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CERTAIN COUNTEREXAMPLES TO THE CONSTRUCTION OF COMBINATORIAL DESIGNS ON INFINITE SETS

WILLIAM J. FRASCELLA

The present note attempts to elaborate the main result of my paper [1]. To this end the following definitions are necessary.*

Definition 1. Let M be some fixed set and F and G families of subsets of M. G is said to be a Steiner cover of F if and only if for every $x \epsilon F$ there is exactly one $y \epsilon G$ such that $x \subset y$.

Definition 2[1]. *Let k be a non-zero cardinal number such that $k \leq \overline{\overline{M}}$. A family F of subsets of M is called a k-tuple family of M if and only if* i) *if $x, y \epsilon F$ such that $x \neq y$ then $x \not\subset y$ and* ii) *if $x \epsilon F$ then $\overline{\overline{x}} = k$.*

As in [1] the result presented here will be given within Zermelo-Fraenkel set theory with the axiom of choice. If x is a set, $\overline{\overline{x}}$ denotes the cardinality of x. If n is a cardinal number then $[x]^{*n} = \{y \subset x : \overline{\overline{y}} * n\}$ where $*$ can stand for the symbols $=, \leq, \geq, <$ or $>$. The expression "$x \subset y$" means "x is a subset of y" improper inclusion not being excluded. If α is an ordinal number ω_α is the smallest ordinal whose cardinality is $\aleph_\alpha$. As usual, we write ω for ω_0. For each ordinal α we define a cardinal number $\mathfrak{a}_\alpha$ by recursion as follows: set $\mathfrak{a}_0 = \aleph_0$. If $\alpha = \beta+1$ then set $\mathfrak{a}_\alpha = 2^{\mathfrak{a}_\beta}$. If α is a limit number then set $\mathfrak{a}_\alpha = \sum_{\beta<\alpha} \mathfrak{a}_\beta$. Also for any ordinal α, $\mathrm{cf}(\alpha)$ represents the smallest ordinal which is cofinal with α.

It is now possible to state the main result of [1] as follows.

Theorem 3. In every set M of cardinality $\mathfrak{a}_\omega$ there is an $\aleph_0$-tuple family F of *M such that there does not exist a family $G \subset [M]^{\aleph_1}$ which is a Steiner cover of F.*

The following will be the principal content of the present note.

Theorem 4. Let α, β and γ be ordinal numbers such that i) $\alpha < \beta < \gamma$, ii) *γ is a limit number,* iii) $\mathrm{cf}(\omega_\gamma) \leq \omega_\alpha < \mathrm{cf}(\omega_\beta)$, iv) *if $\delta < \gamma$ then $\aleph_\delta^{\aleph_\alpha} < \aleph_\gamma$ and*

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v) *for any set* S, $\aleph_\beta < \overline{\overline{S}} < \aleph_\gamma$, *there is a well-ordering of its* $\aleph_\beta$ *subsets* $\{y_\eta\}$ *such that for each* y_η, *if* $x_{\eta'}$ *is an* $\aleph_\alpha$ *subset of* $y_{\eta'}$ *and* $x_{\eta'} \not\subset y_\eta$ $(\eta' < \eta)$ *then there is some* $\aleph_\alpha$ *subset* x^* *of* y_η *which is not contained in any* $x_{\eta'}$ $(\eta' < \eta)$. *Then, in every set* M *of cardinality* $\aleph_\gamma$ *there exists an* $\aleph_\alpha$*-tuple family* F *of* M *such that there does not exist a family* $G \subset [M]^{\aleph_\beta}$ *which is a Steiner cover of* F.

Before proceeding with a proof of Theorem 4 we recall a definition and proposition which was given in [1] and whose proof we do not bother to repeat.

Definition 5.[2] *Let* F *be a family of subsets of a set* M *and* n *a non-zero cardinal number. A family* G *is called a* n*-spoiler of* F *if and only if for every* $x \in F$ *and every* $y \in [M]^n$ *there is a* $z \in G$ *such that* $z \subset x \cup y$.

Proposition 6.[3] *Let* k *and* n *be non-finite cardinal numbers and let* F *be a* k*-tuple family of a non-finite set* M. *Suppose there exists subfamilies* $F_1, F_2 \subset F$ *such that* i) $F_1 \cap F_2 = 0$, ii) F_2 *is an* n*-spoiler of* F_1 and iii) $n^k \overline{\overline{F}}_2 < \overline{\overline{F}}_1$. *Then* F *does not possess a Steiner cover contained in* $[M]^n$.

Proof of Theorem 4. Let M be any set of cardinality $\aleph_\gamma$. On the strength of hypotheses ii) and iv) it will be possible to represent $\aleph_\gamma$ as

(1) $$\aleph_\gamma = \sum_{\xi < \mathrm{cf}(\omega_\gamma)} \aleph_{\alpha_\xi}$$

such that

(2) $\aleph_{\alpha_\xi} < \aleph_\gamma$ *for each* ξ

and

(3) $\aleph_{\alpha_\xi} = \aleph_\eta^{\aleph_\alpha}$ *for each* ξ.

Certainly representation (1) with property (2) is possible solely on the strength of hypothesis ii) and the meaning of the symbol $\mathrm{cf}(\omega_\gamma)$. However in virtue of iv) we know that the sequence $\{\aleph_{\alpha_\xi}^{\aleph_\alpha}\}_{\xi<\mathrm{cf}(\omega_\gamma)}$ must have $\aleph_\gamma$ as its sum. From this it is possible to extract a strictly increasing subsequence whose sum is also $\aleph_\gamma$. This subsequence will satisfy (1), (2) and (3).

Consequently it is possible for each $\xi < \mathrm{cf}(\omega_\gamma)$ to construct a set M_ξ

(4) $M = \bigcup \{M_\xi : \xi < \mathrm{cf}(\omega_\gamma)\}$

(5) $M_{\xi_1} \cap M_{\xi_2} = 0$ *if* $\xi_1 \neq \xi_2$

(6) $\overline{\overline{M}}_{\xi_1} < \overline{\overline{M}}_{\xi_2}$ if $\xi_1 < \xi_2$

and

(7) $\overline{\overline{M}}_\xi = \aleph_{\eta_\xi}^{\aleph_\alpha}$ *for each* $\xi < \mathrm{cf}(\omega_\gamma)$.

It is also possible to require

(8) $\overline{\overline{M}}_\xi > \aleph_\beta$ *for each* $\xi < \mathrm{cf}(\omega_\gamma)$.

Lemma 7. For each $\xi < \mathrm{cf}(\omega_\gamma)$ *there exists an* $\aleph_\alpha$*-tuple family* F_ξ *of* M_ξ *such that* $(\forall y \in [M_\xi]^{\aleph_\beta})(\exists x \in F_\xi)[x \subset y]$.

Proof. Using the axiom of choice the family $[M_\xi]^{\aleph_\beta}$ may be well-ordered (as in v) and expressed as follows

(9) $[M_\xi]^{\aleph_\beta} = \{y_\eta : \eta < \mu\}$

where μ is the cardinality of the family $[M_\xi]^{\aleph_\beta}$. The construction of the family F_ξ will be accomplished by transfinite induction as follows. Let x_0 be any subset of y_0 such that

(10) $\overline{\overline{x}}_0 = \aleph_\alpha$.

Let $\delta < \omega_\mu$ and assume for each $\eta < \delta$ there exists a subset x_η of y_η such that

(11) $\overline{\overline{x}}_\eta = \aleph_\alpha$

and

(12) $\{x_\eta | \eta < \delta\}$ is an $\aleph_\alpha$-tuple family.

Case 1° $(\exists \eta < \delta)\ [x_\eta \subset y_\delta]$

Here define x_δ to be any such $x_\eta (\eta < \delta)$ which is contained in y_δ.

Case 2° $(\forall \eta < \delta)\ [x_\eta \not\subset y_\delta]$

Let $H = \{x_\eta \cap y_\delta | \eta < \delta\}$. Clearly H is a family of subsets of the set y_δ whose cardinality is $\aleph_\beta$. Moreover, since we have

(13) $\overline{\overline{H}} \le \overline{\delta} < \aleph_{\alpha_\xi}^{\aleph_\beta} \le \aleph_\gamma \le \aleph_\gamma^{\aleph_\beta}$

which with assumption v) assures the existence of a subset x^* of y_δ such that

(14) $\overline{\overline{x^*}} = \aleph_\alpha$

and

(15) $x^* \not\subset x_\eta \cap y_\delta$ for all $\eta < \delta$.

Now define $x_\delta = x^*$.

Thus we have defined, by transfinite induction, for each $\eta < \mu$, an $\aleph_\alpha$ - subset x_η of y_η.

Definition 8. Let $F_\xi = \{x_\eta \mid \eta < \mu\}$.

We now show F_ξ satisfies the condition of Lemma 7. Clearly the construction itself shows each member of F_ξ is a subset of M_ξ having cardinality $\aleph_\alpha$. Moreover, suppose

(16) $x, y \in F_\xi$

such that

(17) $x \ne y$.

We may suppose that there exists $\eta_1 < \eta_2 < \omega_\mu$ such that $x = x_{\eta_1}$ and $y = x_{\eta_2}$. Further, we may assume

(18) $x \neq x_\eta$ for all $\eta < \eta_1$

and

(19) $y \neq x_\eta$ for all $\eta < \eta_2$.

By (19) it must be that the construction of $y = x_{\eta_2}$ was made according to Case 2°. Yet (15) and the condition of Case 2° yield

(20) $x_{\eta_2} \not\subset x_{\eta_1}$.

Moreover

(21) $x_{\eta_1} \not\subset x_{\eta_2}$

since if

(22) $x_{\eta_1} \subset x_{\eta_2}$

we would have

(23) $x_{\eta_1} \subset y_{\eta_2}$

which would violate the conditions of Case 2°. Thus F_ξ has the requisite properties and Lemma 7 is established.

Definition 9. $F^\# = \bigcup \{F_\xi \mid \xi < \mathrm{cf}(\omega_\gamma)\}$.

Remark. Since each F_ξ is an $\aleph_\alpha$-tuple family of M_ξ (and therefore of M) and since they are pairwise disjoint it follows that $F^\#$ is an $\aleph_\alpha$-tuple family of M.

Lemma 10. $\overline{\overline{F_\xi}} = \overline{\overline{M_\xi}}$ *for each* $\xi < \mathrm{cf}(\omega_\gamma)$.

Proof. Clearly $\overline{\overline{F_\xi}} \geq \overline{\overline{M_\xi}}$; for otherwise we would have

(24) $\overline{\overline{\bigcup F_\xi}} \leq \overline{\overline{F_\xi}} \cdot \aleph_\alpha < \overline{\overline{M_\xi}}$.

But (24) would allow us to find a subset of M_ξ of cardinality $\aleph_\beta$ which would be disjoint from every member of the family F_ξ. This would contradict the property of F_ξ given in Lemma 7.

To complete the proof of Lemma 10 it only remains to show $\overline{\overline{F_\xi}} \leq \overline{\overline{M_\xi}}$. Since $F_\xi \subset [M_\xi]^{\aleph_\alpha}$ we must have

(25) $\overline{\overline{F_\xi}} \leq \overline{\overline{M_\xi}}^{\aleph_\alpha}$.

But (7) yields

(26) $\overline{\overline{M_\xi}}^{\aleph_\alpha} = (\aleph_{\eta_\xi}^{\aleph_\alpha})^{\aleph_\alpha} = \aleph_{\eta_\xi}^{\aleph_\alpha^2} = \aleph_{\eta_\xi}^{\aleph_\alpha}$

which implies

(27) $\overline{\overline{M_\xi}}^{\aleph_\alpha} = \overline{\overline{M_\xi}}$.

This together with (25) says $\overline{\overline{F_\xi}} \leq \overline{\overline{M_\xi}}$ This completes the proof of Lemma 10.

Lemma 11. $\overline{\overline{F^{\#}}} = \aleph_\gamma$

Proof. This follows from Definition 9, Lemma 10 and the fact that the families F_ξ are disjoint.

Definition 12. $F^* = \{y \epsilon [M]^{\aleph_\alpha} |$ *for each* $\xi < \mathrm{cf}(\omega_\gamma)$, $y \cap M_\xi \epsilon F_\xi\}$.

Remark. It is clear from Definition 12 that the family F^* is in one-one onto correspondence with the generalized Cartesian product set $\prod_{\xi < \mathrm{cf}(\omega_\gamma)} F_\xi$. The association is natural in the sense that to $f \epsilon \prod_{\xi < \mathrm{cf}(\omega_\gamma)} F_\xi$ we let correspond the set $\bigcup\{f(\xi) | \xi < \mathrm{cf}(\omega_\gamma)\}$. Since $\overline{\overline{f(\xi)}} = \aleph_\alpha$ and by hypothesis iii) (i.e. $\mathrm{cf}(\omega_\gamma) \le \omega_\alpha$) it must be that $\overline{\overline{\bigcup\{f(\xi) | \xi < \mathrm{cf}(\omega_\gamma)\}}} = \aleph_\alpha$. Now suppose $x, y \epsilon F^*$ such that $x \neq y$ and $x \subset y$. Thus there exists $f, g \epsilon \prod_{\xi < \mathrm{cf}(\omega_\gamma)} F_\xi$ such that $f \neq g$ and $\bigcup\{f(\xi) | \xi < \mathrm{cf}(\omega_\gamma)\} \subset \bigcup\{g(\xi) | \xi < \mathrm{cf}(\omega_\gamma)\}$. But $f \neq g$ implies the existence of a $\xi_0 < \mathrm{cf}(\omega_\gamma)$ such that $f(\xi_0) \neq g(\xi_0)$. But $f(\xi_0), g(\xi_0) \epsilon F_{\xi_0}$ and the above inclusion forces $f(\xi_0) \subset g(\xi_0)$, contradicting the fact that F_{ξ_0} is a $\aleph_\alpha$-tuple family of M_{ξ_0}. From this it is possible to conclude that F^* is an $\aleph_\alpha$-tuple family of M.

Lemma 13. $\overline{\overline{F^*}} > \aleph_\gamma$.

Proof. By Lemma 10 and the above Remark we obtain

$$(28)\quad \overline{\overline{F^*}} = \overline{\overline{\prod_{\xi < \mathrm{cf}(\omega_\gamma)} F_\xi}} = \prod_{\xi < \mathrm{cf}(\omega_\gamma)} \overline{\overline{M_\xi}}$$

But by (6) the sequence of cardinals $\{\overline{\overline{M_\xi}}\}_{\xi < \mathrm{cf}(\omega_\gamma)}$ is increasing and consequently by a corollary to a theorem of J. König we have

$$(29)\quad \sum_{\xi < \mathrm{cf}(\omega_\gamma)} \overline{\overline{M_\xi}} < \prod_{\xi < \mathrm{cf}(\omega_\gamma)} \overline{\overline{M_\xi}}$$

which with (28) yields

$$(30)\quad \overline{\overline{F^*}} > \sum_{\xi < \mathrm{cf}(\omega_\gamma)} \overline{\overline{M_\xi}} = \aleph_\gamma.$$

Lemma 13 is proved.

Lemma 14. $F^{\#} \cap F^* = 0$.

Proof. Immediate.

Lemma 15. $(\forall y \epsilon [M]^{\aleph_\beta})(\exists \xi < \mathrm{cf}(\omega_\gamma))[\overline{\overline{y \cap M_\xi}} = \aleph_\beta]$.

Proof. Let $y \epsilon [M]^{\aleph_\beta}$. Now suppose to the contrary that

$$(31)\quad (\forall \xi < \mathrm{cf}(\omega_\gamma))[\overline{\overline{y \cap M_\xi}} < \aleph_\beta].$$

But it is clear that

$$(32)\quad y = \bigcup\{y \cap M_\xi | \xi < \mathrm{cf}(\omega_\gamma)\}.$$

But (31) and the hypothesis that $\mathrm{cf}(\omega_\gamma) \leq \omega_\alpha < \mathrm{cf}(\omega\ )$ yields

(33) $\overline{\overline{\bigcup\{y \cap M_\xi | \xi < \mathrm{cf}(\omega_\gamma)\}}} < \aleph_\beta$

which contradicts the fact that $\overline{\overline{y}} = \aleph_\beta$. Thus Lemma 15 is complete.

Lemma 16. $F^\#$ is an $\aleph_\beta$-spoiler of F^*.

Proof. Let $x \epsilon F^*$ and $y \epsilon [M]^{\aleph_\beta}$. Using Lemma 15 there is an $\xi_0 < \mathrm{cf}(\omega_\gamma)$ such that

(34) $\overline{\overline{y \cap M_{\xi_0}}} = \aleph_\beta$.

By Lemma 7 there must exist an $x_0 \epsilon F_{\xi_0}$ such that

(35) $x_0 \subset y \cap M_{\xi_0}$.

But of course this gives an $x_0 \epsilon F^\#$ such that $x_0 \subset y \subset x \cup y$ which shows $F^\#$ to be an $\aleph_\beta$-spoiler of F^*. Lemma 16 is proved.

Lemma 17. $\aleph_\beta^{\aleph_\alpha} \overline{\overline{F^\#}} < \overline{\overline{F^*}}$.

Proof. Since $\aleph_\beta < \aleph_\gamma$, hypothesis iv) guarantees

(36) $\aleph_\beta^{\aleph_\alpha} < \aleph_\gamma$.

But (36) together with Lemma 11 yield

(37) $\aleph_\beta^{\aleph_\alpha} \overline{\overline{F^\#}} = \aleph_\gamma$

which with Lemma 13 establish Lemma 17.

Setting $F = F^\# \cup F^*$ we see that the hypotheses of Proposition 6 are satisfied. Thus the $\aleph_\alpha$-tuple family F of M does not possess any Steiner cover contained in $[M]^{\aleph_\beta}$. This completes the proof of Theorem 4.

NOTES

1. We remark that in the present work our terminology slightly differs from that given in [1]. What in the present note is called a k-tuple family is called, in [1], a k-tuple family (in the wider sense). In [1] we used the simple expression "k-tuple family" for a more restricted concept which plays no role in the present note.
2. This appears as Definition 7 of [1].
3. This appears as Proposition 8 of [1].

REFERENCE

[1] Frascella, W. J., "The non-existence of a certain combinatorial design on an infinite set," *Notre Dame Journal of Formal Logic*, vol. 10 (1969), pp. 317-323.

University of Illinois
Urbana, Illinois