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A PROPER SUBSYSTEM OF S4.04

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It is self-evident that, in the field of modal system S4.04 which has been established in [8]*, the following formula

Ł1 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLppCLMLpp$

is easily provable. It will be proved in this note:

1. that the addition of **Ł1**, as a new axiom, to S4 generates a system, called S4.02, which is a proper extension of S4 and at the same time is properly contained in each of the systems S4.04 and S4.1,
2. that S4.02 neither contains the systems S4.2, S4.3 and S4.3.2 nor is contained in any one of them,
3. and that the addition of **Ł1**, as a new axiom, to each of the systems K1 and Z1 generates the systems which are inferentially equivalent to K1.1 and Z3 respectively.

Proof:

1 Each of the matrices $\mathfrak{M}5$, $\mathfrak{M}7$, $\mathfrak{M}9$ and $\mathfrak{M}11$ which are given in [3], p. 350, verifies S4, but:

(i) $\mathfrak{M}5$ verifies **Ł1**, but falsifies **G1**, *cf.* [2], section **4.2**. Hence, $\mathfrak{M}5$ also falsifies **D1** and **F1**.

(ii) $\mathfrak{M}7$ verifies **F1** and **K1**, but falsifies **Ł1** for $p/3$: $= \mathfrak{C}\mathfrak{C}\mathfrak{C}3L33CLML33 = \mathfrak{C}\mathfrak{C}LC343CLM43 = \mathfrak{C}\mathfrak{C}L23CL13 = \mathfrak{C}LC43C13 = \mathfrak{C}L13 = LC13 = L3 = 4$.

(iii) $\mathfrak{M}9$ verifies **Ł1**, but falsifies **L1**, *cf.* [2], section **4.4**.

(iv) $\mathfrak{M}11$ verifies **Ł1**, but falsifies **N1**, *cf.* [5], p. 383.

2 It follows immediately from the considerations which are given in section **1** that:

*An acquaintance with the papers which are cited in this note and, especially, with the enumeration of the extensions of S4 and their proper axioms given in [3], pp. 247-350, in [4], and in [2], is presupposed.

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(A) System S4.02 is a proper extension of S4 and is properly contained in each of the systems S4.04 and S4.1, since, obviously, in the field of S4, the proper axiom of S4.1, i.e., **N1** implies **Ł1**,

and that

(B) System S4.02 neither contains the systems S4.3.2, S4.3 and S4.2, nor is contained in any one of them.

3 It follows from section **1**, point (ii) that **Ł1** is not a consequence of the system K1. On the other hand, since, in the field of S4, the proper axiom of K1.1, i.e., **J1** implies **Ł1** clearly, S4.02 is contained in K1.1. Hence, in order to prove that {K1; **Ł1**} ⇄ {S4.02; **K1**} ⇄ {S4; **K1**; **Ł1**} ⇄ {S4; **J1**} ⇄ {K1.1}, it suffices to show that **J1** is a consequence of {K1; **Ł1**}. Therefore, let us assume **Ł1** and the system K1, i.e., S4 and the axiom **K1**. Then:

K4	*LMLCpLp*	[S4; **K1**; *cf.* [3], p. 349, and [6], pp. 77-78, section **5**]
Z1	*ℭℭℭCpLpℭpLpCpLpCpLp*	[**Ł1**, *p/CpLp*; **K4**; S1°]
Z2	*ℭℭℭpLpLpℭℭCpLpℭpLpCpLp*	[S4; *cf.* [4], section **1.2.2**, formula *Z20*]
Z3	*ℭℭℭpLpLpCpLp*	[*Z2*; *Z1*; S1°]
Z4	*ℭℭLpqℭLpLq*	[S4°]
Z5	*ℭℭℭpLppℭpLp*	[*Z4*, *p/CpLp*, *q/p*; *Z3*; S2°]
Z6	*ℭℭℭLpqLpℭℭLpqq*	[S4]
J1	*ℭℭℭpLppp*	[*Z6*, *p/CpLp*, *q/p*; *Z5*]

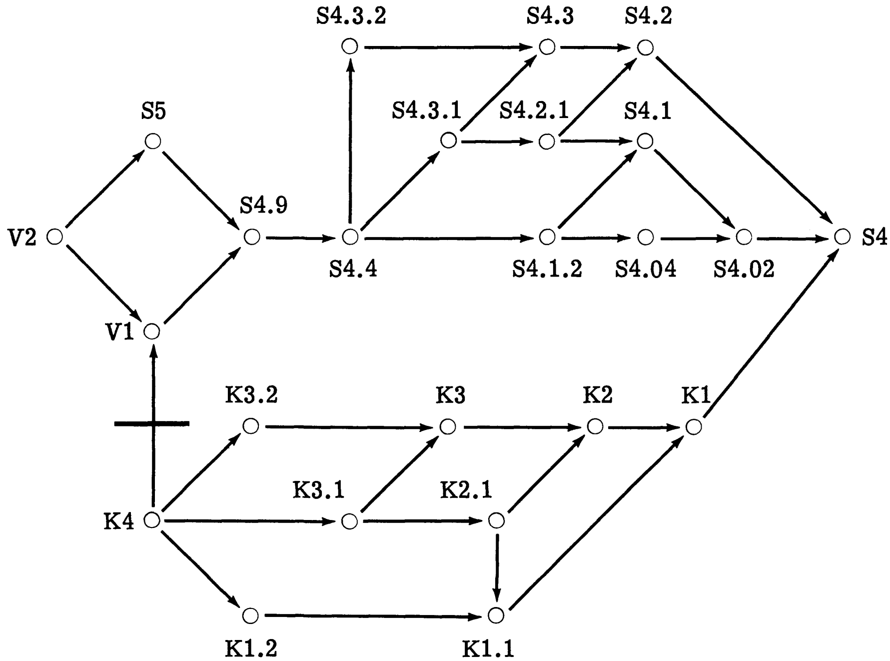
Thus, {K1; **Ł1**} ⇄ {S4; **K1**; **Ł1**} ⇄ {S4; **J1**} ⇄ {K1.1}.

4 Since **Ł1** is an obvious consequence of the system Z3, *cf.* [2], in order to prove that {Z3} ⇄ {S4; **N1**; **Z1**} ⇄ {S4; **Ł1**; **Z1**} ⇄ {S4.02; **Z1**}, it suffices to show that **N1** is a consequence of {S4.02; **Z1**}. Hence, let us assume **Z1** and the system S4.02, i.e., S4 and the axiom **Ł1**. Then:

Z1	*ℭMLCpLpLMCMpLLp*	[**Z1**; S4; *cf.* [2], section **1.1**, formula *Z9*]
Z2	*ℭLMCMpLqLMLCpq*	[S2°; *cf.* [2], section **1.1**, formula *Z10*]
Z3	*ℭMLCpLpLMLCpLp*	[*Z1*; *Z2*, *q/p*; S1°]
N4	*ℭℭℭCpLpℭpLpCpLpCMLCpLpCpLp*	[*Z3*; **Ł1**; S3°]

Since it has been established in [4], section **3**, that, in the field of S4, **N4** can be accepted as the proper axiom of the system S4.1, the proof is complete. Thus, {S4.02; **Z1**} ⇄ {S4; **N1**; **Z1**} ⇄ {Z3}.

5 The deduction presented above shows that the system S4.02 is a fullfledged member of the systems which are contained between S4 and S4.4, and, on the other hand, that the addition of **Ł1**, as a new axiom, to any system belonging to the *K* family or to the *Z* systems does not generate a new system. The investigations concerning the extensions of the system S4 which are given in [1], [7], [4], [2] and in this note allow us to establish the diagram given on p. 383 in which, however, the *Z* systems, *cf.* [7], pp. 354-356, and [2], are omitted. This diagram visualizes the relations occurring among the systems under consideration. In the literature we can find the proofs that each arrow which occurs between two represented in



Diagram

the diagram systems shows that the tail system is a proper extension of the edge system. The bold horizontal line occurring in the diagram indicates that, although system V1 is a proper subsystem of K4, it really does not belong to the family $\mathcal{K}$ of non-Lewis modal systems.

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