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INCOMPLETENESS VIA SIMPLE SETS

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Let P be Peano arithmetic and let Σ_0 be the set of formulas in the language of P which only contain bounded quantifiers. It is well known that if Q is an ω-consistent extension of P, and $Q(x)$ is a Σ_0-formula, then

(1) $Q \vdash (\exists x)\, \phi(x)$ *implies* $Q \vdash \phi(\mathbf{n})$ *for some* $n < \omega$.

What we show here is that by only slightly more complicating the form of ϕ, (1) will fail in every consistent axiomatizable extension of P.* In detail

Theorem: *There is a* Σ_0*-formula* $\phi(x, y, z)$ such that for any consistent *axiomatizable extension* Q *of* P *there is a* $q < \omega$ *such that* $Q \vdash (\exists x)\, (\forall y)\, \phi(x, y, \mathbf{q})$, *but for no* $n < \omega$ *does* $Q \vdash (\forall y)\, \phi\, (\mathbf{n}, y, \mathbf{q})$.

(Note that under these hypotheses (1) above implies our result is the best possible.)

Proof: Let S be the simple set of Post (*cf*. [1] p. 106). We define S in terms of the Kleene predicate T (which enumerates the n-th recursively enumerable set as $\{m : (\exists u)\, T(n, m, u)\}$), the pairing function j, and its first, second inverse k, l.

(2) $F(m, n) \equiv (\exists u)\, [(T(n, m, u) \wedge m > 2n) \wedge (\forall v)\, ((v < j(m, u) \wedge T(n, k(v), l(v)) \rightarrow k(v) \leq 2n)]$

(3) $S(m) \equiv (\exists n)\, F(m, n)$

Let $\phi(y, x), \sigma(y)$ be the intuitive translations of F, S into the language of P and let Q be any consistent axiomatizable extension of P. F is a partial recursive function (in the n to m direction) which is represented in P (á fortiori Q) by

(4) $F(m, n)$ *implies* $Q \vdash \phi(\mathbf{m}, \mathbf{n})$,

and

(5) $Q \vdash (\phi(y, x) \wedge \phi(z, x)) \rightarrow y = z$.

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Let $S' = \{m : Q \vdash \sim \sigma(\mathbf{m})\}$. Now $S \subseteq \{m : Q \vdash \sigma(\mathbf{m})\}$ by (4), $S' \subseteq \omega - S$ by the consistency of Q, S' is recursively enumerable by the axiomatizability of Q, and S' is finite by the simplicity of S. Let $q < \omega$ be greater than any element of S' and define $\theta(y, x)$ to be $\sim \phi(y, x) \wedge y \geq \mathbf{q}$. Thus by our previous remarks we have shown that $Q \vdash (\forall x)\, \theta(\mathbf{m}, x)$ for no $m < \omega$. Since the universal quantifier occurring in θ may be absorbed by $(\forall x)$ our theorem will follow by showing that $Q \vdash (\exists y)\,(\forall x)\, \theta(y, x)$. In order to do this first note that from (2)

(6) $Q \vdash (\phi(y, x) \wedge y \leq 2z) \rightarrow x < z$

and that by induction in P we can prove the following form of the pigeon hole principle (this follows easily by formalizing the usual set theoretic proof that the integers are finite in the sense of Dedekind)

(7) $Q \vdash (\forall y)\,[z \leq y \leq 2z \rightarrow (\exists x)\,(x < z \wedge \phi(y, x))] \rightarrow (\exists x, y, y')\,(z \leq y < y' \leq 2z \wedge x < z \wedge \phi(y, x) \wedge \phi(y', x))$.

Now in Q, $(\forall y)\,(z \leq y \leq 2z \rightarrow \sigma(y))$ implies by (6) that $(\forall y)\,(z \leq y \leq 2z \rightarrow (\exists x)\,(x < z \wedge \phi(y, x)))$ which implies by (7) that $(\exists x, y, y')\,(z \leq y < y' \leq 2z \wedge x < z \wedge \phi(y, x) \wedge \phi(y', x))$ which contradicts (5). Thus $Q \vdash (\exists y)\,(z \leq y \wedge \sim \sigma(y))$. Take $z = \mathbf{q}$ and get $Q \vdash (\exists y)\,(\forall x)\, \theta(y, x)$.

REFERENCES

[1] Rogers, H., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill (1967).

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