\, \mathbf{vrb}(C) . B \,\varepsilon\, \mathbf{vrb}(C) . A \neq B . \supset . \sim (C \,\varepsilon\, \mathbf{vrb})$ [*T2.2.10*]

"Einzelne Buchstaben der aus wenigstens zwei Buchstaben bestehenden Wörter sind keine Wörter."[38]

T2.4.18 $[A] : A \,\varepsilon\, \mathbf{expr} . \sim (\rightharpoonup\{\mathbf{vrb}(A)\}) . \supset . \sim (A \,\varepsilon\, \mathbf{vrb})$ [*T2.4.17, DO2*]

"Ausdrücke, die aus wenigstens zwei Wörtern bestehen, sind keine Wörter."[38]

T2.4.19 $[A] : A \,\varepsilon\, \mathbf{vrb} . \supset . \rightharpoonup\{\mathbf{vrb}(A)\}$ [*T2.2.10, DO2*]
T2.4.20 $[A] : A \,\varepsilon\, A . \supset . A \,\varepsilon\, \mathbf{Kl}(A)$ [*D1, a/A, take* $C = A$]
T2.4.21 $[AB] : A \,\varepsilon\, \mathbf{Kl}(B) . B \,\varepsilon\, B . \supset . A = B$ [*T2.4.20, T2.3.1*]
T2.4.22 $[A] : A \,\varepsilon\, A . \rightharpoonup\{\mathbf{vrb}(A)\} . \supset . A \,\varepsilon\, \mathbf{vrb}$
Hyp(2) $. \supset .$
(3) $A \,\varepsilon\, \mathbf{Kl}(\mathbf{vrb}(A)) .$ [*T2.3.2*, 1]
(4) $[\exists B] . B \,\varepsilon\, \mathbf{vrb}(A) .$ [*T2.2.1*, 1]
(5) $\mathbf{vrb}(A) \,\varepsilon\, \mathbf{vrb}(A) .$ [2, 4, *Ont. 8*, *DO1*]
$A \,\varepsilon\, \mathbf{vrb}$ [*T2.4.21*, 3, 5, *D2*]
T2.4.23 $[A] : A \,\varepsilon\, \mathbf{vrb} . \equiv . \rightharpoonup\{\mathbf{vrb}(A)\} . ! \{\mathbf{vrb}(A)\}$ [*T2.4.22, T2.4.19*]

This is analogous to *Ont. 8*. The next theorem is extremely intuitive but the proof was unbelievably elusive.

T2.4.24 $[AB]:A\,\varepsilon\,\mathbf{vrb}\,.\,B\,\varepsilon\,\mathbf{cnf}(A)\,.\supset.\,B\,\varepsilon\,\mathbf{vrb}$
Hyp(2) . ⊃ .
(3) $\mathbf{vrb}(B) \infty \mathbf{vrb}(A)$. [*T2.2.22*, 2]
(4) $\rightharpoondown\{\mathbf{vrb}(A)\}$. [*T2.4.19*, 1]
(5) $\rightharpoondown\{\mathbf{vrb}(B)\}$. [3, 4, *Ont. 19*]
$B\,\varepsilon\,\mathbf{vrb}$ [*T2.4.22*, 2, 5]

T2.4.25 $[ABa]:a\subset\mathbf{vrb}\,.\,\mathbf{Kl}(a)\,\varepsilon\,\mathbf{vrb}\,.\,A\,\varepsilon\,a\,.\,B\,\varepsilon\,a\,.\supset.\,A = B$
Hyp(4) . ⊃ .
(5) $A\,\varepsilon\,\mathbf{vrb}(\mathbf{Kl}(a))$. [*D1*, 1, 2, 3, *D2*]
(6) $B\,\varepsilon\,\mathbf{vrb}(\mathbf{Kl}(a))$. [*D1*, 1, 2, 4, *D2*]
$A = B$ [*T2.4.17*, 2, 5, 6]

T2.4.26 $[a]:a\subset\mathbf{vrb}\,.\sim(\rightharpoondown\{a\})\,.\supset.\sim(\mathbf{Kl}(a)\,\varepsilon\,\mathbf{vrb})$ [*T2.4.25*]

"Ausdrücke—"der Mensch", "(*p*)", "*f* ∟) Wort"—sind Beispiele von Gegenständen, die Zusammenfassungen von Wörtern, aber Keine Wörter sind."[38]

T2.4.27 $[A]:A\,\varepsilon\,\sim(\mathbf{vrb})\,.\supset.[\exists BC]:B\,\varepsilon\,\mathbf{vrb}(A)\,.\,C\,\varepsilon\,\mathbf{vrb}(A)\,.\,B \neq C$ [*T2.4.22*, *DO2*]

T2.4.28 $[Aa]:A\,\varepsilon\,\mathbf{Kl}(a)\,.\,A\,\varepsilon\,\mathbf{expr}\,.\,a\subset\mathbf{expr}\,.\supset.\,a\subset\mathbf{ingr}(A)$ [*D1*, *D10*]

T2.4.29 $[AB]:B\,\varepsilon\,\mathbf{ingr}(A)\,.\,\mathbf{1vrb}(A)\,\varepsilon\,\mathbf{vrb}(B)\,.\supset.\,\mathbf{1vrb}(A) = \mathbf{1vrb}(B)$ [*D10*, *T2.2.35*]

T2.4.30 $[AB]:B\,\varepsilon\,\mathbf{ingr}(A)\,.\,\mathbf{Uvrb}(A)\,\varepsilon\,\mathbf{vrb}(B)\,.\supset.\,\mathbf{Uvrb}(A) = \mathbf{Uvrb}(B)$ [*Sim.*, *T2.4.29*]

5 *Complexes and Concatenation.* Our aim in this section is to define and investigate a special kind of Klass, *viz.*, Klasses of non-overlapping expressions where the whole Klass is also an expression. We call such Klasses *Complexes*. First we must define "non-overlapping":

D11 $[a]\,.\!\cdot\,.\,\mathbf{disj}(a)\,.\equiv:[ABC]:A\,\varepsilon\,a\,.\,B\,\varepsilon\,a\,.\,C\,\varepsilon\,\mathbf{vrb}(A)\,.\,C\,\varepsilon\,\mathbf{vrb}(B)\,.\supset.\,A = B$
The inscriptions *a* are *disjoint.*

This is the Mereological notion of discreteness specialized to atomic Mereology.[41] The following trivial theorems will be used occasionally.

T2.5.1 $\mathbf{disj}(\wedge)$ [*D11*, *DO4*]
T2.5.2 $[a]:a\subset\mathbf{vrb}\,.\supset.\,\mathbf{disj}(a)$ [*D11*, *T2.2.10*]
T2.5.3 $[A]:\mathbf{disj}(\mathbf{vrb}(A))$ [*D11*, *T2.2.9*, *T2.2.10*]
T2.5.4 $[abABC]:a\subset b\,.\,\mathbf{disj}(b)\,.\,A\,\varepsilon\,a\,.\,B\,\varepsilon\,a\,.\,C\,\varepsilon\,\mathbf{vrb}(A)\,.\,C\,\varepsilon\,\mathbf{vrb}(B)\,.\supset.\,A = B$
Hyp(6) . ⊃ .
(7) $A\,\varepsilon\,b$. [1, 3]
(8) $B\,\varepsilon\,b$. [1, 4]
$A = B$ [*D11*, 2, 7, 8, 5, 6]
T2.5.5 $[ab]:a\subset b\,.\,\mathbf{disj}(b)\,.\supset.\,\mathbf{disj}(a)$ [*T2.5.4*, *D11*]
T2.5.6 $[ab]:\mathbf{disj}(a)\,.\supset.\,\mathbf{disj}(a\cap b)$ [*T2.5.5*]

D12 $[Aa]:A\,\varepsilon\,\mathbf{Cmpl}(a)\,.\equiv.\,A\,\varepsilon\,\mathbf{Kl}(a)\,.\,A\,\varepsilon\,\mathbf{expr}\,.\,a\subset\mathbf{expr}\,.\,\mathbf{disj}(a)$
A is a *Complex* of *a*.

This is the most important definition of general syntax. Leśniewski [18] was the first to define **Cmpl**(*a*), but our definition is much simpler.

Axiom *A8* guarantees the existence of Klasses but the existence of Complexes is, unfortunately, not an easy question. The main difficulty is in establishing the second conjunct of the definiens (we shall refer to it as *D12.2*), *i.e.*, $A \,\varepsilon\, \mathbf{expr}$. This usually requires several long lemmas. The conditions *D12.3* and *D12.4*, *i.e.*, the third and fourth conjuncts of the definiens, are usually immediate, since we often consider Complexes of words and then we can use *T2.4.2* and *T2.5.2*. We now establish the uniqueness of Complexes.

T2.5.7 $[ABa] : A \varepsilon \mathbf{Cmpl}(a) . B \varepsilon \mathbf{Cmpl}(a) . \supset . A = B$ [*D12, T2.3.1*]
T2.5.8 $[A] : A \varepsilon \mathbf{expr} . \supset . A \varepsilon \mathbf{Cmpl}(\mathbf{vrb}(A))$ [*T2.3.2, T2.5.3, T2.4.2, D12*]
T2.5.9 $[ABa] : A \varepsilon \mathbf{Cmpl}(a) . B \varepsilon a . \supset . B \varepsilon \mathbf{ingr}(A)$
Hyp(2) . $\supset$:
(3) $A \varepsilon \mathbf{expr}$. [*D12*, 1]
(4) $B \varepsilon \mathbf{expr}$: [*D12*, 1, 2]
(5) $[C] : C \varepsilon \mathbf{vrb}(B) . \supset . C \varepsilon \mathbf{vrb}(A)$: [*D12, D1*, 1, 2]
$B \varepsilon \mathbf{ingr}(A)$ [*D10*, 4, 3, 5]
T2.5.10 $[Aa] : A \varepsilon \mathbf{vrb}(\mathbf{Cmpl}(a)) . a \subset \mathbf{vrb} . \supset . A \varepsilon a$
Hyp(2) . $\supset$.
(3) $\mathbf{Cmpl}(a) \varepsilon \mathbf{Cmpl}(a)$. [1, *T2.2.5*]
(4) $\mathbf{Cmpl}(a) \varepsilon \mathbf{Kl}(a)$. [3, *D12*]
$A \varepsilon a$ [*T2.3.10*, 4, 2, 1]
T2.5.11 $[A] : A \varepsilon \mathbf{Cmpl}(\mathbf{1vrb}(A)) . \supset . A \varepsilon \mathbf{vrb}$
Hyp(1) . $\supset$.
$[\exists B]$.
(2) $B \varepsilon \mathbf{1vrb}(A)$. [1, *D12, T2.3.4*]
(3) $B = \mathbf{1vrb}(A)$. [2, *T2.2.30*]
(4) $A = B$. [*T2.4.21*, 1, *D12*, 3]
$A \varepsilon \mathbf{vrb}$ [3, 4, *D3, T2.2.9*]

We now define several terms which will be used frequently, often in conjunction with Complexes.

D13 $[AB] : A \varepsilon \mathbf{int}(B) . \equiv . A \varepsilon \mathbf{vrb}(B) . B \varepsilon \mathbf{expr} . A \varepsilon \sim (\mathbf{1vrb}(B)) . A \varepsilon \sim (\mathbf{Uvrb}(B))$
A is a word *interior* to expression B.

D14 $[AB] : A \varepsilon \mathbf{Int}(B) . \equiv . A \varepsilon \mathbf{Kl}(\mathbf{int}(B))$
A is *the interior* of expression B.

D15 $[A] : A \varepsilon \mathbf{expr\text{-}w\text{-}int} . \equiv . A \varepsilon A . [\exists B] . B \varepsilon \mathbf{int}(A)$
A is an *expression with interior*.

T2.5.12 $[A] : A \varepsilon \mathbf{expr\text{-}w\text{-}int} . \equiv . [\exists B] . B \varepsilon \mathbf{int}(A)$ [*D15, D13*]
T2.5.13 $[A] : A \varepsilon \mathbf{expr\text{-}w\text{-}int} . \supset . A \varepsilon \mathbf{expr}$ [*D15, D13*]
T2.5.14 $[AB] : A \varepsilon \mathbf{int}(B) . \supset . B \varepsilon \mathbf{expr}$ [*D13*]
T2.5.15 $[AB] : A \varepsilon \mathbf{Int}(B) . \supset . B \varepsilon \mathbf{expr}$ [*D14, T2.3.4, T2.5.14*]
T2.5.16 $[A] : \mathbf{disj}(\mathbf{int}(A))$ [*T2.5.2, D13, T2.2.9*]
T2.5.17 $[ABC] : A \varepsilon \mathbf{Int}(B) . C \varepsilon \mathbf{Int}(B) . \supset . A = C$ [*D14, T2.3.1*]
T2.5.18 $[AB] : A \varepsilon \mathbf{int}(B) . \supset . \sim (B \varepsilon \mathbf{vrb})$ [*T2.4.17, T2.2.27, D13*]

We now prove several lemmas necessary to establish that the interior of an expression is an expression.

T2.5.19 $[ADE]:D\,\varepsilon\,\mathbf{pr}(E).D\,\varepsilon\,\mathbf{vrb}(A).E\,\varepsilon\,\mathbf{1vrb}(A).\supset.E\,\varepsilon\,\mathcal{N}(\mathbf{1vrb}(A))$

Hyp(3) . ⊃:

(4) $[C]:C\,\varepsilon\,\mathbf{vrb}(A).\supset.\sim(C\,\varepsilon\,\mathbf{pr}(E)):$ [*D3*, 3]

(5) $\sim(D\,\varepsilon\,\mathbf{pr}(E)).$ [2, 4]

$E\,\varepsilon\,\mathcal{N}(\mathbf{1vrb}(A))$ [1, 5]

T2.5.20 $[ADE]:D\,\varepsilon\,\mathbf{pr}(E).D\,\varepsilon\,\mathbf{vrb}(A).\supset.E\,\varepsilon\,\mathcal{N}(\mathbf{1vrb}(A))$ [*T2.5.19*]

T2.5.21 $[AEF]:E\,\varepsilon\,\mathbf{pr}(F).F\,\varepsilon\,\mathbf{vrb}(A).\supset.E\,\varepsilon\,\mathcal{N}(\mathbf{Uvrb}(A))$ [*Sim.*, *T2.5.20*]

T2.5.22 $[ACD]:C\,\varepsilon\,\mathbf{Int}(A).D\,\varepsilon\,\mathbf{vrb}(C).\supset.D\,\varepsilon\,\mathbf{vrb}(A)$

Hyp(2) . ⊃.

$[\exists B].$

(3) $B\,\varepsilon\,\mathbf{int}(A).$

(4) $D\,\varepsilon\,\mathbf{vrb}(B).$ [*D14*, *D1*, 1, 2]

(5) $B\,\varepsilon\,\mathbf{vrb}(A).$ [*D13*, 3]

(6) $B=D.$ [*T2.2.7*, 4, 5]

$D\,\varepsilon\,\mathbf{vrb}(A).$ [5, 6]

T2.5.23 $[ACDEF]:C\,\varepsilon\,\mathbf{Int}(A).D\,\varepsilon\,\mathbf{vrb}(C).F\,\varepsilon\,\mathbf{vrb}(C).$
$D\,\varepsilon\,\mathbf{pr}(E).E\,\varepsilon\,\mathbf{pr}(F).\supset.E\,\varepsilon\,\mathbf{vrb}(C)$

Hyp(5) . ⊃.

(6) $A\,\varepsilon\,\mathbf{expr}.$ [*T2.5.15*, 1]

(7) $A\,\varepsilon\,\mathbf{vrb}(A).$ [*T2.5.22*, 1, 2]

(8) $F\,\varepsilon\,\mathbf{vrb}(A).$ [*T2.5.22*, 1, 3]

(9) $E\,\varepsilon\,\mathbf{vrb}(A).$ [6, *D7*, 7, 8, 4, 5]

(10) $E\,\varepsilon\,\mathcal{N}(\mathbf{1vrb}(A)).$ [*T2.5.20*, 4, 7]

(11) $E\,\varepsilon\,\mathcal{N}(\mathbf{Uvrb}(A)).$ [*T2.5.21*, 5, 8]

(12) $E\,\varepsilon\,\mathbf{int}(A).$ [*D13*, 9, 6, 10, 11]

(13) $E\,\varepsilon\,\mathbf{vrb}(\mathbf{Kl}(\mathbf{int}(A))).$ [*T2.3.9*, 12, 9, *T2.2.9*]

$E\,\varepsilon\,\mathbf{vrb}(C)$ [13, 1, *D14*, *T2.3.1*]

T2.5.24 $[AB]:B\,\varepsilon\,\mathbf{Int}(A).\supset.B\,\varepsilon\,\mathbf{expr}$ [*D7*, *T2.5.23*]

This theorem shows that the interior of an expression is again an expression. This theorem is the vital link for showing that Interiors are Complexes.

T2.5.25 $[AB]:B\,\varepsilon\,\mathbf{Int}(A).\supset.B\,\varepsilon\,\mathbf{Cmpl}(\mathbf{int}(A))$

Hyp(1) . ⊃.

(2) $B\,\varepsilon\,\mathbf{Kl}(\mathbf{int}(A)).$ [1, *D14*]

(3) $B\,\varepsilon\,\mathbf{expr}.$ [1, *T2.5.24*]

(4) $\mathbf{int}(A)\subset\mathbf{expr}.$ [*D13*, *T2.2.9*, *T2.4.2*]

(5) $\mathbf{disj}\{\mathbf{int}(A)\}.$ [*T2.5.2*, *D13*, *T2.2.9*]

$B\,\varepsilon\,\mathbf{Cmpl}(\mathbf{int}(A))$ [*D12*, 2, 3, 4, 5]

D16 $[ABC]:A\,\varepsilon\,\mathbf{Concat}(BC).\equiv.B\,\varepsilon\,\mathbf{prcd}(C).A\,\varepsilon\,\mathbf{Cmpl}(B\cup C)$

A is the *concatenation* of B and C.

The concatenation of B and C consists of the expression B immediately followed by the expression C. Tarski [43] used concatenation as an undefined term and Gödel [9] based his definition on arithmetic. Our definition of concatenation is completely free of arithmetic (as is all of inscriptional syntax). We can form Klasses from arbitrary inscriptions, but only expressions can be concatenated:

T2.5.26 $[ABC] : A\,\varepsilon\,\mathsf{Concat}(BC) . \supset . A\,\varepsilon\,\mathsf{expr}$ [*D16, D12*]
T2.5.27 $[ABC] : A\,\varepsilon\,\mathsf{Concat}(BC) . \supset . B\,\varepsilon\,\mathsf{expr}$ [*D16, D8*]
T2.5.28 $[ABC] : A\,\varepsilon\,\mathsf{Concat}(BC) . \supset . C\,\varepsilon\,\mathsf{expr}$ [*D16, D8*]

In classical syntax concatenation is not commutative, but in inscriptional syntax both $\mathsf{Concat}(BC)$ and $\mathsf{Concat}(CB)$ cannot exist, for if inscription B precedes inscription C then C cannot also precede B.

T2.5.29 $[ABC] : A\,\varepsilon\,\mathsf{Concat}(BC) . \supset . \mathsf{Concat}(CB)\,\varepsilon\,\wedge$ [*D16, T2.4.5, DO4*]
T2.5.29.1 $[ABCD] : A\,\varepsilon\,\mathsf{Concat}(BC) . A\,\varepsilon\,\mathsf{Concat}(BD) . \supset . C = D$ [*Sim., T2.3.8*]

We remark without proof that Concatenation can be defined without the use of Complexes or even Klasses:

$[ABC] :\cdot: A\,\varepsilon\,\mathsf{Concat}(BC) . \equiv :: A\,\varepsilon\,A . B\,\varepsilon\,\mathsf{prcd}(C) . B\,\varepsilon\,\mathsf{ingr}(A) . C\,\varepsilon\,\mathsf{ingr}(A) : [D] :$
$B\,\varepsilon\,\mathsf{prcd}(D) . \supset . \sim(D\,\varepsilon\,\mathsf{prcd}(C)) \therefore [D] \therefore D\,\varepsilon\,\mathsf{vrb}(A) . \supset : D\,\varepsilon\,\mathsf{vrb}(B) . \vee .$
$D\,\varepsilon\,\mathsf{vrb}(C)$

We know of no way to define Kl in terms of Concat. If we took only Klasses of expressions then we could use the ancestral of concatenation.

T2.5.30 $[ABC] : A\,\varepsilon\,\mathsf{Concat}(BC) . \supset . B\,\varepsilon\,\mathsf{ingr}(A) . C\,\varepsilon\,\mathsf{ingr}(A)$
[*D16, D12, D1, D10, T2.5.26, T2.5.27*]
T2.5.31 $[ABCD] : A\,\varepsilon\,\mathsf{Concat}(BC) . D\,\varepsilon\,\mathsf{vrb}(C) . \supset . \sim(D\,\varepsilon\,\mathsf{1vrb}(A))$
Hyp(2) . $\supset$.
$[\exists E]$.
(3) $E\,\varepsilon\,\mathsf{vrb}(B)$. [1, *D16, T2.2.1*]
(4) $E\,\varepsilon\,\mathsf{pr}(D)$. [1, *D16, D8*, 3, 2]
(5) $E\,\varepsilon\,\mathsf{vrb}(A)$. [1, *D16, D12, D1*, 3]
$\sim(D\,\varepsilon\,\mathsf{1vrb}(A))$ [*T2.2.33*, 4, 5]
T2.5.32 $[ABC] : A\,\varepsilon\,\mathsf{Concat}(BC) . \supset . \mathsf{1vrb}(A)\,\varepsilon\,\mathsf{vrb}(B)$
Hyp(1) . $\supset \therefore$
(2) $B\,\varepsilon\,B$. [*1, D16, D8*]
(3) $C\,\varepsilon\,C \therefore$ [*1, D16, D8*]
(4) $[D] \therefore D\,\varepsilon\,\mathsf{vrb}(A) . \supset : [\exists E] . E\,\varepsilon\,B \cup C . D\,\varepsilon\,\mathsf{vrb}(E) \therefore$ [1, *D16, D12, D1*]
(5) $[D] \therefore D\,\varepsilon\,\mathsf{vrb}(A) . \supset : D\,\varepsilon\,\mathsf{vrb}(B) . \vee . D\,\varepsilon\,\mathsf{vrb}(C) \therefore$ [4, 2, 3, *Ont. 21*]
$[\exists D] \therefore$
(6) $D\,\varepsilon\,\mathsf{1vrb}(A)$. [1, *T2.2.27*]
(7) $D\,\varepsilon\,\mathsf{vrb}(A)$: [6, *D3*]
(8) $D\,\varepsilon\,\mathsf{vrb}(B) . \vee . D\,\varepsilon\,\mathsf{vrb}(C)$: [5, 7]
(9) $D\,\varepsilon\,\mathsf{vrb}(B) \therefore$ [8, *T2.5.31*, 1, 6]
$\mathsf{1vrb}(A)\,\varepsilon\,\mathsf{vrb}(B)$ [*T2.2.30*, 6, 9]
T2.5.32.1 $[ABC] : A\,\varepsilon\,\mathsf{Concat}(BC) . \supset . \mathsf{Uvrb}(A)\,\varepsilon\,\mathsf{vrb}(C)$ [*Sim., T2.5.32*]
T2.5.33 $[ABC] : A\,\varepsilon\,\mathsf{Concat}(BC) . \supset . \mathsf{1vrb}(A) = \mathsf{1vrb}(B)$
[*T2.4.29, T2.5.32, T2.5.30*]
T2.5.34 $[ABCDE] : A\,\varepsilon\,\mathsf{Cmpl}(B \cup C) . B \neq C . \mathsf{1vrb}(A)\,\varepsilon\,\mathsf{vrb}(B) .$
$D\,\varepsilon\,\mathsf{vrb}(B) . E\,\varepsilon\,\mathsf{vrb}(C) . \supset . D\,\varepsilon\,\mathsf{pr}(E)$
Hyp(5) . $\supset \therefore$

(6) $\mathbf{disj}(B \cup C)$. } [1, *D12*]
(7) $B \,\varepsilon\, \mathbf{expr}$. }
(8) $E \,\varepsilon \sim(\mathbf{vrb}(B))$ ∴ [6, *D11*, 2, 5]
$[\exists F]$ ∴
(9) $F \,\varepsilon\, \mathbf{Uvrb}(B)$. [7, *T2.2.28*]
(10) $F \,\varepsilon\, \mathbf{vrb}(B)$. [*D5*, 9]
(11) $\mathbf{1vrb}(A) \,\varepsilon\, \mathbf{pr}(E)$. [*T2.2.37*, 1, *D12*, 1, 5, 6]
(12) $E \,\varepsilon\, \mathbf{pr}(F) . \vee . E = F . \vee . F \,\varepsilon\, \mathbf{pr}(E)$: [*T2.2.16*, 5, 10, *T2.2.9*]
(13) $F \,\varepsilon\, \mathbf{pr}(E)$. [12, *D7*, 7, 3, 10, 11, 8, 8, 10]
(14) $\sim (F \,\varepsilon\, \mathbf{pr}(E))$ ∴ [*D5*, 4, 9]
$D \,\varepsilon\, \mathbf{pr}(E)$ [13, 14]

T2.5.35 $[ABC] : A \,\varepsilon\, \mathbf{Cmpl}(B \cup C) . B \neq C . \mathbf{1vrb}(A) \,\varepsilon\, \mathbf{vrb}(B) . \supset . B \,\varepsilon\, \mathbf{prcd}(C)$ [*T2.5.34*, *D8*, *D12*]

Cf. T2.5.32. We now develop several techniques for showing that expressions are identical.

T2.5.36 $[AB] : A \,\varepsilon\, \mathbf{ingr}(B) . B \,\varepsilon\, \mathbf{ingr}(A) . \supset . A = B$ [*D10*, *T2.2.4*]

Martin [30], p. 232 uses this as a definition of identity. *Cf. A1.* There is also a thesis of Mereology analogous to *T2.5.36*.

T2.5.37 $[ABC] : B \,\varepsilon\, \mathbf{expr} . \mathbf{1vrb}(A) = \mathbf{1vrb}(B) . \mathbf{Uvrb}(A) = \mathbf{Uvrb}(B) .$
$C \,\varepsilon\, \mathbf{vrb}(A) . \supset . C \,\varepsilon\, \mathbf{vrb}(B)$

Hyp(4) . ⊃ :
(5) $\mathbf{1vrb}(A) \,\varepsilon\, \mathbf{pr}(C) . \vee . \mathbf{1vrb}(A) = C$: [*T2.2.35*, 4]
(6) $C \,\varepsilon\, \mathbf{pr}(\mathbf{Uvrb}(A)) . \vee . C = \mathbf{Uvrb}(A)$: [*T2.2.36*, 4]
(7) $\mathbf{1vrb}(B) \,\varepsilon\, \mathbf{pr}(C) . \vee . \mathbf{1vrb}(B) = C$: [2, 5]
(8) $C \,\varepsilon\, \mathbf{pr}(\mathbf{Uvrb}(B)) . \vee . \mathbf{Uvrb}(B) = C$: [3, 6]
$C \,\varepsilon\, \mathbf{vrb}(B)$ [7, 8, 1, *D7*, *D5*, *D3*]

T2.5.38 $[AB] : A \,\varepsilon\, \mathbf{expr} . B \,\varepsilon\, \mathbf{expr} . \mathbf{1vrb}(A) = \mathbf{1vrb}(B) .$
$\mathbf{Uvrb}(A) = \mathbf{Uvrb}(B) . \supset . A = B$ [*D10*, *T3.5.37*, *T3.5.36*]

This theorem will frequently be used to show two expressions are identical. The following theorems can be used to establish the hypotheses of *T2.5.38*.

T2.5.39 $[AB] : \mathbf{1vrb}(A) \,\varepsilon\, \mathbf{vrb}(B) . \mathbf{1vrb}(B) \,\varepsilon\, \mathbf{vrb}(A) . \supset . \mathbf{1vrb}(A) = \mathbf{1vrb}(B)$

Hyp(2) . ⊃ .
(3) $\mathbf{1vrb}(A) \,\varepsilon\, \mathbf{1vrb}(A)$. [1]
(4) $\mathbf{1vrb}(B) \,\varepsilon\, \mathbf{1vrb}(B)$. [2]
(5) $\sim(\mathbf{1vrb}(B) \,\varepsilon\, \mathbf{pr}(\mathbf{1vrb}(A)))$. [*D3*, 3, 2]
(6) $\sim(\mathbf{1vrb}(A) \,\varepsilon\, \mathbf{pr}(\mathbf{1vrb}(B)))$. [*D3*, 4, 1]
$\mathbf{1vrb}(A) = \mathbf{1vrb}(B)$ [*T2.2.16*, 1, 2, 5, 6, *T2.2.9*]

T2.5.40 $[AB] : \mathbf{Uvrb}(A) \,\varepsilon\, \mathbf{vrb}(B) . \mathbf{Uvrb}(B) \,\varepsilon\, \mathbf{vrb}(A) . \supset . \mathbf{Uvrb}(A) = \mathbf{Uvrb}(B)$ [*Sim.*, *T2.5.39*]

T2.5.41 $[AB]$ ∴ $A \,\varepsilon\, \mathbf{expr} . B \,\varepsilon\, \mathbf{expr} . \mathbf{1vrb}(A) = \mathbf{1vrb}(B) . \supset :$
$A \,\varepsilon\, \mathbf{ingr}(B) . \vee . B \,\varepsilon\, \mathbf{ingr}(A)$ [*Sim.*, *T2.5.38*]

T2.5.41.1 $[AB] : A \,\varepsilon\, \mathbf{expr} . B \,\varepsilon\, \mathbf{expr} . \mathbf{1vrb}(A) \,\varepsilon\, \mathbf{vrb}(B) . \mathbf{Uvrb}(A) \,\varepsilon\, \mathbf{vrb}(B) . \supset .$
$A \,\varepsilon\, \mathbf{ingr}(B)$. [*Sim.*, *T2.5.38*]

D17 $[AB] : A \,\varepsilon\, \mathbf{hd}(B) . \equiv . A \,\varepsilon\, A . [\exists C] . B \,\varepsilon\, \mathbf{Concat}(AC)$

A is an *initial segment* (*head*) of B.

D18 $[AB]:A\,\varepsilon\,\mathbf{tl}(B)\,.\equiv.\,A\,\varepsilon\,A\,.\,[\exists C]\,.\,B\,\varepsilon\,\mathbf{Concat}(CA)$
A is a *terminal segment* (*tail*) of B.

These are proper heads (and tails), *i.e.*, $A\,\varepsilon\,\mathbf{hd}(B)\,.\supset.\,A \neq B$. These definitions will be useful in mating parentheses.

T2.5.42 $[AB]:A\,\varepsilon\,\mathbf{hd}(B)\,.\supset.\,A\,\varepsilon\,\mathbf{expr}$ [*D17, T2.5.27*]
T2.5.43 $[AB]:A\,\varepsilon\,\mathbf{hd}(B)\,.\supset.\,B\,\varepsilon\,\mathbf{expr}$ [*D17, T2.5.26*]
T2.5.44 $[AB]:A\,\varepsilon\,\mathbf{tl}(B)\,.\supset.\,A\,\varepsilon\,\mathbf{expr}$ [*D18, T2.5.28*]
T2.5.45 $[AB]:A\,\varepsilon\,\mathbf{tl}(B)\,.\supset.\,B\,\varepsilon\,\mathbf{expr}$ [*D18, T2.5.26*]

It is our aim now to show that certain Klasses are heads. A large part of the difficulty will be in first establishing that the Klasses are expressions.

T2.5.46 $[ABCDEF]:B\,\varepsilon\,\mathbf{Kl}(\mathbf{vrb}(A)\cap\mathbf{pr}(C))\,.\,D\,\varepsilon\,\mathbf{vrb}(B)\,.\,F\,\varepsilon\,\mathbf{vrb}(B)\,.$
$D\,\varepsilon\,\mathbf{pr}(E)\,.\,E\,\varepsilon\,\mathbf{pr}(F)\,.\,A\,\varepsilon\,\mathbf{expr}\,.\supset.\,E\,\varepsilon\,\mathbf{vrb}(B)$
Hyp(6) $.\supset.$
(7) $D\,\varepsilon\,\mathbf{vrb}(A)\,.$ [*T2.3.10*, 1, 2, *T2.2.9*]
(8) $F\,\varepsilon\,\mathbf{vrb}(A)\cap\mathbf{pr}(C)\,.$ [*T2.3.10*, 1, 3, *T2.2.9*]
(9) $E\,\varepsilon\,\mathbf{vrb}(A)\,.$ [6, *D7*, 4, 5, 7, 8]
(10) $E\,\varepsilon\,\mathbf{pr}(C)\,.$ [*A5*, 5, 8]
(11) $E\,\varepsilon\,\mathbf{vrb}(\mathbf{Kl}(\mathbf{vrb}(A)\cap\mathbf{pr}(C)))\,.$ [*T2.3.9*, 9, 10, *T2.2.9*]
$E\,\varepsilon\,\mathbf{vrb}(B)$ [1, 11, *T2.3.1*]
T2.5.47 $[ABC]:B\,\varepsilon\,\mathbf{Kl}(\mathbf{vrb}(A)\cap\mathbf{pr}(C))\,.\,A\,\varepsilon\,\mathbf{expr}\,.\supset.\,B\,\varepsilon\,\mathbf{expr}$ [*T2.5.46, D7*]
T2.5.48 $[ABC]:B\,\varepsilon\,\mathbf{Kl}(\mathbf{vrb}(A)\cap(C\cup\mathbf{scd}(C)))\,.\,A\,\varepsilon\,\mathbf{expr}\,.\supset.\,B\,\varepsilon\,\mathbf{expr}$
[*Sim., T2.5.47*]
T2.5.49 $[ABC]\,.\!\cdot\,.\,B\,\varepsilon\,\mathbf{Cmpl}(\mathbf{vrb}(A)\cap\mathbf{pr}(C)):C\,\varepsilon\,\mathbf{int}(A)\,.\lor.\,C\,\varepsilon\,\mathbf{Uvrb}(A):$
$A\,\varepsilon\,\mathbf{expr}:\supset.\,B\,\varepsilon\,\mathbf{hd}(A)$
Hyp(3) $.\supset.\!\cdot.$
(4) $B\,\varepsilon\,\mathbf{expr}\,.$ [*D12*, 1]
(5) $C\,\varepsilon\,\mathbf{vrb}(A)\cap(C\cup\mathbf{scd}(C)):$ [2, *D13, D5*]
(6) $[X]:X\,\varepsilon\,\mathbf{vrb}(B)\,.\supset.\,X\,\varepsilon\,\mathbf{pr}(C):$ [*T2.3.10, T2.2.9*, 1]
$[\exists D]\,.\!\cdot.$
(7) $D\,\varepsilon\,\mathbf{Kl}(\mathbf{vrb}(A)\cap(C\cup\mathbf{scd}(C))\,.$ [5, *T2.1.1*]
(8) $D\,\varepsilon\,\mathbf{expr}\,.$ [3, 7, *T2.5.48*]
(9) $\mathbf{disj}(\mathbf{vrb}(A)\cap(C\cup\mathbf{scd}(C))\,.$ [*T2.5.2, T2.2.9*]
(10) $\mathbf{vrb}(A)\cap(C\cup\mathbf{scd}(C))\subset\mathbf{expr}\,.$ [*T2.4.2, T2.2.9*]
(11) $D\,\varepsilon\,\mathbf{Cmpl}(\mathbf{vrb}(A)\cap(C\cup\mathbf{scd}(C)):$ [*D12*, 7, 8, 9, 10]
(12) $[Y]:Y\,\varepsilon\,\mathbf{vrb}(D)\,.\supset.\,Y\,\varepsilon\,(C\cup\mathbf{scd}(C)):$ [*T2.3.10*, 7, *T2.2.9*]
(13) $[Y]\,.\!\cdot.\,Y\,\varepsilon\,\mathbf{vrb}(D)\,.\supset:Y=C\,.\lor.\,C\,\varepsilon\,\mathbf{pr}(Y)\,.\!\cdot.$ [12, *DO10*, 5, *D9*]
(14) $[XY]:X\,\varepsilon\,\mathbf{vrb}(B)\,.\,Y\,\varepsilon\,\mathbf{vrb}(D)\,.\supset.\,X\,\varepsilon\,\mathbf{pr}(Y):$ [6, 13, *A5*]
(15) $B\,\varepsilon\,\mathbf{prcd}(D)\,.\!\cdot.$ [*D8*, 14, 4, 8]
$[\exists E]:$
(16) $E\,\varepsilon\,\mathbf{Kl}(B\cup D):$ [*T2.1.1*, 15]
(17) $[X]\,.\!\cdot.\,X\,\varepsilon\,\mathbf{vrb}(E)\,.\supset:X\,\varepsilon\,\mathbf{vrb}(B)\,.\lor.\,X\,\varepsilon\,\mathbf{vrb}(D):$
[*D1*, 16, *Ont. 21*, 4, 8]

(18) $[X]:X\varepsilon\,\mathbf{vrb}(E).\supset.X\,\varepsilon\,\mathbf{vrb}(A):$ [17, 1, 7, *D12, T2.2.9, T2.3.10*]

(19) $[X]:X\varepsilon\,\mathbf{vrb}(A).\supset.X\varepsilon\,\mathbf{vrb}(B).\vee.X\varepsilon\,\mathbf{vrb}(D):$ [5, *T2.2.16*, 1, 7, *D1, D12, T2.2.9*]

(20) $[X]:X\varepsilon\,\mathbf{vrb}(A).\supset.X\varepsilon\,\mathbf{vrb}(E):$ [19, 16, *D1*]

(21) $E=A:$ [18, 20, 5, *T2.2.2*]

(22) $A\varepsilon\,\mathbf{Kl}(B\cup D).$ [16, 21]

(23) $\mathbf{disj}(B\cup D).$ [*D11*, 1, 7, *D1, T2.2.16*]

(24) $A\,\varepsilon\,\mathbf{Cmpl}(B\cup D)\therefore$ [*D12*, 22, 3, 4, 8, 23]

$B\varepsilon\,\mathbf{hd}(A)$ [*D17, D16*, 15, 24]

T2.5.50 $[A]:A\,\varepsilon\,\mathbf{expr}.A\,\varepsilon\,\sim(\mathbf{vrb}).\supset.\mathbf{1vrb}(A)\,\varepsilon\,\mathbf{hd}(A)$ [*Sim., T2.5.49*]

T2.5.51 $[A]:A\varepsilon\,\mathbf{expr}.A\varepsilon\,\sim(\mathbf{vrb}).\supset.\mathbf{Uvrb}(A)\,\varepsilon\,\mathbf{tl}(A)$ [*Sim., T2.5.50*]

D19 $[A]\therefore A\varepsilon\,\mathbf{non\text{-}rep}.\equiv:A\varepsilon A:[BC]:B\varepsilon\,\mathbf{vrb}(A).C\,\varepsilon\,\mathbf{vrb}(A).$
$B\varepsilon\,\mathbf{cnf}(C).\supset.B=C$

A is an inscription containing no distinct equiform words, *i.e., non-repetitive*.

D20 $[ABC]:A\varepsilon\,\mathbf{mtch}(BC).\equiv.A\varepsilon A.(\mathbf{vrb}(A)\cap\mathbf{cnf}(B))<(\mathbf{vrb}(A)\cap\mathbf{cnf}(C))$

Inscription *A contains more occurrences* of *C* than of *B*.

D21 $[AB]:.A\varepsilon\,\mathbf{Uprcd}(B).\equiv:A\varepsilon\,\mathbf{vrb}:[C]:A\varepsilon\,\mathbf{pr}(C).\supset.\sim(C\,\varepsilon\,\mathbf{prcd}(B))$
A is the *last word before B*.

These technical terms will be used in PART II of this paper.

T2.5.52 $[ABC]:A\varepsilon\,\mathbf{Concat}(BC).\supset.A\varepsilon\sim(\mathbf{vrb})$
Hyp(1) . ⊃ .

(2) $B\varepsilon\,\mathbf{prcd}(C).$ [*D16*, 1]

(3) $C\,\varepsilon\,\mathbf{expr}.$ [2, *D8*]

$[\exists DE].$

(4) $D\varepsilon\,\mathbf{vrb}(B).$
(5) $E\varepsilon\,\mathbf{vrb}(C).$ [2, 3, *T2.2.1*]

(6) $D\varepsilon\,\mathbf{pr}(E).$ [*D8*, 2, 4, 5]

(7) $D\neq E.$ [*T2.2.14*, 6]

(8) $D\varepsilon\,\mathbf{vrb}(A).$ [1, *T2.5.30*, 4, *D10*]

(9) $E\varepsilon\,\mathbf{vrb}(A).$ [1, *T2.5.30*, 5, *D10*]

$A\,\varepsilon\sim(\mathbf{vrb})$ [*T2.4.17*, 7, 8, 9, 1]

This theorem is a generalization of Tarski's axiom 3, *cf.* [43], p. 173.

T2.5.53 $[ABCD]\therefore\mathbf{Concat}(AB)=\mathbf{Concat}(CD).\equiv:A=C.B=D:$
$\vee.[\exists E].A\,\varepsilon\,\mathbf{Concat}(CE).D\,\varepsilon\,\mathbf{Concat}(EB).$
$\vee.[\exists F].C\,\varepsilon\,\mathbf{Concat}(AF).B\,\varepsilon\,\mathbf{Concat}(FD)$

This is Tarski's axiom 4 which includes the associative property of concatenation. The proof requires many lemmas and will not be included here.

NOTES

1. This is not to say the rules are not part of the system. A formalized system consists of axioms *and* rules. The statement of the rules cannot be in the language, but must be expressed in the metalanguage. The statement of the rules is however, not part of the metatheory.

2. With one minor exception which is stated at the end of *Chapter I.*

3. The definition of syntax varies from author to author. We are in agreement with, among others, Curry [8], p. 36, Carnap [4], p. 10, and Martin [30].

4. We reserve the term "expression" (*cf. Ch. II*) for an uninterrupted string of words. The terms "inscription," "word," and "expression" have no standard usage in the literature.

5. In giving examples we sometimes place more than one on a single line and separate them by commas. The period at the end of the line ends the sentence and, like the commas, is not part of the example. In this particular instance four examples are given on one line.

6. Most presentations of syntax have an empty inscription. Our convention fits nicely with the fact that in Ontology the empty name is not an individual.

7. *Cf.* Markov [24], p. 5.

8. Both of these can be found in Markov [24], p. 12.

9. These special quotation marks are used to avoid setting the inscription under consideration on a separate line.

10. Contradictions have actually arisen by allowing disconnected symbols. *Cf.* Leśniewski [18], p. 78ff., [21], p. 156ff., the reply of von Neumann [32], and the note of Lindenbaum [22].

11. The following example, for which I thank Professor Storrs McCall, is of interest here. In the $\{C,N\}$-calculus define

    (1) $$\mathit{NC}\, p\, q \cdot \equiv \cdot C\, N\, p\, q$$

    Using the thesis $\vdash CNCppCpp$ and definition (1) we have $\vdash \mathit{NC}\ CppCpp$. Now if we "accidentally" separate the ⟨⟨N⟩⟩ and ⟨⟨ C ⟩⟩ in ⟨⟨NC⟩⟩ we get $\vdash NCCppCpp$, which is absurd.

12. *Cf.* Leśniewski [20], n. 3, pp. 295-296, and Tarski [44], p. 169, 174.

13. To decide when two words are equiform is actually a difficult problem in pattern recognition. Syntax does not have the machinery to define equiformity. Using geometrical language two words are equiform if one can be superimposed on the other after translation and dilation.

14. This notation will be used for convenience; it being easiest to point to something to name it.

15. There is no need to specify the order in which the words occur, as the inscription A determines this. There is no inscription consisting of the second word of A followed by the first word of A.

16. This bizarre behavior will not affect later developments as we almost always use equiformity between expressions; most authors only consider equiform expressions. Chomsky [6] uses a notion of equiformity where the discontinuities must occur at the same places.

17. *Cf.* Leśniewski [18], p. 62, Carnap [3], p. 15.

18. Carnap [4], p. 5 shows this is no restriction.

19. The abbreviation comes from the Latin "*verbum*".

20. *Cf.* Leśniewski, [18], pp. 60-61 and Luschei [23], p. 173. Luschei seems to forget an inclusion relationship, or, possibly, he quantifies not over inscriptions, but over expressions.

21. If one objects to variables ranging over names of inscriptions he could introduce the additional primitive term

$$A\ \varepsilon\ \mathbf{inscr}$$

which is read "A is an *inscription*". Then every thesis of **M** of the form

$$[AB\ldots].\quad \varphi(AB\ldots)$$

would be replaced by one of the form

$$[AB\ldots] : A\,\varepsilon\ \mathbf{inscr}\ .\ B\,\varepsilon\ \mathbf{inscr}\ \ldots . \supset .\ \varphi(AB\ldots)$$

22. We shall use the following private convention in writing theses of **M**: If we know a variable is an individual we denote it by a capital Latin letter; if not, or if the variable is a general name, we denote it by a small Latin letter.

23. *Cf.* Sobociński [41], p. 63.

24. We are not writing proofs in a format which would be acceptable to Leśniewski. Since free variables never occur we should probably write:

Dem. $[AB]$:
1. $A\ \varepsilon\ \mathbf{vrb}\,(B). \supset .$
   $[\exists C]$.
2. $C\ \varepsilon\ \mathbf{vrb}\,(A)$.
3. $C = A$.
   $A\ \varepsilon\ \mathbf{vrb}\,(A)$

We abbreviate the proofs purely to save space. For examples and explanations of this proof technique see Lejewski [15].

25. In subsequent definitions we shall merely place the suggested reading beneath the definition without indicating it by quotation marks.

26. We shall not repeatedly make reference to the theorems and definitions of Ontology. *DO6*, which is used here, will not be referred to again although it will occasionally be used. *Ont. 2* was used in step 2 of the last theorem, but not referred to.

27. *Cf.* Sobociński [41], p. 60.

28. *A6* incorporates the following single axiom for an equivalence relation $\sim$:

$$[AB] : A \sim B\ . \equiv : [C] : B \sim C\ . \equiv .\ A \sim C$$

which was discovered independently by Clay [6] and Lejewski [17]. Using this result one can prove that

$$[AB] \therefore A \sim B . \equiv: \varphi(A) . \varphi(B) : [C] : B \sim C . \equiv . A \sim C$$

is a single axiom for weak equivalence, where weak reflexivity is

$$[A] : \varphi(A) . \supset . A \sim A$$

29. Some theses of the Algebra of Classes, which is part of Ontology, were used here. We shall not refer to such theses explicitly.

30. We could also use

    $[CD] : C \,\varepsilon\, \mathbf{vrb}(A) . D \,\varepsilon\, \mathbf{vrb}(B) . (\mathbf{vrb}(A) \cap \mathbf{pr}(C)) \infty (\mathbf{vrb}(B) \cap \mathbf{pr}(D)) . \supset .$
    $C \,\varepsilon\, \mathbf{cnf}(D) . D \,\varepsilon\, \mathbf{cnf}(C).$

    The double conclusion is necessary to get symmetry for **cnf**.

31. Remember that we have no empty inscription.

32. "*Sim. T X*" means the proof is similar to that of *TX*.

33. We intentionally spell "Klass" with a "K" to indicate that we are using the term defined in *D1*. When "class" is used it is used intuitively.

34. *Cf.* Sobociński [40], p. 36. This paper contains a survey of Mereology.

35. The "natural" method used here to motivate *D1* has nothing to do with the way it was actually invented.

36. Since we can prove that Klasses are unique we denote them with capital letters, *i.e.*, we write "**Kl** (a)" rather than "**kl** (a)". Throughout this work we use capitalization to denote uniqueness.

37. *Cf. Ch. I,* **1**, and also Markov [24].

38. Leśniewski [18], p. 61 or 62.

39. But not if incomplete symbols were used.

40. The factor $A \,\varepsilon\, A$, which is required by the rules of definition (*cf.* Leśniewski [19], p. 124), is listed on the same line as $B \,\varepsilon\, \mathbf{pr}\,(A)$ since it follows from it. In general mutually independent conjuncts in the definiens are listed on separate lines by Leśniewski. Space limitations prevent us from doing this.

41. *Cf.* Sobociński [40].

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Seminar in Symbolic Logic
University of Notre Dame
Notre Dame, Indiana

and

Bowling Green State University
Bowling Green, Ohio