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ON WEAK AND STRONG VALIDITY OF RULES FOR THE PROPOSITIONAL CALCULUS

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Harrop [1], [2] has distinguished between rules of inference for propositional calculi which are strongly valid and those which are only weakly valid. He gives an example of a rule which is weakly but not strongly valid with respect to a certain three valued model. Setlur [3] has shown that these notions coincide in the usual model for the classical propositional calculus. We shall show that Setlur's proof is dependent on the definition of rule given by Harrop. In particular, we generalize the notion of rule and then give some examples of rules which are weakly but not strongly valid with respect to the usual model for the classical propositional calculus.

Harrop defines a rule as a metastatement of the form $\alpha_1, \alpha_2, \ldots, \alpha_n \vdash \beta$ where $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta$ are formula schemes, i.e., metalogical variables built up from variables for arbitrary formulas (vafs) and the connectives. By an application of a rule he means a statement of the form $X_1, X_2, \ldots, X_n \vdash Y$ where $X_1, X_2, \ldots, X_n$, Y result from $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta$ by a common substitution. A rule is weakly valid with respect to a certain finite model iff whenever the premises of an application of a rule are valid then the conclusion is also valid. A rule is strongly valid with respect to a certain finite model iff for any assignment of values from the model to the vafs which occur in $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta$, if the premises of the rule are all designated then the conclusion β is also.

We observe that the rule $\alpha_1, \alpha_2, \ldots, \alpha_n \vdash \beta$ is strongly valid with respect to the usual model for the classical propositional calculus iff $C\alpha_1 C\alpha_2 \ldots C\alpha_n\beta$ is a tautology (where truth values are assigned to the vafs which occur in $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta$).

We call $\alpha_1, \alpha_2, \ldots, \alpha_n \vdash \beta$ a *rule with restrictions* iff $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta$ are formula schemes or variables for propositional letters and where there are restrictions imposed on the occurrence (or non-occurrence) of certain propositional letters in some of the $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta$.

We now give some examples of rules with restrictions which are weakly but not strongly valid with respect to the usual model for the classical propositional calculus (henceforth we shall only be concerned with this model):

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I $$CCp\alpha Cp\beta \vdash C\alpha\beta$$

with the restriction that p is a propositional variable which does not occur in the vafs α or β. The derivation of this rule is based on **a**) the law of shortening of the antecedent *CCCrstCst* and on **b**) the law of permutation *CCrCstCsCrt*:

1. $CCp\alpha Cp\beta$

   a, r/p, s/α, $t/Cp\beta = C1\text{-}2$

2. $C\alpha Cp\beta$

   b, r/α, s/p, $t/\beta = C2\text{-}3$

3. $CpC\alpha\beta$

   3, $p/\mathrm{T} = C\mathrm{T}\text{-}4$

4. $C\alpha\beta$

where T is any tautology. Note that it was crucial for our proof that p was a propositional variable and that it did not occur in α or β. Since **a** and **b** are valid and since substitution and detachment are weakly (even strongly) valid rules we have that **I** is weakly valid. **I** is not strongly valid since $CCCp\alpha Cp\beta C\alpha\beta$ is not a tautology.

The rules

$$Cp\alpha \vdash CNp\alpha \qquad \text{and} \qquad C\alpha p \vdash N\alpha$$

where p is a propositional variable not occurring in the vaf α are other examples of weakly valid rules which are not strongly valid.

In conclusion we comment that Setlur's proof fails for rules with restrictions since one cannot substitute arbitrary formulas for the propositional variables occurring in the rules, and this is the heart of his proof.

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