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INTUITIONISTIC NEGATION

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Within Heyting's intuitionistic mathematics there are at least two distinct types of negation. The first is that which Heyting [1] (p. 18) has called "de jure" falsity. If p is a proposition then the negation of p has been proved, $\vdash \sim p$, if it has been shown that the supposition of p leads to a contradiction. That is, $\vdash p \rightarrow F$ where F is any contradiction. Intuitionistically, if p and q are propositions then $\vdash p \rightarrow q$ if a construction has been effected which together with a construction of p would constitute a construction of q. While Heyting holds that only "de jure" negation should play a part in intuitionistic mathematics [1] (p. 18), there has been a second type of negation introduced into Heyting's work which I have called "in absentia" falsity. That is $\vdash \sim p$ if it is certain that p can never be proved. This "in absentia" negation is used explicitly by Heyting in [1] (p. 116, lines 16, 17) and mentioned in [2] (pp. 239-240). In this paper I wish to show that "de jure" falsity and "in absentia" falsity lead to a contradiction in informal intuitionistic mathematics.

Consider the following definitions:

Definition 1 (vide [1], p. 115) A proposition p has been *tested* if $\vdash \sim p \vee \sim\sim p$.

Definition 2 A proposition p has been *decided* if $\vdash p \vee \sim p$.

It is well known that because of the intuitionistic interpretation of disjunction, $\vdash p \vee q$ if and only if at least one of $\vdash p$ or $\vdash q$. Consequently $q \vee \sim q$ does not possess universal intuitionistic validity so long as there are undecided mathematical problems.

Proposition 1 *A decided proposition has been tested.*
Proof: $\vdash p \rightarrow \sim\sim p$.

In a chapter on "Controversial Subjects", Heyting [1] presents some intuitionistic results of Brouwer which if interpreted classically mean that classical mathematics is contradictory.

Proposition 2 (i.e., Theorem 2, [1], p. 118) *It is contradictory, that for every real number (generator)* a, $a \neq 0$ *would imply* $a \not> 0 \vee a \not< 0$.

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The following definitions are necessary:

Definition 3 A *real number generator* (rng) $\{b_n\}$ is an infinitely proceeding sequence (ips) of rational numbers subject to the condition, $\forall k \exists n : | b_{n+j} - b_n | < 1/k$, for all j.

For the intuitionistic interpretation of the universal and existential quantifiers see Heyting, [1] (pp. 102-3) or Myhill, [3] (pp. 281-2). The letters i, j, k, m, n are used for positive integers; a, b, c, d for rng's; and p, q, r for propositions.

Definition 4 $b = c$, b *coincides* with c, if $\forall k \exists n : |b_{n+j} - c_{n+j}| < 1/k$, for all j.
Definition 5 $b \neq c$ if $\sim(b = c)$.
Definition 6 $b > c$ $(c < b)$ if $\exists k, n : b_{n+j} - c_{n+j} > 1/k$, for all j.
Definition 7 $b \not> c$ if $\sim(b > c)$.
Definition 8 $b \equiv c$, b is *identical* with c, if $b_n = c_n$ (rational equality), for all n.

In order that a rng $b \equiv \{b_n\}$ be well defined it is not necessary that each term b_n be known at a specified time. It is sufficient that given any positive integer n an effective procedure is possessed to find b_n. It is thus an effective procedure and not necessarily a (predetermined) law for the components which guarantees the existence of a rng. Of course a law, (e.g.) $b \equiv \{1/n\}$, yields an effective procedure for computing b_n for any n. Other effective procedures are able to take into account further decisions or further knowledge. (e.g.) $b \equiv \{b_n\}$ where $b_1 = 1/2$ is chosen at some time t_1 and b_n, for $n \geq 2$ is chosen at the $(n-1)$th minute after t_1 such that $b_n = b_{n-1}/2$ if it is raining in Wellington and $b_n = b_{n-1}$ if it is not raining in Wellington. Others are absolutely lawless, (e.g.) $c \equiv \{c_n\}$ where $c_1 = j_1 10^{-1}$ and for $n \geq 2$ $c_n = j_n 10^{-n} + \Sigma_{k=1}^{n-1} c_k$ and each j_k is chosen freely from $S \equiv \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

The following discussion shows that an essential part of the proof of Proposition 2 should be rejected because it employs an "in absentia" falsity which leads to an intuitionistic contradiction.

For each i let ω_i be a finite set of mathematical deductions. Let $\sigma_n \equiv \bigcup_{i=1}^{n} \omega_i$ and $\Omega \equiv \bigcup_n \sigma_n$. Let p be some mathematical proposition. Define the rng $b \equiv \{b_n\}$ as follows: $b_n = 2^{-n}$ if σ_n does not contain a deduction of $\sim p$ or of $\sim\sim p$. $b_{n+j} = 2^{-n}$, for all j, if σ_n contains a deduction of $\sim p$ or of $\sim\sim p$. For each n, ω_n is finite so b is well defined.

Troelstra, [4] (p. 212) remarks that since 1945 Brouwer argued from a solipsist situation in which he was concerned with the thoughts of an individual mathematician or a group of mathematicians having all information in common. In the following proposition suppose σ_n contains all deductions made, (a finite number) up until b_n is chosen.

Proposition 3 (vide [1], p. 116) $b(p, \Omega) \neq 0$.
Proof: (i) Assume $b = 0$.
$\therefore \forall m \exists n : |b_n| < 2^{-m}$.
$\therefore \forall m$, $b_m = 2^{-m}$, by induction and definition of b.

(ii) Suppose $\exists m : \sim p \epsilon \sigma_m$.
$\therefore$ $b_{m+j} = 2^{-m}$, for all j, a contradiction.
$\therefore$ $\forall m \sim p \notin \sigma_m$, by $\vdash \sim (\exists x)A(x) \rightarrow (\forall x) \sim A(x)$.
(iii) Similarly $\forall m$, $\sim\sim p \notin \sigma_m$.
(iv) Suppose $\sim p \epsilon \Omega$,
then $\exists m : \sim p \epsilon \sigma_m$, a contradiction.
$\therefore \sim p \notin \Omega$.
(v) Similarly $\sim\sim p \notin \Omega$.
(vi) (iv) and (v) show that p is never tested.
$\therefore \sim (\sim p \vee \sim\sim p)$ by "in absentia" falsity.
$\therefore \sim\sim p \wedge \sim\sim\sim p$ by $\vdash \sim (q \vee r) \rightarrow \sim q \wedge \sim r$, a contradiction.
(vii) $\therefore$ $b \neq 0$.

Consider the following specialisation of the conditions of Proposition 3. Construct the rng $\{c_n\}$ as follows. $c_1 = j_1 10^{-1}$ and for $n \geqslant 2$, $c_n = j_n 10^{-n} + \Sigma_{k=1}^{n-1} c_k$ where each j_n is chosen freely from S. Let $P(c)$ be the proposition "c is rational". Construct the rng $d(c) \equiv \{d_n\}$ as follows. c_1 is chosen first and σ_n is the set of deductions made up until d_n is chosen. c_{n+1} is chosen after d_n and before d_{n+1}. $d_n = 2^{-n}$ if $P(c)$ has not been tested in σ_n. $d_{n+j} = 2^{-n}$, for all j if $P(c)$ is tested in σ_n.

Proposition 4 $\forall c(d \neq 0)$ (vide [1], pp. 118, line 6).
Proof: as for Proposition 3.

Proposition 5 $\forall c(d = 0)$
Proof: It is impossible, under the given construction for c, that either $\sim P(c)$ or $\sim\sim P(c)$ belongs to Ω. Suppose $P(c)$ is tested in σ_m.

(i) Suppose $\sim P(c) \epsilon \sigma_m$. Now impose the first restriction on c, namely, $c_{m+j} = 0$, for all j. Thus $P(c)$, which is a contradiction.
$\therefore \sim P(c) \notin \sigma_m$.

(ii) Suppose $\sim\sim P(c) \epsilon \sigma_m$. Now impose the first restriction on c, namely, $j_{m+j} = (\sqrt{2})j$, for all j, where $(\sqrt{2})j$ is the j-th digit in the decimal expansion of $\sqrt{2}$. Thus $\sim P(c)$, which is a contradiction.
$\therefore \sim\sim P(c) \notin \sigma_m$

(i) and (ii) show that $\sim \exists m : P(c)$ is tested in σ_m.
$\therefore$ $\forall m$ $P(c)$ is not tested in σ_m.
$\therefore$ $\forall m$ $d_m = 2^{-m}$
$\therefore$ $d = 0$.
$\therefore$ $\forall c (d = 0)$.

Proposition 5 could be proved without mentioning restrictions on c by appealing to the intuitionistic fan theorem (vide [1] or [6]) or to the intuitionistic continuity postulate of Kreisel (vide [5]). Using one of these, the supposition, for example, that $\sim P(c) \epsilon \sigma_m$ would imply that all decimal numbers agreeing with c in their first m decimal places would also be irrational, which is also contradictory.

Proposition 5 does not employ the "in absentia" falsity and also proves that $\sim P(c) \vee \sim\sim P(c)$ is never proved in Ω; say it is certain that $\vdash \sim P(c) \vee \sim\sim P(c)$ can never be proved. It seems that Heyting's use of the "in absentia" negation amounts to the following rule of inference. If α is any well formed formula of intuitionistic first order predicate calculus and it is certain that $\vdash \alpha$ can never be proved then $\vdash \sim \alpha$. The previous discussion has shown that this use of the "in absentia" negation leads to a contradiction.

Definition 5 has a stronger intuitionistic counterpart.

Definition 9 b lies apart from c, $b \# c$, if $\exists k, n : |b_{n+j} - c_{n+j}| > 1/k$ for all j.

Given that $\sim (\forall x) A(x) \rightarrow (\exists x) \sim A(x)$ is not an intuitionistic thesis [1] (p. 103), it is clear that $b \# c$ is a stronger condition than $b \neq c$. The "in absentia" negation is also essential to the following:

Proposition 6 (i.e., Theorem 1, [1] p. 117) *It is contradictory that for every real number* a, $a \neq 0$ *would imply* $a \# 0$.

If this proposition is also rejected then, so far as I know, there is no example of a rng b such that $b \neq 0$ while $b \# 0$ has not been proved.

Remark: In the semantic considerations of intuitionistic logic by Beth [7], Grzegorczyk [8] and Kripke [9], only the "in absentia" negation can play a part. Supposing familiarity with [9] and considering only intuitionistic propositional calculus let $\langle G, K, r \rangle$ be an intuitionistic model structure and ϕ a model on $\langle G, K, r \rangle$. Let p and q be propositional letters and F be $q \wedge \sim q$. Then for $H, H' \epsilon K$, $\phi(\sim p, H) = 1$ if for all H' such that $H\, r\, H'$, $\phi(p, H') = 0$. The case $\phi(p \rightarrow F, H) = 1$ reduces to $\phi(\sim p, H) = 1$ because $\phi(F, H') = 0$ for all H' such that $H\, r\, H'$.

A similar result can be extended for any well formed formula of intuitionistic propositional or first order predicate calculus.

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