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COPI'S METHOD OF DEDUCTION AGAIN

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Professor Copi, in the recent 3rd edition of *Symbolic Logic* [1], leaves unchanged the adaptation of Canty's proof of the completeness of **CMD** [2] which he used in the 2nd edition. The object of the present note is twofold. (1) To establish a lemma of the proof which neither author explicitly establishes, and to show that in view of the way in which this lemma needs to be established, the Copi-Canty proof involves a pointless complication. (2) To establish another lemma of the proof, namely that the replacement rules of **CMD** are adequate to deriving from any line in a proof its DNF. The literature, of course, contains various proofs to the effect that any propositional formula can be reduced to DNF within some version of propositional logic. What is proposed here is another proof to the same effect, relating specifically to **CMD**.

(1) Each author uses Metatheorem A below. A proof is supplied.

Metatheorem A: *If* $P_1^s, P_2^s, \ldots, P_n^s \therefore Q^s$ *is a valid argument whose validity depends solely on truth-functional considerations, then* $P_1, P_2, \ldots, P_n \vdash Q$ *in* **RS**.[1]

Proof: In **RS** we can always construct the sequence Σ of wffs $P_1, P_2, \ldots, P_n, Q$. If this sequence can be enlarged to form a sequence $S_1, S_2, \ldots, S_k$, such that every S_i $(1 \leqslant i \leqslant k)$ is either a P_i $(1 \leqslant i \leqslant n)$, or an axiom of **RS**, or is derived from two preceding lines of the sequence by the use of R1, and S_k is Q, then $P_1, P_2, \ldots, P_n \vdash Q$. Now by hypothesis $P_1^s, P_2^s, \ldots, P_n^s \therefore Q^s$ is a truth-functionally valid argument, and therefore $(P_1^s \cdot P_2^s \cdot \ldots \cdot P_n^s) \supset Q^s$ is tautologous. Thus, by the completeness of **RS**, $\vdash (P_1 \cdot P_2 \cdot \ldots \cdot P_n) \supset Q$. Thus there is a sequence Σ' of wffs of **RS**, such that every wff is an axiom of **RS** or follows from two preceding wffs by R1, the last wff of which is $(P_1 \cdot P_2 \cdot \ldots \cdot P_n) \supset Q$. Suppose Σ' prefaced to the sequence Σ to form the sequence Σ''. Σ'' can then be enlarged to from Σ''', where Σ''' is a sequence $S_1, S_2, \ldots, S_k$, as follows.

1. Where the P_i^s's and Q^s are the P_i's and Q as interpreted (normally) within a semantical system $\mathcal{S}$.

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By DR14 of **RS** (P, $Q \vdash P \cdot Q$) we can insert after P_2 the wff $P_1 \cdot P_2$. By DR14 we can insert after P_3 the wff $P_1 \cdot P_2 \cdot P_3$. By the $(n - 1)$th insertion on the strength of DR14 we have $P_1 \cdot P_2 \cdot \ldots \cdot P_n$ added to Σ''. By R1 on this line and $(P_1 \cdot P_2 \cdot \ldots \cdot P_n) \supset Q$ we can add Q, and this sequence is Σ, since every wff in it is a P_i, or an axiom of **RS**, or follows from two preceding lines by R1, and its last line is Q.

Now this proof depends *inter alia* on the completeness result for **RS** in that it infers that $\vdash (P_1 \cdot P_2 \cdot \ldots \cdot P_n) \supset Q$ from the assumption that $P_1^s, P_2^s, \ldots, P_n^s \therefore Q^s$ is valid, and thus the corresponding hypothetical tautologous. Moreover there could be no way of founding Metatheorem A on the completeness result as stated and proved by Copi that did not involve this inference. But the Copi-Canty proof is founded on the completeness result, and so it involves this inference. But once the inference is made, the point of establishing Metatheorem A as a lemma for the proof of the completeness of **CMD** is lost, since the only way the lemma enters into the Copi-Canty proof is as a means of establishing that if $P_1^s, P_2^s, \ldots, P_n^s \therefore Q^s$ is valid then $\vdash (P_1 \cdot P_2 \cdot \ldots \cdot P_n) \supset Q$. But if this is correct, then an even more curious redundancy emerges. For the only point, in the Copi-Canty proof, of establishing that $\vdash (P_1 \cdot P_2 \cdot \ldots \cdot P_n) \supset Q$ is to derive, by the analyticity of **RS**,[2] the conclusion that $P_1^s \cdot P_2^s \cdot \ldots \cdot P_n^s \supset Q^s$ is a tautology, and thus $P_1^s \cdot P_2^s \cdot \ldots \cdot P_n^s \cdot \sim Q^s$ truth-functionally inconsistent. But according to the above analysis the inference from the validity of $P_1^s, P_2^s, \ldots, P_n^s \therefore Q^s$ to the tautologousness of $P_1^s \cdot P_2^s \cdot \ldots \cdot P_n^s \supset Q^s$ must already be made before any appeal to the completeness of **RS** is available. Thus to route the proof through the completeness of **RS** is entirely superfluous.

(2) A proof is supplied of the thesis that the DNF of any line can be derived by **CMD** as an equivalent line such that (where applicable) the disjunctive grouping is by association to the right and (where applicable) the conjunctive grouping is also by association to the right. The proof is by course-of-values induction on the number of occurrences of '$\sim$', '$\cdot$' & '$\vee$' in (interpreted) formulae[3] of the kind used in Chapter 3 of [1], counting recurrences. All rules of inference used are equivalence rules.[4] We suppose all occurrences of '$\supset$' and '$\equiv$' to have been cleared initially by Impl. and Equiv..

α - case: The line is $\sim P$, $P \cdot Q$, or $P \vee Q$, where P and Q are single letters. Any such line is already in DNF, and can trivially be derived from itself by two applications of Taut.. The minimum number of operators for which the first grouping property emerges is 2, and here evidently the non-standard $(P \vee Q) \vee R$, where P, Q and R are single letters, becomes $P \vee (Q \vee R)$ by Assoc. once. The minimum number for which the second grouping property emerges is 2, and here the non-standard $(P \cdot Q) \cdot R$ becomes $P \cdot (Q \cdot R)$ by Assoc. once.

2. Canty makes this derivation depend on the *completeness* of **RS**, but this can only be a *lapsus pennae*.
3. But, for brevity, the superscript 's' is dropped in the exposition.
4. All references to the rules of **CMD** are by the abbreviations of [1], pp. 42-43.

β - case: Suppose the thesis holds for any formula with m ($1 \leqslant m < n$) occurrences of the operators. Any formula with n occurrences will be of the overall form $\sim P$, $P \cdot Q$ or $P \vee Q$, where P and Q are any formulae of the relevant kind, with $(n - 1)$ occurrences between them.

Subcase (i): $\sim P$. From P, by the β-case assumption, we can derive as an equivalent line by **CMD** its DNF

$$P_1 \vee (P_2 \vee (\ldots \vee P_l) \ldots) \tag{1}$$

where the P_i's ($1 \leqslant i \leqslant l$) are as required by the definition of 'DNF'. Then from $\sim P$ there can be derived by $(l - 1)$ applications of De M. as an equivalent line

$$\sim P_1 \cdot (\sim P_2 \cdot (\ldots \cdot \sim P_l) \ldots) \tag{2}$$

The P_i's are either (1) single letters or (2) negations of such or (3) conjunctions of y ($y > 1$) such single letters or negations of single letters. Consider any conjunct $\sim P_j$ of (2) such that P_j is of kinds (1) or (2). If $\sim P_{j_1} = \sim P_1$ then $\sim P_{j_2}$ can be grouped with it by $(2j - 3)$ applications of Assoc. and $(j - 2)$ applications of Com., so that (2) becomes a conjunction with $\sim P_{j_1} \cdot \sim P_{j_2}$ as first conjunct. P_{j_3} can then be extracted from the second conjunct and grouped with $\sim P_{j_1} \cdot \sim P_{j_2}$ to get $(\sim P_{j_1} \cdot \sim P_{j_2}) \cdot \sim P_{j_3}$ by $(2j - 5)$ applications of Assoc. and $(j - 3)$ applications of Com., P_{j_4} then extracted and grouped with this enlarged first conjunct by Assoc. $(2j - 7)$ times and Com. $(j - 4)$ times to get $((\sim P_{j_1} \cdot \sim P_{j_2}) \cdot \sim P_{j_3}) \cdot \sim P_{j_4}$, and so on (except that if in any of these cases P_{j_d} is the last letter of the whole formula, it requires only the same number of uses of Assoc. as would $P_{(j-1)d}$, and one more use of Com.). If $\sim P_{j_1} \neq \sim P_1$, then (2) becomes a conjunction with $\sim P_{j_1}$ as first major conjunct by $(2j - 1)$ applications of Assoc. and $(j - 1)$ applications of Com.; $\sim P_{j_2}$ can then be grouped with it by Assoc. $(2j - 3)$ times and Com. $(j - 2)$, and so on. Thus (2) can be regrouped so that the negations of all P_i's of kinds (1) and (2) form, in conjunction, the first conjunct of the derived formula. All double negations can be cleared by iterated DN. Referring to the first conjunct of this new formula as '$\mathcal{G}$', then from $\sim P$ we have derived as an equivalent line

$$\mathcal{G} \cdot (\sim P_{k_1} \cdot (\sim P_{k_2} \cdot (\ldots \cdot \sim P_{k_r}) \ldots)). \tag{3}$$

$\mathcal{G}$ will not be grouped by association to the right, but the 2nd conjunct of (3) will be so grouped, this being guaranteed by the routine by which the P_j's are extracted. Now each P_k in this formula has a known structure, being of kind (3); let $P_{k_{1_1}}, P_{k_{1_2}}, \ldots, P_{k_{1_s}}$ be the elements of kinds (1) and (2) that in conjunction make up P_{k_1}, grouped, by the β-case assumption, by association to the right; then $\sim P_{k_1}$ is, by $(s - 1)$ applications of De M.,

$$\sim P_{k_{1_1}} \vee (\sim P_{k_{1_2}} \vee (\ldots \vee \sim P_{k_{1_s}}) \ldots)$$

where double negations are then all cleared by iterated DN. A similar analysis holds for $\sim P_{k_2}$ to $\sim P_{k_r}$. Thus from $\sim P$ we can derive as an equivalent line

$$\mathcal{G} \cdot ((\sim P_{k_{1_1}} \vee (\sim P_{k_{1_2}} \vee (\ldots \vee \sim P_{k_{1_s}}) \ldots) \cdot ((\sim P_{k_{2_1}} \vee (\sim P_{k_{2_2}} \vee (\ldots \vee \sim P_{k_{2_t}}) \ldots) \cdot \ldots \cdot (\sim P_{k_{r_1}} \vee (\sim P_{k_{r_2}} \vee (\ldots \vee \sim P_{k_{r_u}}) \ldots) \ldots) \tag{4}$$

By Assoc. once (4) yields as an equivalent line (5) a conjunction with

$$\mathcal{G} \cdot (\sim P_{k_{1_1}} \vee (\sim P_{k_{1_2}} \vee (\ldots \vee \sim P_{k_{1_s}}) \ldots) \tag{6}$$

as first conjunct, and by Dist. (s - 1) times, (6) becomes the DNF

$$\mathcal{G} \cdot \sim P_{k_{1_1}} \vee (\mathcal{G} \cdot \sim P_{k_{1_2}} \vee (\ldots \vee \mathcal{G} \cdot \sim P_{k_{1_s}}) \ldots) \tag{7}$$

Now by Assoc. once we group (7) with the first conjunct of the second major conjunct of (5), i.e. with

$$\sim P_{k_{2_1}} \vee (\sim P_{k_{2_2}} \vee (\ldots \vee \sim P_{k_{2_t}}) \ldots) \tag{8}$$

to get a conjunctive formula (9) with

$$(\mathcal{G} \cdot \sim P_{k_{1_1}} \vee (\mathcal{G} \cdot \sim P_{k_{1_2}} \vee (\ldots \vee \mathcal{G} \cdot \sim P_{k_{1_s}}) \ldots)) \cdot (\sim P_{k_{2_1}} \vee (\sim P_{k_{2_2}} \vee (\ldots \vee \sim P_{k_{2_t}}) \ldots)) \tag{10}$$

as first major conjunct.

If $s = t = 1$ then (10) is already in DNF, and standardly grouped.
If $s = t > 1$, then by Dist. 3 times, Com. twice and Assoc. once, it becomes

$$\sim P_{k_{2_1}} \cdot (\mathcal{G} \cdot \sim P_{k_{1_1}}) \vee ((\sim P_{k_{2_1}} \cdot (\mathcal{G} \cdot \sim P_{k_{1_2}} \vee (\ldots \vee \mathcal{G} \cdot \sim P_{k_{1_s}}) \ldots) \vee ((\mathcal{G} \cdot \sim P_{k_{1_1}} \cdot (\sim P_{k_{2_2}} \vee (\ldots \vee \sim P_{k_{2_t}}) \ldots)) \vee ((\sim P_{k_{2_2}} \vee (\ldots \vee \sim P_{k_{2_t}}) \ldots)) \cdot (\mathcal{G} \cdot \sim P_{k_{1_2}} \vee (\ldots \vee \mathcal{G} \cdot \sim P_{k_{1_s}}) \ldots))))$$

i.e. of the form

$$\mathrm{TR} \vee (\mathrm{TS} \vee (\mathrm{RU} \vee \mathrm{US}))$$

where R, S, T and U are the major disjuncts of the conjuncts of (10) taken in order. TR needs no further manipulation. TS takes on DNF by Dist. (s - 2) times. RU takes on DNF by Dist. (t - 2) times. Repeated use of Assoc. will restore the disjunctive grouping disturbed by the above operations, and also the conjunctive grouping, disturbed from the construction of $\mathcal{G}$ on.[5] If $s = t > 2$ then US merely repeats the problem set by (10). Another such round as that applied to (10) will either finally reduce (10) to DNF or set the same problem again. In general (t - 1) such rounds all told will put (10) in DNF. Repeated Assoc., as before, will restore standard grouping. If $s \neq t$ and $s = 1$ or $t = 1$ the solution is obvious. If $s \neq t$ and $s > 1$ and $t > 1$ then in the manipulation of US, or of the formula which parallels US in some later round, one of U and S, or one of the formulae which parallel U and S in the later round, will reduce to a single conjunction. In this case, where the other formula retains p disjuncts, then Com. (if necessary) and Dist. (p - 1) times will finally put US, or the formula

5. The lemma required here, and invoked five further times below, is not proved, though it is readily provable.

which parallels it in the later round, into DNF. Repeated Assoc. will again restore standard grouping. The DNF of (10) thus obtained can then be grouped by Assoc. with the first conjunct of the second major conjunct of (9), and the above account of the reduction of (10) will apply, *mutatis mutandis*, to this new line. By repeated such Assoc. followed up by the method described for (10), (9) can be reduced to DNF. But (9) was derived from $\sim P$ as an equivalent line, and thus the DNF to which (9) is finally reduced is too. Repeated Assoc. will suffice to restore standard grouping to this DNF.

Subcase (ii): $P \cdot Q$. By the β-case assumption we can from $P \cdot Q$ derive as an equivalent line a formula F which is the conjunction of the DNF of P with the DNF of Q, where each DNF has standard grouping. The analysis of this case will follow that of (10) above.

Subcase (iii): $P \vee Q$. By the β-case assumption we can derive from $P \vee Q$ as an equivalent line a formula F' which is the alternation of the DNF of P with the DNF of Q, and F' is already the DNF of $P \vee Q$. Assoc. will suffice to restore standard grouping.

REFERENCES

[1] Copi, I. M., *Symbolic Logic* (3rd edition), The MacMillan Company, New York (1967), pp. 246-247.

[2] Canty, J. T., "Completeness of Copi's method of deduction," *Notre Dame Journal of Formal Logic*, vol. IV (1963), pp. 142-144.

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