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ATOMISTIC MEREOLOGY II

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4* In this section I will prove that which was mentioned at the beginning of this paper, that the axiom-system in which functor "**el**" occurs as the single primitive mereological notion of atomistic mereology, and which contains only two axioms, namely:

$A\ \ [AB]:\cdot:A\,\varepsilon\,\mathbf{el}(B).\equiv::B\varepsilon B::[Ta]::[C]\therefore C\,\varepsilon\,T.\equiv:[E]:E\,\varepsilon\,a.\supset.E\,\varepsilon\,\mathbf{el}(C):$
$[E]:E\,\varepsilon\,\mathbf{el}(C).\supset.[\exists FG].F\,\varepsilon\,a.G\,\varepsilon\,\mathbf{el}(F).G\,\varepsilon\,\mathbf{el}(E)\therefore B\,\varepsilon\,\mathbf{el}(B).B\,\varepsilon\,a.$
$\supset.A\,\varepsilon\,\mathbf{el}(T)$

and

$V\ \ [A]::A\,\varepsilon\,A.\supset\therefore[\exists B]\therefore B\,\varepsilon\,\mathbf{el}(A):[C]:C\,\varepsilon\,\mathbf{el}(B).\supset.C=B$

is inferentially equivalent to the following four axioms:

$S1\ \ [A]:A\,\varepsilon\,\mathbf{at}(B).\supset.B\,\varepsilon\,B$
$S2\ \ [ABC]:A\,\varepsilon\,\mathbf{at}(B).C\,\varepsilon\,\mathbf{at}(A).\supset.C=A$
$S3\ \ [AB]\therefore A\,\varepsilon\,A.B\,\varepsilon\,B:[C]:C\,\varepsilon\,\mathbf{at}(A).\equiv.C\,\varepsilon\,\mathbf{at}(B):\supset.A=B$
$S4\ \ [Aa]::A\,\varepsilon\,a.\supset\therefore[\exists B]\therefore[\exists E].E\,\varepsilon\,\mathbf{at}(B):[C]:C\,\varepsilon\,\mathbf{at}(B).\equiv.[\exists D].C\,\varepsilon\,\mathbf{at}(D).$
$D\,\varepsilon\,a$

in which Rickey's functor "**at**" occurs, as their single mereological term.

4.1 Let us assume the axioms A and V. Since A is the single axiom of mereology, we have at our disposal all its consequences presented in section **2.** Then:

$D\mathrm{I}\ \ [A]\therefore A\,\varepsilon\,A:[B]:B\,\varepsilon\,\mathbf{el}(A).\supset.B=A:\equiv.A\,\varepsilon\,\mathbf{atm}$
$D\mathrm{II}\,[AB]:A\,\varepsilon\,\mathbf{atm}.A\,\varepsilon\,\mathbf{el}(B).\equiv.A\,\varepsilon\,\mathbf{at}(B)$

Cf. **3.3,** points (A) and (B).

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S1 $[A\,B]: A\,\varepsilon\,\mathbf{at}(B)\,.\supset .\,B\,\varepsilon\,B$ $[D\text{II};\ A1]$

S2 $[A\,B\,C]: A\,\varepsilon\,\mathbf{at}(B)\,.\,C\,\varepsilon\,\mathbf{at}(A)\,.\supset .\,C = A$

PR $[A\,B\,C]\therefore \mathrm{Hp}(2)\,.\supset :$

3. $C\,\varepsilon\,\mathbf{el}(A):$ $[D\text{II};\ 2]$

4. $[B]: B\,\varepsilon\,\mathbf{el}(A)\,.\supset .\,B = A:$ $[D\text{II};\ D\text{I};\ 1]$

$C = A$ $[4;\ 3]$

V1 $[A]: A\,\varepsilon\,A\,.\supset .\,[\exists B]\,.\,B\,\varepsilon\,\mathbf{at}(A)$ $[V;\ A1;\ D\text{I};\ D\text{II};\ Cf.\ \mathbf{3.5.1}]$

D1 $[A\,a]\therefore A\,\varepsilon\,A : [B]: B\,\varepsilon\,a\,.\supset .\,B\,\varepsilon\,\mathbf{el}(A) : [B]: B\,\varepsilon\,\mathbf{el}(A)\,.\supset .\,[\exists E\,F]\,.\,E\,\varepsilon\,a$
$.\,F\,\varepsilon\,\mathbf{el}(E)\,.\,F\,\varepsilon\,\mathbf{el}(B) :\equiv .\,A\,\varepsilon\,\mathbf{Kl}\,(a)$ $[Cf.\ \mathbf{2.1}]$

V3 $[A]: A\,\varepsilon\,A\,.\supset .\,A\,\varepsilon\,\mathbf{Kl}\,(\mathbf{at}(A))$
$[D1;\ D\text{II};\ A2;\ T1;\ V1;\ A4;\ D\text{I};\ A6;\ A7;\ Cf.\ \mathbf{3.5.3}]$

S3 $[A\,B]\therefore A\,\varepsilon\,A\,.\,B\,\varepsilon\,B : [C]: C\,\varepsilon\,\mathbf{at}(A)\,.\equiv .\,C\,\varepsilon\,\mathbf{at}(B) :\supset .\,A = B$

PR $[A\,B]\therefore \mathrm{Hp}(3) :\supset .$

4. $A\,\varepsilon\,\mathbf{Kl}\,(\mathbf{at}(A))\,.$ $[V3;\ 1]$

5. $B\,\varepsilon\,\mathbf{Kl}\,(\mathbf{at}(B))\,.$ $[V3;\ 2]$

6. $B\,\varepsilon\,\mathbf{Kl}\,(\mathbf{at}(A))\,.$ $[E2;\ 3;\ 5]$

$A = B$ $[A3;\ 4;\ 6]$

Z1 $[A\,B\,C]: A\,\varepsilon\,\mathbf{at}(B)\,.\,B\,\varepsilon\,\mathbf{el}(C)\,.\supset .\,A\,\varepsilon\,\mathbf{at}(C)$ $[D\text{II};\ A2]$

Z2 $[A\,C\,a]: A\,\varepsilon\,\mathbf{Kl}\,(a)\,.\,C\,\varepsilon\,\mathbf{at}(A)\,.\supset .\,[\exists D]\,.\,C\,\varepsilon\,\mathbf{at}(D)\,.\,D\,\varepsilon\,a$

PR $[A\,C\,a]: \mathrm{Hp}(2)\,.\supset .$

3. $C\,\varepsilon\,\mathbf{el}(A)\,.$ $[D\text{II};\ 2]$

4. $C\,\varepsilon\,\mathbf{atm}\,.$ $[D\text{II};\ 2]$

$[\exists E\,F]\,.$

5. $E\,\varepsilon\,a\,.$

6. $F\,\varepsilon\,\mathbf{el}(E)\,.$ $[D1;\ 1;\ 3]$

7. $F\,\varepsilon\,\mathbf{el}(C)\,.$

8. $F = C\,.$ $[D\text{I};\ 4;\ 7]$

9. $C\,\varepsilon\,\mathbf{el}(E)\,.$ $[6;\ 8]$

$[\exists D]\,.\,C\,\varepsilon\,\mathbf{at}(D)\,.\,D\,\varepsilon\,a$ $[D\text{II};\ 4;\ 9;\ 5]$

Z3 $[A\,C\,D\,a]: A\,\varepsilon\,\mathbf{Kl}\,(a)\,.\,C\,\varepsilon\,\mathbf{at}(D)\,.\,D\,\varepsilon\,a\,.\supset .\,C\,\varepsilon\,\mathbf{at}(A)$

PR $[A\,C\,D\,a]: \mathrm{Hp}(3)\,.\supset .$

1. $D\,\varepsilon\,\mathbf{el}(A)\,.$ $[D1;\ 1;\ 3]$

$C\,\varepsilon\,\mathbf{at}(A)$ $[Z1;\ 2;\ 4]$

Z4 $[A\,B\,D\,a]\therefore [C\,D]: C\,\varepsilon\,\mathbf{at}(D)\,.\,D\,\varepsilon\,a\,.\supset .\,C\,\varepsilon\,\mathbf{at}(A) : B\,\varepsilon\,a\,.\,D\,\varepsilon\,\mathbf{el}(B)\,.\supset .\,[\exists F]\,.$
$F\,\varepsilon\,\mathbf{el}(D)\,.\,F\,\varepsilon\,\mathbf{el}(A)$

PR $[A\,B\,D\,a]\therefore \mathrm{Hp}(3) :\supset .$

$[\exists F]\,.$

4. $F\,\varepsilon\,\mathbf{at}(D)$ $[T1;\ V1;\ 3]$

5. $F\,\varepsilon\,\mathbf{at}(B)\,.$ $[Z1;\ 4;\ 3]$

6. $F\,\varepsilon\,\mathbf{at}(A)$ $[1;\ 5;\ 2]$

$[\exists F]\,.\,F\,\varepsilon\,\mathbf{el}(D)\,.\,F\,\varepsilon\,\mathbf{el}(A)$ $[D\text{II};\ 4;\ 6]$

Z5 $[A\,B\,a]\therefore [C\,D]: C\,\varepsilon\,\mathbf{at}(D)\,.\,D\,\varepsilon\,a\,.\supset .\,C\,\varepsilon\,\mathbf{at}(A) : B\,\varepsilon\,a\,.\supset .\,B\,\varepsilon\,\mathbf{el}(A)$
$[T1;\ Z4;\ A6]$

Z6 $[A\,B\,a]\therefore [C]: C\,\varepsilon\,\mathbf{at}(A)\,.\supset .\,[\exists D]\,.\,C\,\varepsilon\,\mathbf{at}(D)\,.\,D\,\varepsilon\,a : B\,\varepsilon\,\mathbf{el}(A)\,.\supset .\,[\exists E\,F]\,.$
$E\,\varepsilon\,a\,.\,F\,\varepsilon\,\mathbf{el}(E)\,.\,F\,\varepsilon\,\mathbf{el}(B)$

PR $[ABa] .\therefore . \mathrm{Hp}(2) :\supset:$

$[\exists F]:$

3. $F \varepsilon \mathbf{at}(B) .$ $[T1;\ V1;\ 2]$

4. $F \varepsilon \mathbf{at}(A) .$ $[Z1;\ 3;\ 2]$

$[\exists E] .$

5. $F \varepsilon \mathbf{at}(E) .$

6. $E \varepsilon a .$ $[1;\ 4]$

$[\exists EF] . E \varepsilon a . F \varepsilon \mathbf{el}(E) . F \varepsilon \mathbf{el}(B)$ $[D\mathrm{II};\ 6;\ 5;\ 3]$

D1 $[Aa] .\therefore . A \varepsilon A : [C] : C \varepsilon \mathbf{at}(A) . \equiv . [\exists D] . C \varepsilon \mathbf{at}(D) . D \varepsilon a : \equiv . A \varepsilon \mathbf{Kl}(a)$

$[Z2;\ Z3;\ T1;\ Z5;\ Z6;\ D1]$

S4 $[Aa] : : A \varepsilon a . \supset .\therefore [\exists B] .\therefore [\exists E] . E \varepsilon \mathbf{at}(B) : [C] : C \varepsilon \mathbf{at}(A) . \equiv . [\exists D] . C \varepsilon \mathbf{at}(D) . D \varepsilon a$

PR $[Aa] : : \mathrm{Hp}(1) . \supset .\therefore$

$[\exists B] .\therefore$

2. $B \varepsilon \mathbf{Kl}(a) :$ $[A4;\ 1]$

3. $[C] : C \varepsilon \mathbf{at}(B) . \equiv . [\exists D] . C \varepsilon \mathbf{at}(D) . D \varepsilon a :$ $[\mathbf{D1};\ 2]$

$[\exists E] .$

4. $E \varepsilon \mathbf{at}(B) . .$ $[T1;\ V1;\ 2]$

$[\exists B] .\therefore [\exists E] E \varepsilon \mathbf{at}(B) : [C] : C \varepsilon \mathbf{at}(B) . \equiv . [\exists D] . C \varepsilon \mathbf{at}(D) . D \varepsilon a$ $[4;\ 3]$

D2 $[AB] .\therefore . A \varepsilon A : [C] : C \varepsilon \mathbf{at}(A) . \supset . C \varepsilon \mathbf{at}(B) : \equiv . A \varepsilon \mathbf{el}(B)$

$[V1;\ D\mathrm{II};\ Z1;\ T1;\ A6]$

D3 $[A] : A \varepsilon \mathbf{at}(A) . \equiv . A \varepsilon \mathbf{atm}$ $[D\mathrm{II};\ T1;\ A5]$

Thus, *S1*, *S2*, *S3*, *S4* and, also, **D1**, **D2**, **D3** follow from A and V. Since, *cf.* **4.2**, in the field of $\{S1, S2, S3, S4\}$ the formulas **D1**, **D2** and **D3** will be the definitions of "**Kl**", "**el**" and "**atm**" by "**at**" respectively, it had to be proved here that they are the consequences of $\{A; V\}$. Although I found **D1** doing research on the topic of this paper independently, it should be noticed that R. E. Clay recollects that in 1961 C. Lejewski formulated a similar formula during a casual discussion about mereological atoms. Also, Dr. V. F. Rickey knew of **D1** independently from Lejewski and me. Concerning **D2** *cf.* **3.3**, point (B), and [18], p. 337, postulate $\mathfrak{D}_2^*$. **D3** is due to Rickey.

4.2 Now, let us assume *S1, S2, S3* and *S4*. Then:

V1 $[A] : A \varepsilon A . \supset . [\exists B] . B \varepsilon \mathbf{at}(A)$

PR $[A] : : \mathrm{Hp}(1) . \supset .\therefore$

$[\exists B] .\therefore$

2. $[C] : C \varepsilon \mathbf{at}(B) . \equiv . [\exists D] . C \varepsilon \mathbf{at}(D) . D \varepsilon A :$

$[\exists E] :$ $[S4;\ 1]$

3. $E \varepsilon \mathbf{at}(B) .$

$[\exists D] .$

4. $E \varepsilon \mathbf{at}(D) .$

5. $D \varepsilon A .$ $[2;\ 3]$

6. $D = A .\therefore$ $[T2;\ 5;\ 1]$

$[\exists B] . B \varepsilon \mathbf{at}(A)$ $[4;\ 6]$

D1 $[A\,a] \therefore A\,\varepsilon A : [C] : C\,\varepsilon\,\mathbf{at}(A) . \equiv . [\exists D] . C\,\varepsilon\,\mathbf{at}(D) . D\,\varepsilon\,a : \equiv . A\,\varepsilon\,\mathbf{Kl}(a)$

A4 $[A\,a] : A\,\varepsilon\,a . \supset . [\exists B] . B\,\varepsilon\,\mathbf{Kl}(a)$

PR $[A\,a] :: \mathrm{Hp}(1) . \supset \therefore$

$[\exists B] \therefore$

2. $[C] : C\,\varepsilon\,\mathbf{at}(B) . \equiv . [\exists D] . C\,\varepsilon\,\mathbf{at}(D) . D\,\varepsilon\,a :$

$[\exists E] .$

3. $E\,\varepsilon\,\mathbf{at}(B) .$ — [*S4*; 1] (2, 3)

4. $B\,\varepsilon\,B \therefore$ — [*S1*; 3]

$[\exists B] . B\,\varepsilon\,\mathbf{Kl}(a)$ — [**D1**; 4; 2]

A3 $[A\,B\,a] : A\,\varepsilon\,\mathbf{Kl}(a) . B\,\varepsilon\,\mathbf{Kl}(a) . \supset . A = B$ — [**D1**; *T1*; *S3*]

D2 $[A\,B] \therefore A\,\varepsilon A : [C] : C\,\varepsilon\,\mathbf{at}(A) . \supset . C\,\varepsilon\,\mathbf{at}(B) : \equiv . A\,\varepsilon\,\mathbf{el}(B)$

A1 $[A\,B] : A\,\varepsilon\,\mathbf{el}(\mathrm{B}) . \supset . B\,\varepsilon\,B$

PR $[A\,B] : \mathrm{Hp}(1) . \supset .$

$[\exists C] .$

2. $C\,\varepsilon\,\mathbf{at}(A) .$ — [*T1*; *V1*; 1]

3. $C\,\varepsilon\,\mathbf{at}(B) .$ — [**D2**; 1; 2]

$B\varepsilon B$ — [*S1*; 3]

A2 $[A\,B\,C] : A\,\varepsilon\,\mathbf{el}(B) . B\,\varepsilon\,\mathbf{el}(C) . \supset . A\,\varepsilon\,\mathbf{el}(C)$ — [**D2**]

A5 $[A] : A\,\varepsilon A . \supset . A\,\varepsilon\,\mathbf{el}(A)$ — [**D2**]

A7 $[A\,B] : A\,\varepsilon\,\mathbf{el}(B) . B\varepsilon\,\mathbf{el}(A) . \supset . A = B$ — [*T1*; **D2**; *S3*]

P1 $[A\,B\,C] : A\,\varepsilon\,\mathbf{at}(B) . B\,\varepsilon\,\mathbf{at}(C) . \supset . A\,\varepsilon\,\mathbf{at}(C)$ — [*S2*]

P2 $[A\,B\,C] : A\,\varepsilon\,\mathbf{at}(B) . B\,\varepsilon\,\mathbf{el}(C) . \supset . A\,\varepsilon\,\mathbf{at}(C)$ — [**D2**]

P3 $[A\,B] : A\,\varepsilon\,\mathbf{at}(B) . \supset . A\,\varepsilon\,\mathbf{el}(B)$ — [*T1*; *P1*; **D2**]

P4 $[A\,B\,C] : B\,\varepsilon\,\mathbf{at}(A) . C\,\varepsilon\,\mathbf{el}(B) . \supset . C = B$

PR $[A\,B\,C] : \mathrm{Hp}(2) . \supset .$

$[\exists D] .$

3. $D\,\varepsilon\,\mathbf{at}(C) .$ — [*T1*; *V1*; 2]

4. $D\,\varepsilon\,\mathbf{at}(B) .$ — [*P2*; 3; 2]

5. $D = B .$ — [*S2*; 1; 4]

6. $C\,\varepsilon\,\mathbf{el}(D) .$ — [2; 5]

7. $D\,\varepsilon\,\mathbf{el}(C) .$ — [*P3*; 3]

8. $C = D .$ — [*A7*; 6; 7]

$C = B$ — [8; 5]

V $[A] :: A\,\varepsilon A . \supset \therefore [\exists B] \therefore B\,\varepsilon\,\mathbf{el}(A) : [C] : C\,\varepsilon\,\mathbf{el}(B) . \supset . C = B$ — [*V1*; *P3*; *P4*]

P5 $[A\,B\,a] : A\,\varepsilon\,\mathbf{Kl}(a) . B\,\varepsilon\,a . \supset . B\,\varepsilon\,\mathbf{el}(A)$

PR $[A\,B\,a] \therefore \mathrm{Hp}(2) . \supset :$

3. $[C] : C\,\varepsilon\,\mathbf{at}(B) . \supset . C\,\varepsilon\,\mathbf{at}(A) :$ — [**D1**; 1; 2]

$B\,\varepsilon\,\mathbf{el}(A)$ — [*T1*; **D2**; 2; 3]

P6 $[A\,B\,a] : A\,\varepsilon\,\mathbf{Kl}(a) . B\,\varepsilon\,\mathbf{el}(A) . \supset . [\exists E\,F] . E\,\varepsilon\,a . F\,\varepsilon\,\mathbf{el}(E) . F\,\varepsilon\,\mathbf{el}(B)$

PR $[A\,B\,a] \therefore \mathrm{Hp}(2) . \supset :$

$[\exists F] :$

3. $F\,\varepsilon\,\mathbf{at}(B) .$ — [*T1*; *V1*; 2]

4. $F\,\varepsilon\,\mathbf{at}(A) .$ — [*P2*; 3; 2]

$[\exists E] .$

5. $F\,\varepsilon\,\mathbf{at}(E) .$

6. $E\,\varepsilon\,a :$ — [**D1**; 1; 4] (5, 6)

$[\exists E\,F] . E\,\varepsilon\,a . F\,\varepsilon\,\mathbf{el}(E) . F\,\varepsilon\,\mathbf{el}(B)$ — [*P3*; 6; 5; 3]

P7 $[A\,C\,D\,a] .\cdot. [B] : B \varepsilon a . \supset . B \varepsilon \mathbf{el}(A) : C \varepsilon \mathbf{at}(D) . D \varepsilon a . \supset . C \varepsilon \mathbf{at}(A)$ [*P2*]

P8 $[A\,C\,a] .\cdot. [B] : B \varepsilon \mathbf{el}(A) . \supset . [\exists E\,F] . E \varepsilon a . E \varepsilon \mathbf{el}(E) . F \varepsilon \mathbf{el}(B) : C \varepsilon \mathbf{at}(A) . \supset .$
$[\exists D] . C \varepsilon \mathbf{at}(D) . D \varepsilon a$

PR $[A\,C\,a] .\cdot. \mathrm{Hp}(2) : \supset :$

$[\exists E\,F] :$

3. $E \varepsilon a .$ [*P3*; 1; 2]
4. $F \varepsilon \mathbf{el}(E) .$
5. $F \varepsilon \mathbf{el}(C) .$

$[\exists V] .$

6. $V \varepsilon \mathbf{at}(F) .$ [*T1*; *V1*; 4]
7. $V \varepsilon \mathbf{at}(E) .$ [*P2*; 6; 4]
8. $V \varepsilon \mathbf{at}(C) .$ [*P2*; 6; 5]
9. $V = C :$ [*S2*; 2; 8]

$[\exists D] . C \varepsilon \mathbf{at}(D) . D \varepsilon a$ [7; 9; 3]

D1 $[A\,a] .\cdot. A \varepsilon A : [B] : B \varepsilon a . \supset . B \varepsilon \mathbf{el}(A) : [B] : B \varepsilon \mathbf{el}(A) . \supset . [\exists E\,F] . E \varepsilon a .$
$F \varepsilon \mathbf{el}(E) . F \varepsilon \mathbf{el}(B) : \equiv . A \varepsilon \mathbf{Kl}(a)$ [*P7*; *P8*; **D1**; *T1*; *P5*; *P6*]

A8 $[A\,a] : A \varepsilon a . \supset A \varepsilon \mathbf{el}(\mathbf{Kl}(a))$

PR $[A\,a] : \mathrm{Hp}(1) . \supset .$

$[\exists B] .$

2. $B \varepsilon \mathbf{Kl}(a) .$ [*A4*; 1]
3. $B = \mathbf{Kl}(a) .$ [*T3*; *A3*; 2]
4. $A \varepsilon \mathbf{el}(B) .$ [*D1*; 2; 1]

$A \varepsilon \mathbf{el}(\mathbf{Kl}(a))$ [4; 3]

P9 $[A\,B\,a] .\cdot. B \varepsilon a : [C] : C \varepsilon a . \supset . C \varepsilon \mathbf{el}(A) : \supset . A \varepsilon A$ [*A1*]

P10 $[A\,B\,T\,a] : : A \varepsilon \mathbf{el}(B) .\cdot. [C] .\cdot. C \varepsilon T . \equiv : [B] : B \varepsilon a . \supset . B \varepsilon \mathbf{el}(C) : [B] :$
$B \varepsilon \mathbf{el}(C) . \supset . [\exists E\,F] : E \varepsilon a . F \varepsilon \mathbf{el}(E) . F \varepsilon \mathbf{el}(B) .\cdot. B \varepsilon a . \supset . A \varepsilon \mathbf{el}(T)$

PR $[A\,B\,T\,a] : : \mathrm{Hp}(3) .\cdot. \supset .\cdot.$

4. $[C] .\cdot. C \varepsilon \mathbf{Kl}(a) . \equiv : [B] : B \varepsilon a . \supset . B \varepsilon \mathbf{el}(C) :$
$[B] : B \varepsilon \mathbf{el}(C) . \supset . [\exists E\,F] . E \varepsilon a . F \varepsilon \mathbf{el}(E) .$
$F \varepsilon \mathbf{el}(B) .\cdot.$ [*D1*; *P9*; 3]
5. $[C] : C \varepsilon \mathbf{Kl}(a) . \equiv . C \varepsilon T :$ [2; 4]
6. $B \varepsilon \mathbf{el}(\mathbf{Kl}(a)) .$ [*A8*; 3]
7. $B \varepsilon \mathbf{el}(T) .$ [*E2*; 5; 6]

$A \varepsilon \mathbf{el}(T)$ [*A2*; 1; 7]

P11 $[A\,B] :\cdot: B \varepsilon B : : [T\,a] : : [C] .\cdot. C \varepsilon T . \equiv : [V] : V \varepsilon a . \supset . V \varepsilon \mathbf{el}(C) : [V] :$
$V \varepsilon \mathbf{el}(C) . \supset . [\exists E\,F] . E \varepsilon a . F \varepsilon \mathbf{el}(E) . F \varepsilon \mathbf{el}(V) .\cdot. B \varepsilon \mathbf{el}(B) . B \varepsilon a . \supset .$
$A \varepsilon \mathbf{el}(T) : : \supset . A \varepsilon \mathbf{el}(B)$

PR $[A\,B] :\cdot: \mathrm{Hp}\,(2) : : \supset .\cdot.$

3. $B \varepsilon \mathbf{el}(B) :$ [*A5*; 1]
4. $[C] : C \varepsilon \mathbf{el}(B) . \supset . [\exists E\,F] . E \varepsilon \mathbf{el}(B) . F \varepsilon \mathbf{el}(E) .$
$F \varepsilon \mathbf{el}(C) :$ [*T1*; *A5*; 3]
5. $B = \mathbf{Kl}(\mathbf{el}(B)) .\cdot.$ [*D1*; *A3*; 1; 4]
6. $[C] .\cdot. C \varepsilon \mathbf{Kl}(\mathbf{el}(B)) . \equiv : [V] : V \varepsilon \mathbf{el}(B) . \supset . V \varepsilon \mathbf{el}(C) :$
$[V] : V \varepsilon \mathbf{el}(C) . \supset . [\exists E\,F] . E \varepsilon \mathbf{el}(B) . F \varepsilon \mathbf{el}(E) .$
$F \varepsilon \mathbf{el}(V) .\cdot.$ [*D1*; *P9*; *A1*; 3]

7.	$A\,\varepsilon\,\mathbf{el}(\mathbf{Kl}(\mathbf{el}(B)))\,.$	[2; 6; 3]
	$A\,\varepsilon\,\mathbf{el}(B)$	[$E1$; 5; 7]

A $[AB]::A\,\varepsilon\,\mathbf{el}(B)\,.\equiv::B\,\varepsilon\,B::[Ta]::[C]\,.\because.\,C\,\varepsilon\,T\,.\equiv:[B]:B\,\varepsilon\,a\,.\supset.\,B\,\varepsilon\,\mathbf{el}$ $(C):[B]:B\,\varepsilon\,\mathbf{el}(C):\supset.[\exists EF]\,.\,E\,\varepsilon\,a\,.\,F\,\varepsilon\,\mathbf{el}(E)\,.\,F\,\varepsilon\,\mathbf{el}(B)\,.\because.\,B\,\varepsilon\,\mathbf{el}(B)\,.\,B\,\varepsilon\,a\,.$ $.\,A\,\varepsilon\,\mathbf{el}(T)$ [$A1$; $P10$; $P11$]

D3 $[A]:A\,\varepsilon\,\mathbf{at}(A)\,.\equiv.\,A\,\varepsilon\,\mathbf{atm}$

$P12$ $[AB]:A\,\varepsilon\,\mathbf{atm}\,.\,B\,\varepsilon\,\mathbf{el}(A)\,.\supset.\,B=A$

PR $[AB]:\mathrm{Hp}(2)\,.\supset.$

	$[\exists C]\,.$	
3.	$C\,\varepsilon\,\mathbf{at}(B)\,.$	[$T1$; $V1$; 2]
4.	$C\,\varepsilon\,\mathbf{at}(A)\,.$	[$P2$; 3; 2]
5.	$C=A\,.$	[**D3**; $S2$; 1; 4]
6.	$A\,\varepsilon\,\mathbf{el}(B)\,.$	[$P3$; 3; 5]
	$B=A$	[$A7$; 2; 6]

$P13$ $[A]\,.\because.\,A\,\varepsilon\,A:[B]:B\,\varepsilon\,\mathbf{el}(A)\,.\supset.\,B=A:\supset.\,A\,\varepsilon\,\mathbf{atm}$

PR $[A]\,.\because.\,\mathrm{Hp}(2):\supset.$

	$[\exists C]\,.$	
3.	$C\,\varepsilon\,\mathbf{at}(A)\,.$	[$V1$; 1]
4.	$C=A\,.$	[$P3$; 2; 3]
	$A\,\varepsilon\,\mathbf{atm}$	[**D3**; 3; 4]

DI $[A]\,.\because.\,A\,\varepsilon\,A:[B]:B\,\varepsilon\,\mathbf{el}(A)\,.\supset B=A:\equiv.\,A\,\varepsilon\,\mathbf{atm}$ [$T1$; $P12$; $P13$]

DII $[AB]:A\,\varepsilon\,\mathbf{atm}\,.\,A\,\varepsilon\,\mathbf{el}(B)\,.\equiv.\,A\,\varepsilon\,\mathbf{at}(B)$ [**D3**; $P2$; $V1$; $S2$; $P3$]

Thus, A, V, $D1$, DI and DII follow from $\{S1, S2, S3, S4\}$. Since, *cf.* **4.1**, in the field of $\{A, V\}$ the formulas $D1$, DI and DII are the definitions of "**Kl**", "**atm**" and "**at**" by "**el**" respectively, it had to be proved here that they are the consequences of $\{S1, S2, S3, S4\}$.

4.3 It follows from **4.1** and **4.2** that $\{A, V, D1, D\mathrm{I}, D\mathrm{II}\} \rightleftarrows \{S1, S2, S3, S4,$ **D1**, **D2**, **D3**$\}$. It shows that the systems $\{A, V\}$ and $\{S1, S2, S3, S4\}$ are inferentially equivalent.

5. In this section it will be shown that 1) the system of atomistic mereology is consistent, and that 2) in the axiom-systems $\{A, V\}$ and $\{S1, S2, S3, S4\}$ the axioms belonging to one of these axiomatizations are mutually independent.

5.1 Below, in **5.3** and **5.4**, in order to obtain the desired proofs we shall have to use some systems. Namely:

System **A**: Leśniewski's ontology extended by an additional axiom:

L $\mathfrak{A}\,\varepsilon\,\mathfrak{A}\,.\,\mathfrak{B}\,\varepsilon\,\mathfrak{B}\,.\sim(\mathfrak{A}=\mathfrak{B})$

(where "$\mathfrak{A}$" and "$\mathfrak{B}$" are the name constants), i.e., by an assumption that there are at least two different objects.

System **B**: System $\{A, V\}$ of atomistic mereology.

System **C**: System $\{A\}$ of (general) mereology, *cf*. **2**, extended by three additional axioms. Namely, axiom L and

$K1$ $[A] : A \varepsilon A . \supset . [\exists B] . B \varepsilon \mathbf{atm}$
$K2$ $[A B] : A \varepsilon \mathbf{atm} . B \varepsilon \mathbf{atm} . \supset . A = B$

Obviously, $K1$ and $K2$ assume that if in mereology there is an object, then in its field there is also one and only one object which is a mereological atom. Concerning the formulation of $K1$ and $K2$ it should be remarked that an expression "**atm**" is used merely as a convenient abbreviation. In $K1$ and $K2$ the formulas in which "**atm**" occurs can be substituted by the formulas which do not contain this defined term, *cf.* DI in **4.1**.

Concerning the systems **A**, **B** and **C** it should be noticed that 1) the consistency of **B** will be proved in **5.2** below, and that 2) the systems **A** and **C** are also consistent, but the easy proofs of their consistency are omitted in this paper. Moreover, we have to notice that the rules of procedure of every theory which is based on Leśniewski's ontology are exactly the same as the rules of the latter system. Hence, obtaining a needed interpretation of a system under investigation in the field of **A**, **B** or **C** we can only be concerned with the syntactical forms of the involved formulas.

5.2 The consistency of atomistic mereology. In [8] Lejewski has proved that the system of (general) mereology is consistent, since its single axiom A has an interpretation in Leśniewski's protothetic. Using the same interpretation and the same mode of reasoning which are given in [8] we shall show that the proper axiom of atomistic mereology, viz. axiom V, possesses the same property. Namely, let us understand the nominal variables of mereology as propositional variables, "ε"—as the functor of conjunction, and "**el**"—as the functor of assertion, *cf.* [8], p. 326. Moreover, due to definition $Df1$, *cf.* **1**, the ontological functor "=" can also be interpreted as the functor of conjunction. Then, the following definition and the theses which are valid in the field of protothetic:

$D1$ $[p] . p \equiv \mathbf{as}(p)$
$Z1$ $[p] . p \supset p$
$Z2$ $[q r] : r . \mathbf{as}(q) . \supset . r . q$
$Z3$ $[f q] : [r] . f(r) . \supset . f(q)$

imply at once in protothetic:

V' $[p] : : p . p . \supset \therefore [\exists q] \therefore q . \mathbf{as}(p) : [r] : r . \mathbf{as}(q) . \supset . r . q$

Since in Lejewski's interpretation V' corresponds to the axiom V and the same interpretation verifies axiom A, the consistency of atomistic mereology is proven. It should be remarked that in [5] Clay discusses some other models for atomistic mereology.

5.3 The mutual independency of the axioms A and V. (a) Assume system **A** and introduce the following definition:

$Df\mathcal{A}$ $[A B] : A \varepsilon B . \equiv . A \varepsilon \mathbf{el}_1(B)$

in its field. Then, the following formula which corresponds to axiom V:

V'' $[A] : : A \varepsilon A . \supset \therefore [\exists B] \therefore B \varepsilon \mathbf{el}_1(A) : [C] : C \varepsilon \mathbf{el}_1(B) . \supset_{\circ} C = B$ $\quad [Df\mathcal{A};\ T2]$

is provable in the field of ontology. On the other hand formula:

$A1'$ $[AB]:A\,\varepsilon\,\mathbf{el}_1(B).\supset.B\,\varepsilon\,B$

fails in the field of system A. Namely, by $Df\mathscr{A}$, $A1'$ is inferentially equivalent to:

$Z1$ $[AB]:A\,\varepsilon\,B.\supset.B\,\varepsilon\,B$

Hence, by L and $Z1$, we have:

$Z2$ $\mathfrak{A}\cup\mathfrak{B}\,\varepsilon\,\mathfrak{A}\cup\mathfrak{B}$

But, in the field of ontology $Z2$ implies at once:

$Z3$ $[CD]:C\,\varepsilon\,\mathfrak{A}\cup\mathfrak{B}.D\,\varepsilon\,\mathfrak{A}\cup\mathfrak{B}.\supset.C=D$

Whence,

$Z4$ $\mathfrak{A}=\mathfrak{B}$ $[L;\ Z3]$

which contradicts L. Since $A1'$ corresponds to $A1$ which is a consequence of A, the latter formula fails too in the field of system **A**. Therefore, A is not a consequence of V. (b) Assume system **C**. Hence we have at our disposal axiom A and all its consequences given in section **2**. Moreover, we can use the definitions DI and DII introduced in **4.1**. Then:

$Z1$ $[ABV]:A\,\varepsilon\,A.B\,\varepsilon\,\mathbf{atm}.\sim(B\,\varepsilon\,\mathbf{at}(A)).V\,\varepsilon\,\mathbf{el}(B).\supset.\sim(V\,\varepsilon\,\mathbf{el}(A))$ $[D\mathrm{I};\ D\mathrm{II}]$

$Z2$ $[ABC]:B\,\varepsilon\,\mathbf{atm}.B\,\varepsilon\,\mathbf{ex}(\mathrm{A}).C\,\varepsilon\,\mathbf{el}(\mathrm{A}).C\,\varepsilon\,\mathbf{atm}.\supset.\sim(C\,\varepsilon\,\mathbf{atm})$ $[D\mathrm{I};\ K2;\ A14]$

$Z3$ $[AB]::A\,\varepsilon\,A.B\,\varepsilon\,\mathbf{atm}.\supset{:.}\,B\,\varepsilon\,\mathbf{at}(A).\vee:[C]:C\,\varepsilon\,\mathbf{el}(A).\supset.\sim(C\,\varepsilon\,\mathbf{atm})$ $[T1;\ D3;\ Z1;\ Z2]$

$Z4$ $[BCDE]:D\,\varepsilon\,\mathbf{ex}(B).E\,\varepsilon\,\mathbf{at}(D).C\,\varepsilon\,\mathbf{el}(B).\supset.\sim(C\,\varepsilon\,\mathbf{atm})$ $[D\mathrm{II};\ A13;\ Z2]$

$Z5$ $[AB]::A\,\varepsilon\,A.B\,\varepsilon\,B.\sim(A\,\varepsilon\,\mathbf{el}(B)).\supset{.\!\cdot\!.}\,[\exists D]{.\!\cdot\!.}\,D\,\varepsilon\,D:[K]:K\,\varepsilon\,\mathbf{el}(D).\supset.\sim(K\,\varepsilon\,\mathbf{atm})$

PR $[AB]{:\cdot:}\,\mathrm{Hp}(3).\supset::$

$[\exists C]::$

4. $C\,\varepsilon\,\mathbf{ex}(B){.\!\cdot\!.}$ $[A6;\ D3;\ 1;\ 2;\ 3]$

$[\exists V]{.\!\cdot\!.}$

5. $V\,\varepsilon\,\mathbf{atm}{.\!\cdot\!.}$ $[T1;\ K1;\ D\mathrm{I};\ 4]$

6. $V\,\varepsilon\,\mathbf{at}(C).\vee:[K]:K\,\varepsilon\,\mathbf{el}(C).\supset.\sim(K\,\varepsilon\,\mathbf{atm}){.\!\cdot\!.}$ $[T1;\ Z3;\ 4;\ 5]$

7. $[K]:K\,\varepsilon\,\mathbf{el}(B).\supset.\sim(K\,\varepsilon\,\mathbf{atm}):\vee:[K]:K\,\varepsilon\,\mathbf{el}(C).\supset.\sim(K\,\varepsilon\,\mathbf{atm})..$ $[6;\ Z4;\ 4]$

$[\exists D]{.\!\cdot\!.}\,D\,\varepsilon\,D:[K]:K\,\varepsilon\,\mathbf{el}(D).\supset.\sim(K\,\varepsilon\,\mathbf{atm})$ $[7;\ 2;\ 4]$

$Z6$ $[AB]::A\,\varepsilon\,A.B\,\varepsilon\,B.\sim(A=B).\supset{.\!\cdot\!.}\,[\exists D]{.\!\cdot\!.}\,D\,\varepsilon\,D:[K]:K\,\varepsilon\,\mathbf{el}(D).\supset.\sim(K\,\varepsilon\,\mathbf{atm})$ $[A7;\ Z5]$

$Z7$ $[\exists D]{.\!\cdot\!.}\,D\,\varepsilon\,D:[K]:K\,\varepsilon\,\mathbf{el}(D).\supset.[\exists N].N\,\varepsilon\,\mathbf{el}(K).\sim(N=K)$ $[T1;\ Z6;\ L;\ D\mathrm{I}]$

Since $Z7$ is a negation of V and system **C** is consistent, it proves that axiom A does not imply V. Hence, points (a) and (b) show that the axioms A and V are mutually independent.

5.4 The mutual independency of the axioms *S1*, *S2*, *S3* and *S4*. (a) Assume system **A** and introduce the following definition in its field:

Df $\mathscr{B}$ $[AB]:A\,\varepsilon\,B\,.\equiv.\,A\,\varepsilon\,\mathbf{at}_1(B)$

Then, the formulas:

S2' $[ABC]:A\,\varepsilon\,\mathbf{at}_1(B)\,.\,C\,\varepsilon\,\mathbf{at}_1(A)\,.\supset.\;C = A$ $[Df\ \mathscr{B};\ T2]$
S3' $[AB]\,.\!\cdot\,.\,A\,\varepsilon\,A\,.\,B\,\varepsilon\,B:[C]:C\,\varepsilon\,\mathbf{at}_1(A)\,.\equiv.\,C\,\varepsilon\,\mathbf{at}_1(B):\supset.A = B$ $[Df\ \mathscr{B};\ T2]$
S4' $[Aa]::A\,\varepsilon\,a\,.\supset\!\cdot\,.\,[\exists B]\,.\!\cdot\,.\,[\exists E]\,.\,E\,\varepsilon\,\mathbf{at}_1(B):[C]:C\,\varepsilon\,\mathbf{at}_1(B)\,.\equiv.\,[\exists D]\,.$
$C\,\varepsilon\,\mathbf{at}_1(D)\,.\,D\,\varepsilon\,a$ $[Df\ \mathscr{B};\ T1;\ T4]$

which corresponds to the axioms *S2*, *S3* and *S4* respectively are provable in ontology. On the other hand in the field of system **A** formula:

S1' $[AB]:A\,\varepsilon\,\mathbf{at}_1(B)\,.\supset.\,B\,\varepsilon\,B$

which corresponds to the axiom *S1* fails for exactly the same reason for which formula *A1'* is rejected in **5.3**, point (a). Hence, *S1* is not a consequence of *S2*, *S3* and *S4*. (b) Assume system **B**, i.e., the axioms *A* and *V* of atomistic mereology. Hence, we have at our disposal all theorems given in sections **2** and **4.1**. Now, add the following definition:

Df $\mathscr{C}$ $[AB]:A\,\varepsilon\,A\,.\,[\exists F]\,.\,F\,\varepsilon\,\mathbf{at}(A)\,.\,F\,\varepsilon\,\mathbf{at}(B)\,.\equiv.\,A\,\varepsilon\,\mathbf{at}_2(B)$

to this system. Then:

S1' $[AB]:A\,\varepsilon\,\mathbf{at}_2(B)\,.\supset.\,B\,\varepsilon\,B$ $[Df\ \mathscr{C};\ D\mathrm{II};\ A1]$
Z1 $[ABC]\,.\!\cdot\,.\,[C]:C\,\varepsilon\,\mathbf{at}_2(A)\,.\supset.\;C\varepsilon\,\mathbf{at}_2(B):D\,\varepsilon\,\mathbf{el}(A)\,.\supset.\,[\exists F]\,.\,F\,\varepsilon\,\mathbf{el}(D)\,.$
$F\,\varepsilon\,\mathbf{el}(B)$
PR $[ABD]\,.\!\cdot\,.\,\mathrm{Hp}(2):\supset.$
3. $D\,\varepsilon\,\mathbf{at}_2(A)\,.$ $[T1;\ A1;\ V;\ D\mathrm{I};\ D\mathrm{II};\ A2;\ Df\ \mathscr{C};\ 2]$
4. $D\,\varepsilon\,\mathbf{at}_2(B)\,.$ $[1;\ 3]$
$[\exists F]\,.\,F\,\varepsilon\,\mathbf{el}(D)\,.\,F\,\varepsilon\,\mathbf{el}(B)$ $[Df\ \mathscr{C};\ D\mathrm{II};\ 4]$

S3' $[AB]\,.\!\cdot\,.\,A\,\varepsilon\,A\,.\;B\varepsilon\,B:[C]:C\,\varepsilon\,\mathbf{at}_2(A)\,.\equiv.\,C\,\varepsilon\,\mathbf{at}_2(B):\supset.A = B$ $[Z1;\ A6;\ A7]$
Z2 $[A]:A\,\varepsilon\,A\,.\supset.\,[\exists B]\,.\,B\,\varepsilon\,\mathbf{at}_2(A)$ $[T1;\ V1;\ A5;\ D\mathrm{I};\ D\mathrm{II};\ Df\ \mathscr{C}]$
Z3 $[BCa]:B\,\varepsilon\,\mathbf{Kl}(a)\,.\;C\varepsilon\,\mathbf{at}_2(B)\,.\supset.\,[\exists D]\,.\,C\,\varepsilon\,\mathbf{at}_2(D)\,.\,D\,\varepsilon\,a$
$[T1;\ Df\ \mathscr{C};\ D\mathrm{II};\ D1;D\mathrm{I}]$
Z4 $[BCDa]:B\,\varepsilon\,\mathbf{Kl}(a)\,.\,C\,\varepsilon\,\mathbf{at}_2(D)\,.\,D\,\varepsilon\,a\,.\supset.\,C\,\varepsilon\,\mathbf{at}_2(B)$
PR $[BCDa]:\mathrm{Hp}(3)\,.\supset.$
4. $D\,\varepsilon\,\mathbf{el}(B)\,.$ $[D1;\ 1;\ 3]$
$[\exists F]\,.$
5. $F\,\varepsilon\,\mathbf{at}(C)\,.$
6. $F\,\varepsilon\,\mathbf{at}(B)\,.$ $[Df\ \mathscr{C};\ 2]$
7. $F\,\varepsilon\,\mathbf{at}(B)\,.$ $[D\mathrm{II};\ A2;\ 6;\ 4]$
$C\,\varepsilon\,\mathbf{at}_2(B)$ $[Df\ \mathscr{C};\ T1;\ 2;\ 5;\ 7]$

S4' $[Aa]::A\,\varepsilon\,a\,.\supset\!\cdot\,.\,[\exists B]\,.\!\cdot\,.\,[\exists E]:E\,\varepsilon\,\mathbf{at}_2(B):[C]:C\,\varepsilon\,\mathbf{at}_2(B)\,.\equiv.\,[\exists D]\,.$
$D\,\varepsilon\,\mathbf{at}_2(D)\,.\,D\,\varepsilon\,a$ $[A4;\ T1;\ Z2;\ Z3;\ Z4]$

Thus, the formulas *S1'*, *S3'* and *S4'* which correspond to the axioms *S1*, *S3* and *S4* respectively are provable in atomistic mereology. On the other hand formula:

$S2'$ $[A B C] : A \, \varepsilon \, \mathbf{at}_2(B) \, . \, C \, \varepsilon \, \mathbf{at}_2(A) \, . \supset . \, C = A$

is not generally valid in the field of the latter system. Namely, in its field we have:

$Z5$ $[A B C] : B \, \varepsilon \, \mathbf{ex}(C) \, . \, A \, \varepsilon \, \mathbf{at}_2(B) \, . \, C \, \varepsilon \, \mathbf{at}_2(A) \, . \supset . \sim (C = A)$

PR $[A B C] \therefore \mathrm{Hp}(3) \, . \supset :$

$[\exists F] \, .$

4. $F \, \varepsilon \, \mathbf{el}(A) \, .$ [$Df\ \mathcal{C}$; DII; 2]

5. $F \, \varepsilon \, \mathbf{el}(B) \, .$

6. $\sim (B \, \varepsilon \, \mathbf{ex}(A)) :$ [$T1$; $D3$; 1; 2; 5; 4]

7. $B \, \varepsilon \, \mathbf{ex}(A) \, . \vee . \sim (C = A) :$ [1]

$\sim (C = A)$ [7; 6]

which proves that $S2$ is not a consequence of $S1$, $S3$ and $S4$.

(c) Assume system **C**. Hence, we have at our disposal axiom A and all its consequences given in section **2.** Additionally, we add the definitions DI, DII and:

$Df\ \mathfrak{D}$ $[A B] : A \, \varepsilon \, \mathbf{atm} \, . \, B \, \varepsilon \, B \, . \equiv . \, A \, \varepsilon \, \mathbf{at}_3(B)$

to this system. Then:

$S1'$ $[A B] : A \, \varepsilon \, \mathbf{at}_3(B) \, . \supset . \, B \, \varepsilon \, B$ [$Df\ \mathfrak{D}$]

$S2'$ $[A B C] : A \, \varepsilon \, \mathbf{at}_3(B) \, . \, C \, \varepsilon \, \mathbf{at}_3(A) \, . \supset . \, C = A$ [$Df\ \mathfrak{D}$; $K2$]

$S4'$ $[A a] :: A \, \varepsilon \, a \, . \supset \therefore [\exists B] \therefore [\exists E] \, . \, E \, \varepsilon \, \mathbf{at}_3(B) : [C] : C \, \varepsilon \, \mathbf{at}_3(B) \, . \equiv . \, [\exists D] \, .$
$C \, \varepsilon \, \mathbf{at}_3(D) \, . \, D \, \varepsilon \, a$

PR $[A a] :: \mathrm{Hp}(1) \, . \supset ::$

$[\exists B] ::$

2. $B \, \varepsilon \, \mathbf{Kl}(a) \therefore$ [$A4$; 1]

3. $[C] : C \, \varepsilon \, \mathbf{at}_3(B) \, . \equiv . \, [\exists D] \, . \, C \, \varepsilon \, \mathbf{at}_3(D) \, . \, D \, \varepsilon \, a \therefore$
[$T1$; $A5$; $Df\ \mathfrak{D}$; DI; 2]

$[\exists E] \, .$

4. $E \, \varepsilon \, \mathbf{at}_3(B) ::$ [$T1$; $K1$; $Df\ \mathfrak{D}$; 2]

$[\exists B] \therefore [\exists E] \, . \, E \, \varepsilon \, \mathbf{at}_3(B) : [C] : C \, \varepsilon \, \mathbf{at}_3(B) \, . \equiv . \, [\exists D] \, .$
$C \, \varepsilon \, \mathbf{at}_3(D) \, . \, D \, \varepsilon \, a$ [5; 3; 4]

Thus formulas $S1'$, $S2'$ and $S4'$ which correspond to the axioms $S1$, $S2$ and $S4$ respectively are provable in **C**. On the other hand formula

$S3'$ $[A B] \therefore A \, \varepsilon \, A \, . \, B \, \varepsilon \, B : [C] : C \, \varepsilon \, \mathbf{at}_3(A) \, . \equiv . \, C \, \varepsilon \, \mathbf{at}_3(B) : \supset . \, A = B$

which corresponds to the axiom $S3$ fails in the field of system **C**, since we have:

$Z1$ $\mathfrak{A} \, \varepsilon \, \mathfrak{A} \, . \, \mathfrak{B} \, \varepsilon \, \mathfrak{B} \, . \supset : [C] : C \, \varepsilon \, \mathbf{at}_3(\mathfrak{A}) \, . \equiv . \, C \, \varepsilon \, \mathbf{at}_3(\mathfrak{B}) : \sim (\mathfrak{A} = \mathfrak{B})$ [L; $Df\ \mathfrak{D}$]

Thus, $S3$ is not a consequence of $S1$, $S2$ and $S4$. I like to note here that I owe to Professor R. E. Clay a suggestion concerning the proof presented above.

(d) Assume system **A** and add the following definition

$Df\ \mathcal{E}\quad [A\,B]: A\,\varepsilon\,A\,.\,A = B\,.\equiv.\,A\,\varepsilon\,\mathbf{at}_4(B)$

to it. Then:

$S1'\quad [A\,B]: A\,\varepsilon\,\mathbf{at}_4(B)\,.\supset.\,B\,\varepsilon\,B$ $\quad[T1;\ Df\ \mathcal{E};\ Df1]$

$S2'\quad [A\,B\,C]: A\,\varepsilon\,\mathbf{at}_4(B)\,.\,C\,\varepsilon\,\mathbf{at}_4(A)\,.\supset.\,C = A$ $\quad[Df\ \mathcal{E}]$

$S3'\quad [A\,B]\,\therefore A\,\varepsilon\,A\,.\,B\,\varepsilon\,B:[C]: C\,\varepsilon\,\mathbf{at}_4(A)\,.\equiv.\,C\,\varepsilon\,\mathbf{at}_4(B):\supset.\,A = B$ $\quad[Df1;\ Df\ \mathcal{E}]$

which correspond to the axioms *S1*, *S2* and *S3* respectively are provable in **A**. But:

$Z1\quad [A\,B\,K]\,\therefore A\,\varepsilon\,A\,.\,B\,\varepsilon\,B\,.\sim(A = B):[C\,D]: C\,\varepsilon\,\mathbf{at}_4(D)\,.\,D\,\varepsilon\,A\cup B\,.$
$\quad\supset.\,C\,\varepsilon\,\mathbf{at}_4(K):\supset.\sim([C\,D]: C\,\varepsilon\,\mathbf{at}_4(D)\,.\,D\,\varepsilon\,A\cup B\,.\supset.\,C\,\varepsilon\,\mathbf{at}_4(K))$

PR $[A\,B\,K]\,\therefore \mathrm{Hp}(4):\supset.$

5. $A\,\varepsilon\,\mathbf{at}_4(A)\,.$ $\quad[Df1;\ Df\ \mathcal{E};\ 1]$
6. $B\,\varepsilon\,\mathbf{at}_4(B)\,.$ $\quad[Df1;\ Df\ \mathcal{E};\ 2]$
7. $A = K\,.$ $\quad[Df\ \mathcal{E};\ Df3;\ 1;\ 4;\ 5]$
8. $B = K\,.$ $\quad[Df\ \mathcal{E};\ Df3;\ 2;\ 4;\ 6]$
9. $A = B\,.$ $\quad[7;\ 8]$

$\sim([C\,D]: C\,\varepsilon\,\mathbf{at}_4(D)\,.\,D\,\varepsilon\,A\cup B\,.\supset.\,C\,\varepsilon\,\mathbf{at}_4(K))$ $\quad[3;\ 9]$

$Z2\quad [A\,B]::A\,\varepsilon\,A\,.\,B\,\varepsilon\,B\,.\sim(A = B)\,.\supset\therefore[\exists Va]\,\therefore V\varepsilon\,a:[K]:[\exists E]\,.$
$\quad E\,\varepsilon\,\mathbf{at}_4(K)\,.\supset.\sim([C]: C\,\varepsilon\,\mathbf{at}_4(K)\,.\equiv.\,[\exists D]\,.\,C\,\varepsilon\,\mathbf{at}_4(D)\,.\,D\,\varepsilon\,a)$

PR $[A\,B]::\mathrm{Hp}(3)\,.\supset\therefore$

4. $A\,\varepsilon\,A\cup B:$ $\quad[Df3;\ 1]$
5. $[K]:[\exists E]\,.\,E\,\varepsilon\,\mathbf{at}_4(K)\,.\supset.\sim([C]: C\,\varepsilon\,\mathbf{at}_4(K)\,.\equiv.\,[\exists D]\,.$
$\quad C\,\varepsilon\,\mathbf{at}_4(D)\,.\,D\,\varepsilon\,A\cup B)\,\therefore$ $\quad[Z1;\ 1;\ 2;\ 3]$

$[\exists Va]\,\therefore V\varepsilon\,a:[K]:[\exists E]\,.\,E\,\varepsilon\,\mathbf{at}_4(K)\,.\supset.\sim([C]:$
$C\,\varepsilon\,\mathbf{at}_4(K)\,.\equiv.\,[\exists D]\,.\,C\,\varepsilon\,\mathbf{at}_4(D)\,.\,D\,\varepsilon\,a)$ $\quad[4;\ 5]$

$Z3\quad [\exists A\,a]\,\therefore A\,\varepsilon\,a:[B]:[\exists E]\,.\,E\,\varepsilon\,\mathbf{at}_4(B)\,.\supset.\sim([C]: C\,\varepsilon\,\mathbf{at}_4(B)\,.\equiv.\,[\exists D]\,.$
$\quad C\,\varepsilon\,\mathbf{at}_4(D)\,.\,D\,\varepsilon\,a)$ $\quad[Z2;\ L]$

Since *Z3* is a negation of *S4*, and system **A** is consistent, it proves that *S4* is not a consequence of *S1*, *S2* and *S3*.

The points (a)-(d) show that the axioms *S1*, *S2*, *S3* and *S4* are mutually independent.

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