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THE CONSISTENCY OF THE AXIOMS OF ABSTRACTION AND EXTENSIONALITY IN A THREE-VALUED LOGIC

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The Abstraction Axiom I want to consider is the following one, which is based on the Łukasiewicz three-valued logic.

$$(*)\ (Sy)(Ax)(x \,\varepsilon\, y \leftrightarrow \phi(x, z_1, \ldots, z_n))$$

where ϕ is either a propositional constant or constructed from atomic wffs $u \,\varepsilon\, v$ by using $\sim$, &, A. The connectives and quantifiers of the logic can be represented as follows:

	$p \,\&\, q$			$\sim p$	$p \vee q$			$p \rightarrow q$			$p \leftrightarrow q$			$p \supset q$		
p/q	1	$\frac{1}{2}$	0		1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0	0	1	1	1	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	1
0	0	0	0	1	1	$\frac{1}{2}$	0	1	1	1	0	$\frac{1}{2}$	0	1	1	1

$(Ax) fx$ has the minimum value of the values of fx. $(Sx) fx$ has the maximum value of the values of fx.

Th. Skolem has produced models, in [1] and in [2] for an Abstraction Axiom the same as (*) except that ϕ may not be constructed using quantifiers A and S. He shows that the Axiom of Extensionality is also valid in his model in [2]. The procedure we use for constructing the model roughly follows the lines of P. C. Gilmore's paper (see [3]), where he constructed a model for his partial set theory PST′.

1. To construct the model, we need to extend the wffs used above to express (*) by adding some terms, some of which will be used as the domain of the model. We give the formation rules for terms and wffs as follows:

1. If x and y are set variables, then $x \,\varepsilon\, y$ is an atomic wff.
2. Any combination of wffs using $\sim$, $\rightarrow$, A are wffs.
3. A propositional constant (i.e., 1, $\frac{1}{2}$ or 0) is an atomic wff.

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4. A propositional constant or a wff constructed from atomic wffs using only $\sim$, &, A is a standard wff.
5. If P is a standard wff and x is a set variable, then $\{x : P\}$ is a term.
6. If $\{x : P\}$ and $\{x : Q\}$ are terms and y is a set variable, then $\{x : P\} \varepsilon y$, $y \varepsilon \{x : P\}$, $\{x : P\} \varepsilon \{x : Q\}$ are atomic wffs.

We will use $a, b, c, \ldots$ for constant terms. We construct a model for (*) with domain the set D of all constant terms $\{x : P\}$, i.e., P either has no free variables at all or has x as its only free variable. Non-constant terms can be defined from these as follows: associate with any term $\{x : P(x, z_1, \ldots, z_k)\}$, for which $z_1, \ldots, z_k$ are the only free variables of the term, the function which for constant terms $a_1, \ldots, a_k$ of D takes as value the constant term $\{x : P(x, a_1, \ldots, a_k)\}$ of D.

Let any specification of values for all the constant atomic wffs of the form $x \varepsilon y$, where x and y range over the domain D, be called a *structure on* D. Let $V[M](P)$ denote the value of the constant wff P given by the structure M on D. Also let $V[M](1) = 1$, $V[M](0) = 0$ and $V[M](\frac{1}{2}) = \frac{1}{2}$. Define $M_1 \leqslant M_2$ for two structures M_1 and M_2 on D if, for every constant atomic wff P, if $V[M_1](P) = 1$ then $V[M_2](P) = 1$ and if $V[M_1](P) = 0$ then $V[M_2](P) = 0$. Define the structure M_0, such that, for all constant atomic wffs P, $V[M_0](P) = \frac{1}{2}$. Then $M_0 \leqslant M$, for any structure M on D. Here, '$\leqslant$' defines a partial ordering on the set of structures, since (i) $M \leqslant M$, (ii) if $M_1 \leqslant M_2$ and $M_2 \leqslant M_3$ then $M_1 \leqslant M_3$ and (iii) if $M_1 \leqslant M_2$ and $M_2 \leqslant M_1$ then $M_1 = M_2$ (i.e., M_1 and M_2 are the same structure). From now on, when mentioning values of wffs in a structure it is automatically assumed that the wffs are constant ones, i.e., they have no free variables.

Lemma 1 *Let M and M' be two structures on D such that $M \leqslant M'$. Then, for any standard wff P, if $V[M](P) = 1$ then $V[M'](P) = 1$ and if $V[M](P) = 0$ then $V[M'](P) = 0$.*

Proof. By induction on the wff evaluation procedure. This means that we start at the values of the substitution instances of all the atomic wffs and build up the value of P from these values according to the connectives and quantifiers in the Łukasiewicz logic. If P is an atomic wff or a propositional constant, the lemma holds.

(i) Let $V[M](\sim Q) = 1$, then $V[M](Q) = 0$. By the induction hypothesis, $V[M'](Q) = 0$ and hence $V[M'](\sim Q) = 1$. Similarly, if $V[M](\sim Q) = 0$, then $V[M'](\sim Q) = 0$.

(ii) Let $V[M](Q \& R) = 1$, then $V[M](Q) = V[M](R) = 1$. By induction hypothesis, $V[M'](Q) = V[M'](R) = 1$ and hence $V[M'](Q \& R) = 1$. Similarly, if $V[M](Q \& R) = 0$, then $V[M'](Q \& R) = 0$.

(iii) Let $V[M]((Ax)Q) = 1$, then $V[M](Q(x)) = 1$ for all x. By induction hypothesis, $V[M'](Q(x)) = 1$ for all x and hence $V[M']((Ax)Q) = 1$. Similarly, if $V[M]((Ax)Q) = 0$, then $V[M']((Ax)Q) = 0$.

The model is the limit of a sequence of structures $M_0 \leqslant M_1 \leqslant M_2 \leqslant \ldots \leqslant M_\mu \leqslant \ldots$, on D. M_0 is defined above, i.e., $V[M_0](P) = \frac{1}{2}$ for all atomic wffs P. Assuming M_μ defined for some ordinal μ, $M_{\mu+1}$ is defined as follows. For all standard wffs P,

$$V[M_{\mu+1}](a\,\varepsilon\{x : P(x)\}) = V[M_\mu](P(a)).$$

For a limit ordinal μ, for all atomic wffs P, if $V[M_\nu](P) = 1$ for some $\nu < \mu$, then $V[M_\mu](P) = 1$; if $V[M_\nu](P) = 0$ for some $\nu < \mu$ then $V[M_\mu](P) = 0$; and if $V[M_\nu](P) = \frac{1}{2}$ for all $\nu < \mu$ then $V[M_\mu](P) = \frac{1}{2}$.

In the definition of M_μ for a limit ordinal μ, it was assumed that if $V[M_\nu](P) = 1$ (or 0) for some $\nu < \mu$, then $V[M_\tau](P) = 1$ (or 0) for all τ such that $\nu \leqslant \tau < \mu$. The construction of M_μ needs to be coupled with lemma 2 (below) so that when M_μ is formed the assumption above will be satisfied. That is, lemma 2 is proved for each structure M_μ as it is constructed.

I will give some examples in M_1, M_2 and M_3. Since standard wffs include the propositional constants 0 and 1, by definition of M_1, $V[M_1](a\,\varepsilon\{x : 1\}) = 1$ and $V[M_1](a\,\varepsilon\{x : 0\}) = 0$. Let $\{x : 1\}$ be called $\cup$ and $\{x : 0\}$ be called $\vee$. Hence $V[M_1](\vee\,\varepsilon\,\cup) = 1$ and $V[M_1](\cup\,\varepsilon\,\cup) = 1$. Using these two we can construct wffs taking values 1 or 0 in M_2. For example,

$$V[M_2](\cup\,\varepsilon\{x : \vee\,\varepsilon x\}) = 1 = V[M_2](\vee\,\varepsilon\{x : \sim x\,\varepsilon\, x\})$$
$$V[M_2](\cup\,\varepsilon\{x : x\,\varepsilon\, x\}) = 1 = V[M_2](\vee\,\varepsilon\{x : \sim\cup\,\varepsilon\, x\})$$

Let $\{c\}$ be $\{x : (Ay)(\sim y\,\varepsilon\, x \vee y\,\varepsilon\, c \,\&.\sim y\,\varepsilon\, c \vee y\,\varepsilon\, x\}$. Then

$$V[M_2](\vee\,\varepsilon\{\vee\}) = 1 = V[M_2](\cup\,\varepsilon\{\cup\})$$
$$V[M_2](\cup\,\varepsilon\{\vee\}) = 0 = V[M_2](\vee\,\varepsilon\{\cup\})$$

Some examples in M_3 are the following:

$$V[M_3](\{\vee\}\,\varepsilon\{x : \vee\,\varepsilon\, x\}) = 1 = V[M_3](\{\cup\}\,\varepsilon\{x : \cup\,\varepsilon\, x\})$$
$$V[M_3](\{x : \vee\,\varepsilon\, x\}\,\varepsilon\{x : \cup\,\varepsilon\, x\}) = 1 = V[M_3](\{\vee\}\,\varepsilon\{x : \sim x\,\varepsilon\, x\})$$

Lemma 2 *$M_\nu \leqslant M_\mu$, for all $\nu \leqslant \mu$.*

Proof. By transfinite induction on μ. The induction hypothesis: $M_\nu \leqslant M_\tau$ for all $\nu \leqslant \tau$, for all $\tau < \mu$.

(i) $\mu = 0$: $M_0 \leqslant M_0$.

(ii) μ is a successor ordinal: Let $V[M_\nu](a\,\varepsilon\{x : P\}) = 1$. There is a $\eta < \nu$ such that $V[M_\eta](P(a)) = 1$ by the method of construction of the structures. Since $\eta \leqslant \mu - 1$, $M_\eta \leqslant M_{\mu-1}$ by the induction hypothesis. Hence $V[M_{\mu-1}](P(a)) = 1$. By the construction of M_μ, $V[M_\mu](a\,\varepsilon\{x : P\}) = 1$. Similarly, if $V[M_\nu](a\,\varepsilon\{x : P\}) = 0$, then $V[M_\mu](a\,\varepsilon\{x : P\}) = 0$.

(iii) μ is a limit ordinal: Let $\nu < \mu$. Let $V[M_\nu](a\,\varepsilon\{x : P\}) = 1$ (=0). Then $V[M_\mu](a\,\varepsilon\{x : P\}) = 1$ (=0) by definition of M_μ. Let $\nu = \mu$. Then $M_\nu \leqslant M_\mu$.

Lemma 3 *There is an ordinal λ of the second number class such that $M_\lambda = M_{\lambda+1}$.*

Proof. The increasing chain of structures $M_0 \leqslant M_1 \leqslant \ldots \leqslant M_\mu \leqslant \ldots$ can be regarded as two increasing chains of subsets of the denumerable set of all atomic wffs of the form $a\,\varepsilon\, b$. One chain is of those atomic wffs taking the value 1 and the other is of those taking the value 0. If $M_\nu = M_{\nu+1}$, then $M_\nu = M_\mu$ for all ordinals μ, $\nu \leqslant \mu$, since, by the method of construction, there is no way of changing the values of any atomic wffs. There is a denumerable set of ordinals μ such that $M_\mu \neq M_{\mu+1}$. But the set of all ordinals of the second number class is non-denumerable and hence for some λ in this class, $M_\lambda = M_{\lambda+1}$.

Theorem 1 *$v \varepsilon \{x : P\} \leftrightarrow P(v)$ is valid in M_λ, for all standard wffs P.*

Proof. Let $V[M_\lambda](a \varepsilon \{x : P\}) = 1$. Let ν be the least ordinal such that $V[M_\nu](a \varepsilon \{x : P\}) = 1$. ν is a successor ordinal. Hence $V[M_{\nu-1}](P(a)) = 1$. Since $\nu - 1 \leqslant \lambda$, $M_{\nu-1} \leqslant M_\lambda$, by lemma 2. Hence $V[M_\lambda](P(a)) = 1$ since P is standard, by lemma 1. Similarly, if $V[M_\lambda](a \varepsilon \{x : P\}) = 0$, then we have that $V[M_\lambda](P(a)) = 0$. Let $V[M_\lambda](P(a)) = 1$, then $V[M_{\lambda+1}](a \varepsilon \{x : P\}) = 1$. Since $M_\lambda = M_{\lambda+1}$, $V[M_\lambda](a \varepsilon \{x : P\}) = 1$. Similarly, if $V[M_\lambda](P(a)) = 0$, then $V[M_\lambda](a \varepsilon \{x : P\}) = 0$.

Theorem 2 *The Abstraction Axiom (*) is valid in M_λ.*

Proof. By Theorem 1, for any standard wff P, $v \varepsilon \{x : P\} \leftrightarrow P(v)$ is valid in M_λ. Hence, $(Sy)(Ax)(x \varepsilon y \leftrightarrow P(x, z_1, \ldots, z_n))$ is valid in M_λ, for all wffs P which are propositional constants or constructed from atomic wffs of the form $x \varepsilon y$ by using only $\sim$, &, A.

2. The next task is to prove that the Axiom of Extensionality is valid in M_λ.

Let P be a standard wff such that $V[M_\lambda](P) = 1$ or 0. Let $\nu(P)$ be the least ordinal such that $V[M_{\nu(P)}](P) = 1$ or 0. Form the set of all substitution instances of all the atomic wffs of P which take the value 1 or 0 in $M_{\nu(P)}$. Call this *the dependent set of P*, D(P).

Lemma 4 *Let $P(a)$ be a standard wff such that $V[M_\lambda](P(a)) = 1$ or 0. If, for each $Q(a) \varepsilon$ D$(P(a))$, $V[M_\lambda](Q(b)) = V[M_\lambda](Q(a))$, then $V[M_\lambda](P(b)) = V[M_\lambda](P(a))$.*

Proof. By induction on the wff evaluation procedure. Let $P(a)$ be an atomic wff such that $V[M_\lambda](P(a)) = 1$ or 0. Then D$(P(a)) = \{P(a)\}$. Hence $V[M_\lambda](P(b)) = V[M_\lambda](P(a))$.

(i) Let $P(a)$ be $\sim R(a)$. Since D$(\sim R(a))$ = D$(R(a))$, for each $Q(a) \varepsilon$ D$(R(a))$, $V[M_\lambda](Q(b)) = V[M_\lambda](Q(a))$. By the induction hypothesis, $V[M_\lambda](R(b)) = V[M_\lambda](R(a))$. Hence $V[M_\lambda](P(b)) = V[M_\lambda](P(a))$.

(ii) Let $P(a)$ be $R(a)$ & $S(a)$ and $V[M_\lambda](R(a) \,\&\, S(a)) = 1$. Then $V[M_\lambda](R(a)) = 1$ and $V[M_\lambda](S(a)) = 1$. Since $\nu(R(a)) \leqslant \nu(R(a) \,\&\, S(a))$, D$(R(a)) \subseteq$ D$(R(a) \,\&\, S(a))$. Hence, for each $Q(a) \varepsilon$ D$(R(a))$, $V[M_\lambda](Q(b)) = V[M_\lambda](Q(a))$. By the induction hypothesis, $V[M_\lambda](R(b)) = V[M_\lambda](R(a))$. Similarly, $V[M_\lambda](S(b)) = V[M_\lambda](S(a))$. Hence $V[M_\lambda](P(b)) = V[M_\lambda](P(a))$.

(iii) Let $P(a)$ be $R(a)$ & $S(a)$ and $V[M_\lambda](R(a) \,\&\, S(a)) = 0$. Then, as above, $V[M_\lambda](P(b)) = V[M_\lambda](P(a))$.

(iv) Let $P(a)$ be $(Az)\,R(a, z)$ and $V[M_\lambda]((Az)\,R(a, z)) = 1$. Then $V[M_\lambda](R(a, z)) = 1$, for all z. Since $\nu(R(a, z)) \leqslant \nu((Az)\,R(a, z))$ for all z, then D$(R(a, z)) \subseteq$ D$((Az)\,R(a, z))$ for all z. Hence, for each $Q(a) \varepsilon$ D$(R(a, z))$, $V[M_\lambda](Q(b)) = V[M_\lambda](Q(a))$. By the induction hypothesis, $V[M_\lambda](R(b, z)) = V[M_\lambda](R(a, z))$. Since this holds for all z, $V[M_\lambda](P(b)) = V[M_\lambda](P(a))$.

(v) Let $P(a)$ be $(Az)\,R(a,z)$ and $V[M_\lambda]((Az)\,R(a,z)) = 0$. Then, as above, $V[M_\lambda](P(b)) = V[M_\lambda](P(a))$.

Let P be an atomic wff (not 1 or 0) such that $V[M_\lambda](P) = 1$ or 0. Define the *corresponding standard wff of P*, $\mathrm{C}(P)$, as follows: Let P have the form $a\,\varepsilon\{x : Q(x)\}$. Then $\mathrm{C}(P)$ is $Q(a)$.

Let P be a standard wff such that $V[M_\lambda](P) = 1$ or 0. Let P have dependent set, $\mathrm{D}(P)$. We define a *general dependent set of P*, $\mathrm{GD}(P)$, as follows:

(i) The dependent set $\mathrm{D}(P)$ of P is a $\mathrm{GD}(P)$.

(ii) If $V[M_\lambda](R) = 1$ or 0 and R is an atomic wff (not 1 or 0), then $\mathrm{D}(\mathrm{C}(R))$ is a $\mathrm{GD}(R)$.

(iii) Let $S \subseteq \mathrm{GD}(P)$, then $(\mathrm{GD}(P) \cap \overline{S}) \cup \bigcup_{Q\varepsilon S} \mathrm{GD}(Q)$ is a $\mathrm{GD}(P)$.

This assumes $V[M_\lambda](Q) = 1$ or 0, for all $Q\,\varepsilon\,S$. Note that lemma 5 (below) should be coupled with the definition of a general dependent set so that the assumption can be made before the construction of the general dependent sets $\mathrm{GD}(Q)$.

Lemma 5 *Let P be a standard wff such that* $V[M_\lambda](P) = 1$ *or* 0. *If* $Q\,\varepsilon\,\mathrm{GD}(P)$, *then,* $V[M_\lambda](Q) = 1$ *or* 0.

Proof. By induction on the stages of construction of $\mathrm{GD}(P)$ for all standard wffs such that $V[M_\lambda](P) = 1$ or 0.

(i) By definition of $\mathrm{D}(P)$, if $Q\,\varepsilon\,\mathrm{D}(P)$ then $V[M_\lambda](Q) = 1$ or 0.

(ii) If $Q\,\varepsilon\,\mathrm{D}(\mathrm{C}(R))$, where R is an atomic wff (not 1 or 0) and $V[M_\lambda](R) =$ 1 or 0, then $V[M_\lambda](Q) = 1$ or 0.

(iii) Let $S \subseteq \mathrm{GD}(P)$ and $T\,\varepsilon\,(\mathrm{GD}(P) \cap \overline{S}) \cup \bigcup_{Q\varepsilon S} \mathrm{GD}(Q)$. If $T\,\varepsilon\,\mathrm{GD}(Q)$, for some $Q\,\varepsilon\,S$, then by the induction hypothesis for $\mathrm{GD}(Q)$, $V[M_\lambda](T) = 1$ or 0. If $T\,\varepsilon\,\mathrm{GD}(P) \cap \overline{S}$, then, by the induction hypothesis for $\mathrm{GD}(P)$, $V[M_\lambda](T) = 1$ or 0.

Lemma 6 *Let P be an atomic wff such that* $V[M_\lambda](P) = 1$ *or* 0. *If* $\mathrm{GD}(P)$ *is not* $\mathrm{D}(P)$ *then, for each* $Q\,\varepsilon\,\mathrm{GD}(P)$, $V[M_{\nu(P-1)}](Q) = 1$ *or* 0.

Proof. By transfinite induction on the ordinals $\nu(P)$. The induction hypothesis is that the lemma holds for all atomic wffs Q such that $\nu(Q) < \nu(P)$.

(i) $\nu(P) = 0$: P is 1 or 0. The only $\mathrm{GD}(P)$ is of the form $\mathrm{D}(P)$. Hence the lemma holds vacuously.

(ii) $\nu(P)$ is a successor ordinal: Use induction on the stages of construction of $\mathrm{GD}(P)$.

(ia) $\mathrm{D}(P)$ is not used as a general dependent set in this lemma.

(iia) If $V[M_\lambda](R) = 1$ or 0, R is an atomic wff (not 1 or 0) and if $Q\,\varepsilon\,\mathrm{D}(\mathrm{C}(R))$, then $V[M_{\nu(R-1)}](Q) = 1$ or 0. In the process of construction of general dependent sets of P, R is either P itself or is a member of a $\mathrm{GD}(P)$. If R is P itself, then $V[M_{\nu(P-1)}](Q) = 1$ or 0. If R is a member of a $\mathrm{GD}(P)$, then, by the induction hypothesis, $V[M_{\nu(P-1)}](R) = 1$ or 0 or

$V[M_{\nu(P)}](R) = 1$ or 0, the latter being the case when R is a member of GD (P). Hence $\nu(R) \leqslant \nu(P)$ and if $Q \,\varepsilon\, \mathrm{D}(\mathrm{C}(R))$ then $V[M_{\nu(P-1)}](Q) = 1$ or 0.

(iii) Let $S \subseteq \mathrm{GD}(P)$. By the induction hypothesis for $\mathrm{GD}(P)$, $V[M_{\nu(P-1)}](Q) = 1$ or 0, for all $Q \,\varepsilon\, S$. By the induction hypothesis for the ordinals, the lemma holds for any $\mathrm{GD}(Q)$ except for $\mathrm{D}(Q)$. Let $T \,\varepsilon\, (\mathrm{GD}(P) \cap \overline{S}) \cup \bigcup_{Q \varepsilon S} \mathrm{GD}(Q)$. If $T \,\varepsilon\, \mathrm{GD}(Q)(\mathrm{GD}(Q) \neq \mathrm{D}(Q))$, for some $Q \,\varepsilon\, S$, then $V[M_{\nu(P-1)}](T) = 1$ or 0. If $T \,\varepsilon\, \mathrm{GD}(Q)$, where $\mathrm{GD}(Q)$ is $\mathrm{D}(Q)$, for some $Q \,\varepsilon\, S$, then, since $\mathrm{D}(Q)$ is $\{Q\}$, $T \,\varepsilon\, \mathrm{GD}(P)$. By the induction hypothesis for $\mathrm{GD}(P)$, the lemma holds. If $T \,\varepsilon\, \mathrm{GD}(P) \cap \overline{S}$, then, again, the lemma holds.

Lemma 7 *Let $P(a)$ be a standard wff such that $V[M_\lambda](P(a)) = 1$ or 0. Consider any general dependent set D' of $P(a)$, such that, in the process of construction,* (ii) *is not applied to any atomic wff of form $c \,\varepsilon\, a$. If, for all $Q(a) \,\varepsilon\, D'$, $V[M_\lambda](Q(b)) = V[M_\lambda](Q(a))$, then $V[M_\lambda](P(b)) = V[M_\lambda](P(a))$.*

Proof. By induction on the stages of construction of general dependent sets of all standard wffs $P(a)$ such that $V[M_\lambda](P(a)) = 1$ or 0, and such that (ii) is not applied to any atomic wff of form $c \,\varepsilon\, a$.

(i) Let $D' = \mathrm{D}(P(a))$. Then, by lemma 4, the lemma holds.

(ii) Let $D' = \mathrm{D}(\mathrm{C}(P(a)))$, where $P(a)$ is an atomic wff. We need only consider $P(a)$ in the form $a \,\varepsilon\, \{x : Q\}$. Hence $\mathrm{C}(P(a))$ is $Q(a)$. $V[M_\lambda](Q(a)) = 1$ or 0. By the lemma condition, if $R(a) \,\varepsilon\, \mathrm{D}(\mathrm{C}(P(a)))$ then $V[M_\lambda](R(b)) = V[M_\lambda](R(a))$. Hence, by lemma 4, $V[M_\lambda](Q(b)) = V[M_\lambda](Q(a))$. Therefore, $V[M_\lambda](b \,\varepsilon\, \{x : Q\}) = V[M_\lambda](a \,\varepsilon\, \{x : Q\})$. Hence $V[M_\lambda](P(b)) = V[M_\lambda](P(a))$.

(iii) Let $S \subseteq D'$ and for each $Q(a) \,\varepsilon\, S$, let the lemma hold for D' and the GD $(Q(a))$. By the condition of the lemma, for all $T(a) \,\varepsilon\, (D' \cap \overline{S}) \cup \bigcup_{Q(a) \varepsilon S} \mathrm{GD}(Q(a))$, $V[M_\lambda](T(b)) = V[M_\lambda](T(a))$. Since $\mathrm{GD}(Q(a)) \subseteq (D' \cap \overline{S}) \cup \bigcup_{Q(a) \varepsilon S} \mathrm{GD}(Q(a))$, for all $Q(a) \,\varepsilon\, S$, by induction hypothesis, $V[M_\lambda](Q(b)) = V[M_\lambda](Q(a))$, for all $Q(a) \,\varepsilon\, S$. Also, for all $T(a) \,\varepsilon\, D' \cap \overline{S}$, $V[M_\lambda](T(b)) = V[M_\lambda](T(a))$. Hence, if $U(a) \,\varepsilon\, D'$, $V[M_\lambda](U(b)) = V[M_\lambda](U(a))$. By induction hypothesis for D', $V[M_\lambda](P(b)) = V[M_\lambda]P(a)$.

Lemma 8 *If $V[M_\lambda](a \,\varepsilon\, c) = 1$ or 0 then $a \,\varepsilon\, c$ has a general dependent set without any wffs of the form $a \,\varepsilon\, b$ for any b, except a. The general dependent sets so constructed are such that* (ii) *is not applied to any atomic wffs of form $a' \,\varepsilon\, a$.*

Proof. Let the wff $a \,\varepsilon\, c$ be W. The proof is by transfinite induction on $\nu(W)$. The induction hypothesis is that the lemma holds for all wffs $a \,\varepsilon\, d$ (call it X) such that $\nu(X) < \nu(W)$.

(i) $\nu(W) = 1$: Let $V[M_1](a \,\varepsilon\, c) = 1$ or 0. Let a and c be different. Then $V[M_0](\mathrm{C}(a \,\varepsilon\, c))$ is 1 or 0. Hence $\mathrm{D}(\mathrm{C}(a \,\varepsilon\, c)) = \{1\}$ or $\{0\}$. This satisfies the lemma. If a is c, then $\mathrm{D}(a \,\varepsilon\, c) = \{a \,\varepsilon\, c\}$ satisfies the lemma.

(ii) $\nu(W)$ is a successor ordinal >1: Let $V[M_{\nu(W)}](a \,\varepsilon\, c) = 1$ or 0. If a is c, then $\mathrm{D}(a \,\varepsilon\, c) = \{a \,\varepsilon\, c\}$ satisfies the lemma. If a and c are different, $V[M_{\nu(W-1)}](Z(a)) = 1$ or 0, where $Z(a)$ is $\mathrm{C}(W)$. Hence, $\mathrm{D}(Z(a))$ is a general dependent set of W and has a subset S of all atomic wffs of the form $a \varepsilon b$, where b is not a. For all Q, if $Q \varepsilon S$, then $V[M_{\nu(W-1)}](Q) = 1$ or 0. Hence, by induction hypothesis, all these wffs $Q \varepsilon S$ have general dependent sets $\mathrm{GD}(Q)$ without wffs of the above form. Form the set $(\mathrm{D}(Z(a)) \cap \overline{S}) \cup \bigcup_{Q \varepsilon S} \mathrm{GD}(Q)$, which has no atomic wffs of the above form. This is a general dependent set of W which satisfies the lemma.

Lemma 9 *If $a \varepsilon c \leftrightarrow a \varepsilon d$ has value 1 in M_λ, for all a, then $c \varepsilon c \leftrightarrow d \varepsilon d$ has the value 1 in M_λ.*

Proof. Call $c \varepsilon c$, W. Let $V[M_\lambda](W) = 1$ or 0. By lemma 8, W has a general dependent set D' without atomic wffs of certain forms and constructed in a certain way. For the sake of lemma 8 the right hand c of $c \varepsilon c$ is regarded as different from the left hand c. So (ii) is applied in forming a general dependent set of $c \varepsilon c$, but apart from this one instance all the usual conditions apply. By lemma 6, all members of D' have the value 1 or 0 in $M_{\nu(W-1)}$, since, by lemma 8, D' can be constructed so that it is not $\mathrm{D}(W)$. Hence W is not a member of D'. Hence D' has atomic wffs containing c, only of the form $a \varepsilon c$ or not at all. By condition of the lemma, if $Q(c) \varepsilon D'$ then $V[M_\lambda](Q(d)) = V[M_\lambda](Q(c))$. By lemma 7, $V[M_\lambda](d \varepsilon c) = V[M_\lambda](c \varepsilon c)$. Since (ii) was applied to $c \varepsilon c$ in forming the general dependent set D', the substitution of d for c occurs only in the left hand c of $c \varepsilon c$. By the condition of the lemma, $V[M_\lambda](d \varepsilon d) = V[M_\lambda](d \varepsilon c)$ and hence $V[M_\lambda](d \varepsilon d) = V[M_\lambda](c \varepsilon c)$. Similarly by letting $d \varepsilon d$ be W and substituting c for d, $V[M_\lambda](c \varepsilon c) = V[M_\lambda](d \varepsilon d)$. Hence the lemma is proved.

Theorem 3 *The Axiom of Extensionality is valid in M_λ.*

Proof. The Axiom of Extensionality is the following:

$$(Av)(v \varepsilon x \leftrightarrow v \varepsilon y) \supset (Az)(x \varepsilon z \leftrightarrow y \varepsilon z)$$

We will prove: if $v \varepsilon c \leftrightarrow v \varepsilon d$ is valid in M_λ, then $c \varepsilon z \leftrightarrow d \varepsilon z$ is valid in M_λ. Let $V[M_\lambda](c \varepsilon c') = 1$ or 0. By lemma 8, $c \varepsilon c'$ has a general dependent set D' without any wffs of the form $c \varepsilon b$, for any b except c. Hence the only occurrences of c in D' are of the forms $a \varepsilon c$ (a is not c) and $c \varepsilon c$. Because of the condition of the theorem and because of lemma 9, if $Q(c) \varepsilon D'$, then $V[M_\lambda](Q(d)) = V[M_\lambda](Q(c))$. By lemma 7, $V[M_\lambda](d \varepsilon c') = V[M_\lambda](c \varepsilon c')$. Hence $c \varepsilon z \leftrightarrow d \varepsilon z$ is valid in M_λ and the theorem is shown.

REFERENCES

[1] Skolem, Th., "A set theory based on a certain three-valued logic," *Mathematica Scandinavia*, vol. 8 (1960), pp. 127-136.

[2] Skolem, Th., "Studies on the axiom of comprehension," *Notre Dame Journal of Formal Logic*, vol. 4 (1963), pp. 162-170.

[3] Gilmore, P. C., "The consistency of partial set theory without extensionality," *IBM Research Report*, RC 1973, Dec. 21, 1967.

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