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SOLUTIONS TO FOUR MODAL PROBLEMS OF SOBOCIŃSKI

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In Sobociński's [1], [2], and [3] several questions are left open, among them

Q1. Is K1.1.1 a proper extension of K1.1?
Q2. Is K2.2 a proper extension of K2.1?
Q3. Is S4.1.1 a proper extension of S4.1?
Q4. Is S4.1.3 a proper extension of S4.1.2?

All four questions are here settled in the negative, a familiarity with these three papers being presupposed. We shall also assume a nodding acquaintance with Kripkean relational models and with the fact that α is a thesis of S4 if and only if for every S4 model $\mathfrak{F} = (\mathfrak{G}, \mathfrak{K}, \mathfrak{R})$, i.e., in which $\mathfrak{R}$ is a reflexive and transitive relation on $\mathfrak{K}$, $\phi(\alpha, \mathfrak{G}) = 1$ for each valuation ϕ on $\mathfrak{F}$.

Ad Q1 and *Q2*. We shall show that

$$CLCLCLCCpLqLCpLqCpLqCpLqCLCLCpLqLqCLCLCNpLqLqLq$$

is validated by every S4 model and thus is a thesis of S4, from which it follows that Grzegorczyk's axiom $CLCLCpLqLqCLCLCNpLqLqLq$ is a thesis of K1.1 and K2.2.

Suppose ϕ is a valuation on an S4 model $(\mathfrak{G}, \mathfrak{K}, \mathfrak{R})$ such that

$$\phi(CLCLCpLqLqCLCLCNpLqLqLq, \mathfrak{G}) = 0,$$

from which

$$\phi(LCLCpLqLq, \mathfrak{G}) = 1 \quad (1)$$
$$\phi(LCLCNpLqLq, \mathfrak{G}) = 1 \quad (2)$$
$$\phi(Lq, \mathfrak{G}) = 0. \quad (3)$$

The task is now to show that

$$\phi(LCLCLCCpLqLCpLqCpLqCpLq, \mathfrak{G}) = 0, \quad (4)$$

and this will complete the proof. From (1) we get

$$\phi(CLCpLqLq, \mathfrak{G}) = 1$$

and from this, together with (3),

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$$\phi(LCpLq,\mathfrak{G}) = 0.$$

Hence there exists an $\mathfrak{H}_1 \epsilon \mathfrak{K}$ such that $\mathfrak{G}\mathfrak{R}\mathfrak{H}_1$ and

$$\phi(CpLq,\mathfrak{H}_1) = 0. \tag{5}$$

Now suppose for *reductio* that there exists an $\mathfrak{H} \epsilon \mathfrak{K}$ such that $\mathfrak{H}_1\mathfrak{R}\mathfrak{H}$ and

$$\phi(CLCCpLqLCpLqCpLq,\mathfrak{H}) = 0,$$

from which

$$\phi(LCCpLqLCpLq,\mathfrak{H}) = 1 \tag{6}$$

$$\phi(Lq,\mathfrak{H}) = 0. \tag{7}$$

By (2) and (7)

$$\phi(LCNpLq,\mathfrak{H}) = 0,$$

and hence there exists an $\mathfrak{H}_2 \epsilon \mathfrak{K}$ such that $\mathfrak{H}\mathfrak{R}\mathfrak{H}_2$ and

$$\phi(p,\mathfrak{H}_2) = \phi(Lq,\mathfrak{H}_2) = 0 \tag{8}$$

from which it follows that

$$\phi(CpLq,\mathfrak{H}_2) = 1.$$

But from (6) we also have

$$\phi(CCpLqLCpLq,\mathfrak{H}_2) = 1$$

and so

$$\phi(LCpLq,\mathfrak{H}_2) = 1.$$

However, by (1), (8), and the fact that $\mathfrak{G}\mathfrak{R}\mathfrak{H}_2$ by the transitivity of $\mathfrak{R}$

$$\phi(LCpLq,\mathfrak{H}_2) = 0$$

and we have a contradiction. Hence

$$\phi(CLCCpLqLCpLqCpLq,\mathfrak{H}) = 1$$

for all $\mathfrak{H} \epsilon \mathfrak{K}$ such that $\mathfrak{H}_1\mathfrak{R}\mathfrak{H}$, from which it follows that

$$\phi(LCLCCpLqLCpLqCpLq,\mathfrak{H}_1) = 1. \tag{9}$$

From (5) and (9) we therefore have

$$\phi(CLCLCCpLqLCpLqCpLqCpLq,\mathfrak{H}_1) = 0$$

and thus (4).

In this regard, it is interesting to note that *CMCMCpMqMqCMCMCNpMqMqMq*, the formula obtained simply by replacing all *L*'s by *M*'s in Grzegorczyk's axiom, can serve in place of *CMLpLMp* as the proper axiom of S4.2. For it is an easy matter to verify that both

$$CCMLpLMpCMCMCpMqMqCMCMCNpMqMqMq$$

and

$$CCMCMCpMNCppMNCppCMCMCNpMNCppMNCppMNCppCMLpLMp$$

are validated by every S4 model and are thus theses of S4. Consequently, this variation on Grzegorczyk's axiom is a thesis of S4.2, while S4.2 results from its addition to S4.

Ad Q3 and *Q4*. Let β be the result of substituting $p/CpLp$ and $q/CNpLNp$ in $KCLCLCpLppCMLppCLCLCqLqqCMLqq$. We shall show that $C\beta CLCL$-$CpLpLpCMLpLp$ is validated by every S4 model and so is a thesis of S4, from which it obviously follows that $CLCLCpLpLpCMLpLp$ is a thesis of S4.1 and S4.1.2. Suppose ϕ is a valuation on an S4 model $(\mathfrak{G}, \mathfrak{K}, \mathfrak{R})$ such that

$$\phi(CLCLCpLpLpCMLpLp, \mathfrak{G}) = 0,$$

from which

$$\phi(LCLCpLpLp, \mathfrak{G}) = 1 \quad (1)$$

$$\phi(MLp, \mathfrak{G}) = 1 \quad (2)$$

$$\phi(Lp, \mathfrak{G}) = 0. \quad (3)$$

The task is now to show that

$$\phi(\beta, \mathfrak{G}) = 0, \quad (4)$$

and this will complete the proof. By (3) there exists an $\mathfrak{H}_1 \in \mathfrak{K}$ such that $\mathfrak{G}\mathfrak{R}\mathfrak{H}_1$ and

$$\phi(p, \mathfrak{H}_1) = 0. \quad (5)$$

From (1) and (5) we then have

$$\phi(CLCpLpLp, \mathfrak{H}_1) = 1$$

$$\phi(Lp, \mathfrak{H}_1) = 0$$

and so

$$\phi(LCpLp, \mathfrak{H}_1) = 0;$$

whence

$$\phi(p, \mathfrak{H}_2) = 1 \quad (6)$$

for some $\mathfrak{H}_2 \in \mathfrak{K}$ such that $\mathfrak{H}_1\mathfrak{R}\mathfrak{H}_2$. Since $\mathfrak{G}\mathfrak{R}\mathfrak{H}_2$ by the transitivity of $\mathfrak{R}$, it follows from (5) and (6) that

$$\phi(Lp, \mathfrak{G}) = \phi(LNp, \mathfrak{G}) = 0$$

and hence that

$$\phi(CpLp, \mathfrak{G}) = 0 \text{ or } \phi(CNpLNp, \mathfrak{G}) = 0. \quad (7)$$

By (2) there exists an $\mathfrak{H}_3 \in \mathfrak{K}$ such that $\mathfrak{G}\mathfrak{R}\mathfrak{H}_3$ and

$$\phi(Lp, \mathfrak{H}_3) = 1,$$

from which we easily get

$$\phi(LCpLp, \mathfrak{H}_3) = \phi(LCNpLNp, \mathfrak{H}_3) = 1$$

and hence

$$\phi(MLCpLp,\mathfrak{G}) = \phi(MLCNpLNp,\mathfrak{G}) = 1. \tag{8}$$

Next, suppose for *reductio* that there exists an $\mathfrak{H} \epsilon \mathfrak{K}$ such that $\mathfrak{G}\mathfrak{R}\mathfrak{H}$ and

$$\phi(CLCCpLpLCpLpCpLp,\mathfrak{H}) = 0.$$

Then we have

$$\phi(LCCpLpLCpLp,\mathfrak{H}) = 1 \tag{9}$$
$$\phi(Lp,\mathfrak{H}) = 0. \tag{10}$$

By (10) there exists an $\mathfrak{H}_4 \epsilon \mathfrak{K}$ such that $\mathfrak{H}\mathfrak{R}\mathfrak{H}_4$ and

$$\phi(p,\mathfrak{H}_4) = 0, \tag{11}$$

from which, together with (1) and the fact that $\mathfrak{G}\mathfrak{R}\mathfrak{H}_4$ by the transitivity of $\mathfrak{R}$, we have

$$\phi(CLCpLpLp,\mathfrak{H}_4) = 1$$
$$\phi(Lp,\mathfrak{H}_4) = 0$$

and so

$$\phi(LCpLp,\mathfrak{H}_4) = 0. \tag{12}$$

But we also have

$$\phi(CpLp,\mathfrak{H}_4) = 1$$

because of (11), and this, together with (12), yields

$$\phi(CCpLpLCpLp,\mathfrak{H}_4) = 0$$

contrary to (9). Hence

$$\phi(CLCCpLpLCpLpCpLp,\mathfrak{H}) = 1$$

for all $\mathfrak{H} \epsilon \mathfrak{K}$ such that $\mathfrak{G}\mathfrak{R}\mathfrak{H}$, and so

$$\phi(LCLCCpLpLCpLpCpLp,\mathfrak{G}) = 1. \tag{13}$$

Finally, suppose for *reductio* that there exists an $\mathfrak{H} \epsilon \mathfrak{K}$ such that $\mathfrak{G}\mathfrak{R}\mathfrak{H}$ and

$$\phi(CLCCNpLNpLCNpLNpCNpLNp,\mathfrak{H}) = 0,$$

from which

$$\phi(LCCNpLNpLCNpLNp,\mathfrak{H}) = 1 \tag{14}$$
$$\phi(Np,\mathfrak{H}) = 1 \tag{15}$$
$$\phi(LNp,\mathfrak{H}) = 0. \tag{16}$$

By (1) and (15) we have

$$\phi(CLCpLpLp,\mathfrak{H}) = 1$$
$$\phi(Lp,\mathfrak{H}) = 0$$

and so

$$\phi(LCpLp,\mathfrak{H}) = 0.$$

Hence there exists an $\mathfrak{H}_5 \epsilon \mathfrak{K}$ such that $\mathfrak{H}\mathfrak{R}\mathfrak{H}_5$ and

$$\phi(Np,\mathfrak{H}_5) = \phi(Lp,\mathfrak{H}_5) = 0. \tag{17}$$

But then there exists an $\mathfrak{H}_6 \in \mathfrak{K}$ such that $\mathfrak{H}_5 \mathfrak{R} \mathfrak{H}_6$ and

$$\phi(p,\mathfrak{H}_6) = 0, \tag{18}$$

from which, together with (1) and the fact that $\mathfrak{G}\mathfrak{R}\mathfrak{H}_6$, we have

$$\phi(Lp,\mathfrak{H}_6) = 0$$
$$\phi(CLCpLpLp,\mathfrak{H}_6) = 1$$

and so

$$\phi(LCpLp,\mathfrak{H}_6) = 0.$$

Hence there exists an $\mathfrak{H}_7 \in \mathfrak{K}$ such that $\mathfrak{H}_6 \mathfrak{R} \mathfrak{H}_7$ and

$$\phi(p,\mathfrak{H}_7) = 1. \tag{19}$$

Now by (14) we have

$$\phi(CCNpLNpLCNpLNp,\mathfrak{H}_5) = 1.$$

But

$$\phi(CNpLNp,\mathfrak{H}_5) = 1$$

by (17), and so

$$\phi(LCNpLNp,\mathfrak{H}_5) = 1.$$

Hence

$$\phi(CNpLNp,\mathfrak{H}_6) = 1,$$

from which, together with (18), we have

$$\phi(LNp,\mathfrak{H}_6) = 1$$

and so

$$\phi(Np,\mathfrak{H}_7) = 1$$

contrary to (19). Hence

$$\phi(CLCCNpLNpLCNpLNpCNpLNp,\mathfrak{H}) = 1$$

for all $\mathfrak{H} \in \mathfrak{K}$ such that $\mathfrak{G}\mathfrak{R}\mathfrak{H}$, and so

$$\phi(LCLCCNpLNpLCNpLNpCNpLNp,\mathfrak{G}) = 1. \tag{20}$$

From (7), (8), (13), and (20) it follows that either

$$\phi(CLCLCCpLpLCpLpCpLpCMLCpLpCpLp,\mathfrak{G}) = 0$$

or

$$\phi(CLCLCCNpLNpLCNpLNpCNpLNpCMLCNpLNpCNpLNp,\mathfrak{G}) = 0,$$

and therefore (4).

It has come to my attention that the questions settled in this note have been resolved independently by Krister Segerberg using a somewhat different strategy.

REFERENCES

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