Notre Dame Journal of Formal Logic
Volume XII, Number 2, April 1971

DUALITY IN FINITE MANY-VALUED LOGIC

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1. *Introduction* The notion of duality is a familiar one in the two-valued propositional calculus. In view of the one-many correspondence between a truth-function and well-formed formulas (proposition-letter formulas), a well-formed formula can have many different duals—the well-formed formulas that have the truth-table of the dual truth-function. We can obtain a dual (called the "principal dual" in Church [1]) of a well-formed formula in a mechanical way by replacing each occurrence of a connective in the well-formed formula by the occurrence of its dual connective. One of the important principles of duality is that if two well-formed formulas are truth-functionally equivalent, then their duals are also truth-functionally equivalent. The interest in duality lies in the fact that by appealing to this principle of duality, we can assert a dual equivalence (or theorem) corresponding to a given equivalence. For example, we can assert the dual equivalence

$$p \vee (p \wedge q) \Leftrightarrow p$$

when we have asserted the equivalence

$$p \wedge (p \vee q) \Leftrightarrow p .$$

Given a truth-function $f(x_1, x_2, \ldots, x_n)$ in the two-valued propositional calculus, its dual is defined to be the truth-function $\neg f(\neg x_1, \neg x_2, \ldots, \neg x_n)$ where '$\neg$' is the singulary connective 'negation'. As there is only one negation connective in the two-valued propositional calculus, the dual of a truth-function (alternately, the principal dual of a well-formed formula) can be uniquely determined. We notice that the definition of duality is such that

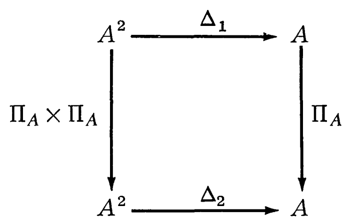
(i) the identity and the negation functions are self-dual;
(ii) the dual of the dual of a truth-function is the original truth-function (alternately, the principal dual of the principal dual of a well-formed formula is the same as the well-formed formula);
(iii) the semantic notions of tautology and contradiction are mutually dual.

Received July 5, 1969

In this paper an attempt is made to generalize the notion of duality to many-valued logic requiring that the definition of duality must satisfy conditions (i), (ii) and (iii) above as is the case in two-valued propositional calculus. It is suggested that a natural generalization of the definition of duality must be such that it is possible to define the dual of a product connective[1] of two binary connectives in a natural way.

In many-valued logic, there are many negation connectives. Intuitively one would expect that corresponding to each definition of negation, one would have a different definition of the dual of a truth-function (alternately, a different definition of the principal dual of a well-formed formula). Though this is the case not every definition of negation in many-valued logic would result in a definition of duality that satisfies conditions (i), (ii), (iii) and the possibility of defining the dual of a product connective in a natural manner.

2. *Diagrammatic representation of the notion of duality* Let A be a non-empty finite set of truth-values. As negation is a singulary connective it can be considered to be a mapping $A \longrightarrow A$. The conditions (i), (ii), and (iii) in section 1, which are required to be satisfied by a definition of duality, restrict the negation connective to be a permutation mapping Π_A of order 2 from A onto A. Consider the natural product map $\Pi_A \times \Pi_A$: $A^2 \longrightarrow A^2$ defined by $\langle a_1, a_2 \rangle \longrightarrow \langle \Pi_A a_1, \Pi_A a_2 \rangle$. It is clear that this product map is one-one and onto. A binary connective Δ_1 can be considered to be a mapping $A^2 \longrightarrow A$. To say that a binary connective Δ_2 is dual to a binary connective Δ_1 is to say that the diagram below is commutative.



Equivalently, for any $\langle a_1, a_2 \rangle \in A^2$:

$$\Pi_A \Delta_1 (\langle a_1, a_2 \rangle) = \Delta_2 (\Pi_A \times \Pi_A (\langle a_1, a_2 \rangle))$$
$$= \Delta_2 (\langle \Pi_A a_1, \Pi_A a_2 \rangle)$$

As Π_A is a permutation mapping, the truth-function of the binary connective Δ_2 is uniquely defined.

We are able to define, relative to a permutation Π of order 2, a unique dual connective Δ_2 corresponding to each binary connective Δ_1. Consequently we can determine the principal dual of any well-formed formula

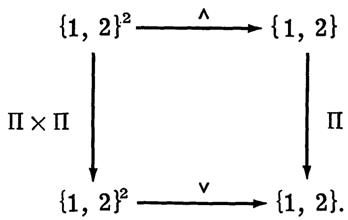
1. The product connective has been defined by Wajsberg in [2], and later on in this paper in section 3.

with arbitrary binary connectives.[2] For example, the binary connectives '$\wedge$' and '$\vee$' defined by the tables

$\wedge$	1	2
1	1	2
2	2	2

$\vee$	1	2
1	1	1
2	1	2

are dual to each other relative to the permutation Π: $\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ as is evident by the commutative diagram



The binary connectives '$\wedge_4$' and '$\vee_4$' defined by the tables

$\wedge_4$	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	4
4	4	4	4	4

$\vee_4$	1	2	3	4
1	1	1	1	1
2	1	2	2	2
3	1	2	3	3
4	1	2	3	4

are dual to each other relative to the permutation Π: $\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}$. The binary connective '$\wedge_4^d$' defined by the table

$\wedge_4^d$	1	2	3	4
1	1	1	3	4
2	1	2	3	4
3	3	3	3	3
4	4	4	3	4

is dual to the connective $\wedge_4$ relative to the permutation Π: $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}$. The table for $\wedge_4^d$ is determined by using the equality

$$\wedge_4^d(\langle \Pi a_1, \Pi a_2\rangle) = \Pi \wedge_4(\langle a_1, a_2\rangle),$$

where Π is the permutation $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}$, for each $\langle a_1, a_2\rangle \in \{1, 2, 3, 4\}^2$. Corresponding to the equivalence "$(p \wedge_4 q) \wedge_4 q$ is truth-functionally equivalent to $(p \wedge_4 q)$" we have the two dual equivalences "$(p \vee_4 q) \vee_4 q$ is truth-functionally equivalent to $(p \vee_4 q)$," and "$(p \wedge_4^d q) \wedge_4^d q$ is truth-functionally equivalent to $(p \wedge_4^d q)$," relative to the permutations $\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}$ respectively.

The binary connectives '$\wedge$' and '$\vee$' in the two-valued propositional

2. To define the duals of m-argument connectives ($m > 2$), we can consider the product map $\Pi \times \Pi \times \ldots \times \Pi$ and the appropriate commutative diagram.

calculus have been generalized by Post in [3] to the binary connectives $\wedge_n$ and $\vee_n$ in n-valued logic. He has considered truth-values to be counting numbers and the connectives $\wedge_n$ and $\vee_n$ are defined as follows:

$$\wedge_n (\langle a_1, a_2\rangle) = \max \{a_1, a_2\},$$

$$\vee_n (\langle a_1, a_2\rangle) = \min \{a_1, a_2\} \quad .$$

The connectives $\wedge_n$ and $\vee_n$ are dual to each other relative to the permutation of order 2

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ n & n-1 & n-2 & n-3 & \dots & 1 \end{pmatrix}$$

Denoting this permutation as $\approx_n$, Post has shown that $\wedge_n$ and $\vee_n$ have the property that using $\approx_n$ one can be defined in terms of the other, in view of the equivalences

$\approx_n(\approx_n p \vee_n \approx_n q)$ is truth-functionally equivalent to $p \wedge_n q$,
$\approx_n(\approx_n p \wedge_n \approx_n q)$ is truth-functionally equivalent to $p \vee_n q$.

The definition of duality relative to the permutation $\approx_n$ is a natural generalization of the definition in two-valued logic in the sense that the connectives defined by the dual operations of min and max are dual relative to this permutation. It can be shown that relative to any other permutation of order 2 these two connectives (i.e., $\wedge_n$ and $\vee_n$) are not dual to each other.

In the following sections, after defining the product connective of two connectives, we mention that there is a natural way of defining the dual of a product connective. The definition relative to the permutation $\approx_n$ is a natural generalization of duality to many-valued logic in that it is possible to define the dual of a product connective in a natural manner.

3. *Product connective of two binary connectives* Among the binary connectives of the two-valued propositional calculus only six have the property that some of the well-formed formulas in any one connective (many occurrences of the same connective permitted) are tautologies. The six connectives are implication, counter-implication, equivalence, Sheffer stroke (alternative denial: not both), Peirce stroke (joint denial: neither nor) and the constant function 1 (Truth). Kalicki has studied, in [4] and [5], matrices with a finite number of truth-values, that describe a single binary connective Δ. He denotes the set of all Δ-formulas that are Δ-tautologies by $S(M)$, where M is the matrix that describes the binary connective Δ. For any finite matrix M, Kalicki has given an effective method for deciding whether the set $S(M)$ is empty or not.

For arbitrary finite matrices M, N, describing binary connectives Δ_1 and Δ_2, respectively, Wajsberg has defined in [2] a finite matrix P, called the product matrix of M and N, that describes a binary connective Δ. A product matrix P has the property $S(P) = S(M) \cap S(N)$, that is, the set of tautologies in the product connective Δ is precisely the intersection of the set of tautologies in the connective Δ_1 and the set of tautologies in the connective Δ_2.

The Cartesian product $A_1 \times A_2$, where A_1 and A_2 are respectively the sets of truth-values over which the connectives Δ_1 and Δ_2 are defined, is the set of truth-values over which the connective Δ is defined. Entries in the product matrix are also ordered pairs; the entry at the intersection of a row headed by $\langle v_i, v_j \rangle$ and a column headed by $\langle v_k, v_r \rangle$ is $\langle \Delta_1 (\langle v_i, v_k \rangle), \Delta_2 (\langle v_j, v_r \rangle) \rangle$. The set of designated values[3] of the product matrix is the subset $B_1 \times B_2$ of $A_1 \times A_2$, where B_1 and B_2 are the sets of designated values of the matrices M and N respectively.

Considering a function $\sigma : A^2 \times A^2 \longrightarrow A^2$ defined by the equality

$$\sigma(\langle a_1, a_2 \rangle, \langle a_1', a_2' \rangle) = (\langle a_1, a_1' \rangle, \langle a_2, a_2' \rangle),$$

the product connective of two connectives Δ_1 and Δ_2, denoted by $\Delta_1 * \Delta_2$, may be considered to be the composite $(\Delta_1 \times \Delta_2) \circ \sigma$.

We now illustrate the construction of the product matrix of the binary connectives, implication and counter-implication.

$\Rightarrow$	1	2
*1	1	2
2	1	1

$\Leftarrow$	1	2
*1	1	1
2	2	1

$\Rightarrow * \Leftarrow$	⟨1,1⟩	⟨1,2⟩	⟨2,1⟩	⟨2,2⟩
*⟨1,1⟩	⟨1,1⟩	⟨1,1⟩	⟨2,1⟩	⟨2,1⟩
⟨1,2⟩	⟨1,2⟩	⟨1,1⟩	⟨2,2⟩	⟨2,1⟩
⟨2,1⟩	⟨1,1⟩	⟨1,1⟩	⟨1,1⟩	⟨1,1⟩
⟨2,2⟩	⟨1,2⟩	⟨1,1⟩	⟨1,2⟩	⟨1,1⟩

4. *Dual of a product connective* A natural way of defining the dual, Δ^d, of a product connective Δ where $\Delta = \Delta_1 * \Delta_2$ is to define it as the product of the dual connectives Δ_1^d and Δ_2^d, that is, $\Delta^d = \Delta_1^d * \Delta_2^d$. This way of defining the dual of a product connective is consistent with the fact that the semantic notions of tautology and contradiction are mutually dual.

We now show that the definition of duality relative to the permutation

$$\Pi_A : \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ n & n-1 & n-2 & \dots & 1 \end{pmatrix}$$

enables us to define the dual of a product connective in a natural manner as the product of dual connectives.

The dual of a connective Δ_1 (alternatively Δ_2) may be obtained by composing the maps Π_A, Δ_1 (alternatively Δ_2) and $\Pi_A \times \Pi_A$. This is clear from the commutative diagram

3. In two-valued logic, the two truth-values denote Truth and Falsity. In many-valued logic the set of designated values is the generalization of the single truth-value Truth. A well-formed formula is defined to be a tautology if the truth-function of the well-formed formula has its range of values restricted to the set of designated values. Usually, designated values of a connective are marked with an asterisk adjacent to the row designators in the matrix that describes the connective.

$$\begin{array}{ccc} S = A \times A & \xrightarrow{\Delta_i} & A \\ \Pi_S = \Pi_A \times \Pi_A \downarrow & & \downarrow \Pi_A \\ S = A \times A & \xrightarrow{\Delta_i^{\mathrm{d}}} & A \end{array} \qquad i = 1, 2$$

We observe that since Π_A is of order 2,

$$\begin{aligned} \Delta_i^{\mathrm{d}} &= \Pi_A \circ \Delta_i \circ (\Pi_A \times \Pi_A)^{-1} \\ &= \Pi_A \circ \Delta_i \circ (\Pi_A \times \Pi_A). \end{aligned}$$

Also the dual of the connective $\Delta_1 \times \Delta_2$ may be obtained by composing the maps Π_S, $\Delta_1 \times \Delta_2$, and $\Pi_S \times \Pi_S$. This is clear from the commutative diagram

$$\begin{array}{ccc} S \times S & \xrightarrow{\Delta_1 \times \Delta_2} & S \\ \Pi_S \times \Pi_S \downarrow & & \downarrow \Pi_S \\ S \times S & \xrightarrow{(\Delta_1 \times \Delta_2)^{\mathrm{d}}} & S \end{array}$$

We observe that $(\Delta_1 \times \Delta_2)^{\mathrm{d}} = \Pi_S \circ (\Delta_1 \times \Delta_2) \circ (\Pi_S \times \Pi_S)^{-1} = \Pi_S \circ (\Delta_1 \times \Delta_2) \circ (\Pi_S \times \Pi_S)$. By straightforward computation we can verify the identity $(\Delta_1 \times \Delta_2)^{\mathrm{d}} = \Delta_1^{\mathrm{d}} \times \Delta_2^{\mathrm{d}}$ and infer

$$\begin{aligned} (\Delta_1 * \Delta_2)^{\mathrm{d}} &= ((\Delta_1 \times \Delta_2) \circ \sigma)^{\mathrm{d}} \\ &= (\Delta_1 \times \Delta_2)^{\mathrm{d}} \circ \sigma \\ &= (\Delta_1^{\mathrm{d}} \times \Delta_2^{\mathrm{d}}) \circ \sigma \\ &= \Delta_1^{\mathrm{d}} * \Delta_2^{\mathrm{d}} \end{aligned}$$

It is to be noted that as long as we define duality relative to a permutation of order 2, the dual of a product connective may be defined in a natural manner as above. Also, not every connective in many-valued logic is a product connective. This is evident from the fact that the set of truth-values over which a product connective is defined is a Cartesian product. But this need not concern us because we are only suggesting that a product connective being a many-valued connective, any generalization of the notion of duality to many-valued logic must be such that it is possible to define in a natural manner the dual of a product connective.

5. *Conclusion* This paper suggests that among the different negations in many-valued logic, from the point of view of a natural generalization of the notion of duality to many-valued logic, the negation function

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ n & n\text{-}1 & n\text{-}2 & \dots & 1 \end{pmatrix}$$

is a natural choice. Of course only specific criteria for the generalization of the notion of duality to many-valued logic have been considered.

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