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A CLASS OF MODELS FOR INTERMEDIATE LOGICS

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Let α be an ordinal, $c(\alpha)$ its cardinality and B a $c(\alpha)$-field of sets, with union $+$, intersection $\cdot$ and complementation $'$. By $\mathsf{L}_\alpha(B)$ we denote the set of weakly decreasing functions from α into B. A lattice structure is defined on $\mathsf{L}_\alpha(B)$ by putting

$$(f+g)(\kappa) = f(\kappa) + g(\kappa)$$
$$(f \cdot g)(\kappa) = f(\kappa) \cdot g(\kappa)$$

for all $\kappa \leq \alpha$ and $f, g \in \mathsf{L}_\alpha(B)$. There is a zero 0 in $\mathsf{L}_\alpha(B)$ and a one 1. As is well-known $\mathsf{L}_\alpha(B)$ is not complemented for $\alpha > 1$. However a relatively pseudocomplemented structure can be defined on $\mathsf{L}_\alpha(B)$.

Definition: For $f, g \in \mathsf{L}_\alpha(B)$ let $f \rightarrow g$ be defined by

$$(f \rightarrow g)(\kappa) = \sum_{\rho \leq \kappa} f(\rho)' \cdot \prod_{\sigma < \rho} g(\sigma) + g(\kappa)$$

Remarks:

1. The void product $\prod_{\sigma < 1} g(\sigma)$ is put equal to $1 \in B$.
2. Notice the following recursive relation

$$(f \rightarrow g)(\kappa + 1) = f(\kappa + 1)' \cdot (f \rightarrow g)(\kappa) + g(\kappa + 1)\ .$$

Theorem 1: *If $f, g \in \mathsf{L}_\alpha(B)$, then $f \rightarrow g \in \mathsf{L}_\alpha(B)$.*

Proof. By the assumed nature of B the $(f \rightarrow g)(\kappa)$ are in B for all $\kappa \leq \alpha$. If $\tau < \kappa$ then

$$\begin{aligned}(f \rightarrow g)(\kappa) &= \sum_{\rho \leq \kappa} f(\rho)' \cdot \prod_{\sigma < \rho} g(\sigma) + g(\kappa)\\ &= \sum_{\rho \leq \tau} f(\rho)' \cdot \prod_{\sigma < \rho} g(\sigma) + \sum_{\tau < \rho \leq \kappa} f(\rho)' \cdot \prod_{\sigma < \rho} g(\sigma) + g(\kappa)\\ &\leq \sum_{\rho \leq \tau} f(\rho)' \cdot \prod_{\sigma < \rho} g(\sigma) + g(\tau)\\ &= (f \rightarrow g)(\tau)\end{aligned}$$

i.e. $f \rightarrow g$ is weakly decreasing.

Theorem 2: *$\langle \mathsf{L}_\alpha(B), +, \cdot, \rightarrow \rangle$ is relatively pseudocomplemented.*

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Proof. First of all $f \cdot (f \to g) \leq g$ for

$$f \cdot (f \to g)(\kappa) = f(\kappa) \cdot \sum_{\rho \leq \kappa} f(\rho)' \cdot \prod_{\sigma < \rho} g(\sigma) + f(\kappa)\, g(\kappa) \ .$$

If $\rho \leq \kappa$ then $f(\rho) \geq f(\kappa)$, so $f(\rho)' \leq f(\kappa)'$, consequently $f \cdot (f \to g)(\kappa) = f(\kappa) \cdot g(\kappa) \leq g(\kappa)$ for all $\kappa \leq \alpha$. Next suppose $f \cdot h \leq g$, then

$$h(\kappa) \leq \prod_{\lambda \leq \kappa} (g(\lambda) + f(\lambda)') \text{ for all } \kappa \leq \alpha \ .$$

Let $x \in h(\kappa)$, then consider

$$\mathsf{A}(x) = \{\lambda;\ \lambda \leq \kappa \ \&\ x \in f(\lambda)'\}$$

If $\mathsf{A}(x) = \phi$, then $x \in f(\lambda)$ for all $\lambda \leq \kappa$, so $x \in f(\kappa)$. Since $x \in g(\kappa) + f(\kappa)'$, it follows $x \in g(\kappa)$, so $x \in (f \to g)(\kappa)$. If $\mathsf{A}(x) \neq \phi$, then there is a least λ_0 in $\mathsf{A}(x)$, so $x \in f(\lambda_0)'$, whereas $x \in f(\lambda)$ for all $\lambda < \lambda_0$. Because $x \in g(\lambda) + f(\lambda)'$ for these λ, it follows $x \in g(\lambda)$ for all $\lambda < \lambda_0$, hence

$$x \in \prod_{\sigma < \lambda_0} g(\sigma)$$

and consequently

$$x \in f(\lambda_0)' \cdot \prod_{\sigma < \lambda_0} g(\sigma) \ ,$$

so $x \in (f \to g)(\kappa)$. So, if $f \cdot h \leq g$, then $h \leq f \to g$, which completes the proof that $\mathsf{L}_\alpha(B)$ is relatively pseudocomplemented.

Remarks:

3. The pseudocomplement of $f \in \mathsf{L}_\alpha(B)$ assumes a very simple form:

$$(f \to 0)(\kappa) = f(1)' \ .$$

Notice, that $((f \to 0) \to 0)(\kappa) = f(1) \geq f(\kappa)$, hence reciprocity of complement does not occur in general (i.e. for $\alpha > 1$).

4. If $f = 1$ and $f \to g = 1$, then also $g = 1$, so every $\mathsf{L}_\alpha(B)$ is a model for intuitionistic logic, with meet, join, relative pseudocomplement and pseudocomplement as interpretations of conjunction, disjunction, implication and negation respectively.

Let $\mathsf{D}(f,g)$ stand for $(f \to g) + (g \to f)$.

Theorem 3: $\mathsf{D}(f,g) = 1$ *in* $\mathsf{L}_\alpha(B)$.

Proof. If $\mathsf{D}(f,g) \neq 1$ then there is an x and a $\kappa \leq \alpha$ such that $x \in \mathsf{D}(f,g)(\kappa)'$. Now

$$\mathsf{D}(f,g)(\kappa)' = \prod_{\rho<\kappa} (f(\rho) + \sum_{\sigma<\rho} g(\sigma)') \cdot \sum_{\sigma<\kappa} g(\sigma)' \cdot \prod_{\rho<\kappa} (g(\rho) + \sum_{\sigma<\rho} f(\sigma)') \cdot \sum_{\sigma<\kappa} f(\sigma)' \ .$$

Let $\sigma(x)$ be the least σ such that $x \in g(\sigma)'$ and $\tau(x)$ the least σ such that $x \in f(\sigma)'$. Then since $x \in \mathsf{D}(f,g)(\kappa)'$ it follows

$$x \in f(\sigma(x)) + \sum_{\sigma<\sigma(x)} g(\sigma)' \ ,$$

hence $x \in f(\sigma(x))$, and consequently $\tau(x) > \sigma(x)$. On the other hand

$$x \in g(\tau(x)) + \sum_{\sigma<\tau(x)} f(\sigma)' \ ,$$

so $x \in g(\tau(x))$, consequently $\sigma(x) > \tau(x)$. Hence the assumption: for some $\mathsf{D}(f,g)(\kappa) \neq 1$ is contradictory, so $\mathsf{D}(f,g)(\kappa) = 1$ for all κ.

This theorem shows that the $\mathsf{L}_\alpha(B)$ are models of intermediate logics. A particular case arises when $B = \{0, 1\}$, because in that case $\mathsf{L}_\alpha(B)$ is an $\alpha + 1$ chain, in which implication has the form

$$f \to g = \begin{cases} 1 \text{ if } f \leq g \\ g \text{ if } f > g \end{cases}$$

This can be seen as follows: if $(f \to g) \neq 1$, then there is an x such that

$$x \in (f \to g)(\kappa)' \text{ for some } \kappa \leq \alpha .$$

So

$$x \in g(\kappa)' \cdot \prod_{\rho \leq \kappa} (f(\rho) + \sum_{\sigma < \rho} g(\sigma)') .$$

If $\sigma(x)$ is the least σ such that $x \in g(\sigma)'$, then it follows $x \in f(\sigma(x))$. However also $x \in g(\sigma(x))'$, hence if $f \leq g$, we have also $x \in f(\sigma(x))'$, a contradiction, so there can be no κ such that $(f \to g) \neq 1$, hence $f \to g = 1$.

If $f > g$, then there is a $\kappa \leq \alpha$, such that $f(\kappa) > g(\kappa)$. Then $f(\lambda) = 1$ for all $\lambda \leq \kappa$ and $g(\mu) = 0$ for all $\mu \geq \kappa$. By definition of $f \to g$ then follows $(f \to g)(\rho) = g(\rho)$ for all $\rho \leq \alpha$, so $f \to g = g$.

If α is finite, say n, and $B = \{0, 1\}$ then $\mathsf{L}_\alpha(B) = \langle (0, \ldots, 0), (0, \ldots, 0, 1), \ldots, (1, \ldots, 1) \rangle$ is a chain of $n + 1$ elements. The relations of these chains to Peirce's law is interesting (cf. [1] and also [2] and [3]). Let

$$\mathsf{P}(f_1, f_2) = ((f_2 \to f_1) \to f_2) \to f_2$$

and let its iterates be defined by

$$\mathsf{P}(f_1, \ldots, f_{n+1}) = \mathsf{P}(\mathsf{P}(f_1, \ldots, f_n), f_{n+1}) ,$$

then $\mathsf{P}(f_1, \ldots, f_n)$ is equal to 1 on $\mathsf{L}_m(\{0, 1\})$ for $m < n$ and different from 1 for $m \geq n$. This result is not typical for $\mathsf{L}_m(\{0, 1\})$ and it can be shown for arbitrary $\mathsf{L}_m(B)$. In order to do so, we calculate the function $\mathsf{P}(f,g)$. A convenient description of $\mathsf{P}(f,g)(k + 1)$ results from the following theorems.

Theorem 4: $(g \to f)(k) = f(k) + ((g \to f) \to g)(k)'$, *for all finite* $k \geq \alpha$.

Proof. Put $g \to f = r$ and $r \to g = t$, then we have to prove $r(k) = f(k) + t(k)'$. First of all $r(1) = f(1) + g(1)'$ and $t(1) = g(1) + r(1)' = g(1) + g(1) f(1)' = g(1)$, so

$$r(1) = f(1) + t(1)' .$$

Next suppose $r(k) = f(k) + t(k)'$, then first notice $r(k + 1) = f(k + 1) + g(k + 1)' \cdot r(k) = f(k + 1) + g(k + 1)' \cdot f(k) + g(k + 1)' \cdot t(k)'$, hence

$$g(k + 1)' \cdot t(k)' \leq r(k + 1) ,$$

which we use in the following reduction

$$
\begin{aligned}
f(k+1) + t(k+1)' &= f(k+1) + g(k+1)' \cdot (r(k+1) + t(k)') \\
&= f(k+1) + g(k+1)' \cdot (f(k+1) + g(k+1)' \cdot (r(k) + t(k)')) \\
&= f(k+1) + g(k+1)' \cdot r(k) + g(k+1)' \cdot t(k)' \\
&= r(k+1) + g(k+1)' \cdot t(k)' \\
&= r(k+1) .
\end{aligned}
$$

Theorem 5: *$t(k+1) = g(k+1) + r(k)'$ for all finite $k < \alpha$.*

Proof.

$$
\begin{aligned}
t(k+1) &= g(k+1) + r(k+1)' \cdot t(k) \\
&= g(k+1) + f(k+1)' \cdot (g(k+1) + r(k)') \cdot t(k) \\
&= g(k+1) + t(k) \cdot f(k+1)' \cdot r(k)' \\
&= g(k+1) + t(k) \cdot f(k)' \cdot g(k) + r(k-1)' \\
&= g(k+1) + t(k) \cdot r(k)' \\
&= g(k+1) + r(k)' \cdot
\end{aligned}
$$

Theorem 6: *$\mathsf{P}(f,g)\,(k+1) = g(k+1) + (g \to f)\,(k)$ for all finite $k < \alpha$.*

Proof. We use the fact that $\mathsf{P}(f,g)\,(k) \geq (g \to f)\,(k)$, which is easily established. We again use r for $g \to f$ and t for $r \to g$. Then

$$
\begin{aligned}
\mathsf{P}(f,g)\,(k+1) &= g(k+1) + t(k+1)' \cdot \mathsf{P}(f,g)\,(k) \\
&= g(k+1) + g(k+1)' \cdot r(k) \cdot \mathsf{P}(f,g)\,(k) \\
&= g(k+1) + r(k) \cdot \mathsf{P}(f,g)\,(k) \\
&= g(k+1) + r(k) .
\end{aligned}
$$

Concerning the iterates of Peirce's law we have the following

Theorem 7: *$\mathsf{P}(f_1, \ldots, f_n)\,(n-1) = 1$ for all finite $n \leq \alpha$.*

Proof. Evidently $\mathsf{P}(f_1, f_2)\,(1) = 1$. Further

$$
\begin{aligned}
\mathsf{P}(f_1, \ldots, f_{n+1})\,(n) &= f_{n+1}\,(n) + (f_{n+1} \to \mathsf{P}(f_1, \ldots, f_n))\,(n-1) \\
&\geq \mathsf{P}(f_1, \ldots, f_n)\,(n-1) .
\end{aligned}
$$

So if $\mathsf{P}(f_1, \ldots, f_n)\,(n-1) = 1$, then also $\mathsf{P}(f_1, \ldots, f_{n+1})\,(n) = 1$.

The above theorem expresses the fact that $\mathsf{P}(f_1, \ldots, f_n) = 1$ on any $\mathsf{L}_m(B)$ where $m < n$. We next derive a formula for $\mathsf{P}(f_1, \ldots, f_n)\,(n)$ from which it will be clear that $\mathsf{P}(f_1, \ldots, f_n) \neq 1$ on $\mathsf{L}_n(B)$, and consequently on all $\mathsf{L}_m(B)$ where $m \geq n$.

Theorem 8: *$\mathsf{P}(f_1, \ldots, f_n)\,(n) = \sum_{k=1}^{n} f_k\,(k) + \sum_{k=2}^{n} f_k\,(k-1)'$ for all finite n, satisfying $2 \leq n \leq \alpha$.*

Proof. Evidently $\mathsf{P}(f_1, f_2)\,(2) = f_2\,(2) + f_1\,(1) + f_2\,(1)'$. If the formula holds for n, then it also holds for $n+1$, because

$$
\begin{aligned}
\mathsf{P}(f_1, \ldots, f_{n+1})\,(n+1) &= f_{n+1}\,(n+1) + (f_{n+1}\,\mathsf{P}(f_1, \ldots, f_n))\,(n) \\
&= f_{n+1}\,(n+1) + \mathsf{P}(f_1, \ldots, f_n)\,(n) + f_{n+1}\,(n)' \\
&= \sum_{k=1}^{n+1} f_k\,(k) + \sum_{k=2}^{n+1} f_k\,(k-1)' .
\end{aligned}
$$

From this formula it is clear how to choose the f_k so as to obtain

$\mathsf{P}(f_1, \ldots, f_n)$ $(n) \neq 1$, and such a choice is possible for any $\mathsf{L}_m(B)$, where $m \geq n$ and B arbitrary.

Remark:

5. The sequence $\mathsf{P} = \{\mathsf{P}(f_1, \ldots, f_n)\}$ $(n < \omega)$ is weakly increasing and no member is equal to 1 on $\mathsf{L}_\omega(B)$. One should notice that the behaviour of P is best discussed in the context of infinitary logics (cf. e.g. [4]). Because we consider non-classical logics a few modifications are required, because for example the infinite disjunction

$$\bigvee\!\!\!\bigvee_n F_n$$

is not introduced by means of negation and infinite conjunction, and we should add the axiom

$$F_k \to \bigvee\!\!\!\bigvee_n F_n$$

and the rule

$$\text{if } F_1 \to G, F_2 \to G, \ldots, F_k \to G, \ldots \text{ then}$$
$$\bigvee\!\!\!\bigvee_n F_n \to G.$$

Instead of the usual axioms for classical propositional logic one should accept an intermediate set, e.g. the intuitionistic system with $\mathsf{D}(f, g)$ added. For such logics the infinite disjunctions are interpreted as unions in the following way in the case of P:

$$\left[\bigcup_n \mathsf{P}(f_1, \ldots, f_n)\right] (k) = \bigcup_n [\mathsf{P}(f_1, \ldots, f_n)\ (k)].$$

Then it is clear from Theorem 7 that

$$\bigcup_n \mathsf{P}(f_1, \ldots, f_n) = 1$$

on all $\mathsf{L}_\alpha(B)$ where $\alpha \leq \omega$. So on $\mathsf{L}_\omega(B)$ the infinite disjunction of the iterates of Peirce's law is valid while no finite iterate is.

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