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AFFINE GEOMETRY HAVING A SOLID AS PRIMITIVE

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INTRODUCTION In this dissertation we consider the problem of constructing a system of geometry devoid of such geometrical primitives as points, lines, and surfaces; and admitting as primitives only solids or "chunks". In other words can we start with some solid as a primitive term and from there add appropriate axioms to obtain a geometrical system equivalent to a well known geometrical system such as Euclidean geometry or affine geometry?

Part of the answer was given by A. Tarski (in [8]) who solved the problem for ordinary Euclidean geometry. His solution was as follows:

(1) He begins with an axiomatization of the relation "*A* is part of *B*" (This deductive system is due to Leśniewski (see [5]) and is called Mereology (see Appendix A)).

(2) He adds to Mereology the primitive term sphere and uses only the part relation and sphere to define the notion point-class and the notion of equidistance among point-classes.

(3) Then he adds the ordinary axioms of Euclidean geometry (based on point and equidistance as primitive—see M. Pieri [6]) replacing the primitive terms by the defined terms above.

(4) Finally if we interpret sphere as open ball in ordinary Euclidean geometry there is a bijective correspondence between point-classes and points; and Tarski's system is equivalent to Euclidean geometry.

We shall consider a generalization of Tarski's result in which we solve the above problem for the class of affine geometries which are equivalent to finite dimensional vector spaces over an ordered field.

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Our method of proof will be as follows: In section 1 we add the primitive term parallelepiped to Mereology, define point-class and between for point-classes, and add an axiom system for affine geometry (based on point and between as primitive) with the primitive terms replaced by the defined terms. In section 2 we show that every affine geometry which satisfies the axioms given in section 1 can be given a vector space structure over an ordered field (see Appendix B). This result will be used throughout the remainder of this paper. In section 3 we interpret our system based on Mereology into affine geometry. In the remaining sections we prove that affine geometry is a model for our system. Thus in section 4 we show that there is a bijective correspondence between point-classes and points of the affine geometry by getting a geometrical characterization of each of the terms used in defining point-class. In section 5 we prove that our correspondence preserves betweenness for point-class and betweenness for points and finally in section 6 we use the correspondence and the results of section 5 to show that the axioms of our system are provable in affine geometry. Since our system is constructed to be a model of affine geometry the two systems are equivalent which is the result we wish to obtain in this paper.

The notation we shall use in our proof is due to Peano-Russell. Each conjunct is preceded by one or more dots which replace the use of parentheses. Each proof begins with a line which looks as follows:
PF [ ]:Hp( ) . ⊃ . where **PF** is followed by a quantifier containing a list of variables which occur in the hypotheses followed by Hp( ) which indicates the number of conjuncts which appear in the hypotheses followed by the logical symbol for implication. While all other notation is explained in the text we note here that the justification for each step of a proof is given at the right margin after each step and the use of the symbol →← indicates that the justification involves reasoning by contradiction.

§1. In this section we give a formal construction of our system $(\mathfrak{A}^n, \mathfrak{F})$. Since we use the formal system of Mereology in which to express our system a brief development of Mereology is given in Appendix A. However, in order to read what follows in the section, it is sufficient to make the following interpretations:

(1) Interpret "**el**(B)" as "the power set of B".
(2) Interpret "ε" as "∊" - the epsilon of set theory.
(3) Interpret "**Kl**(a)" as "the union of members of a".
(4) Interpret "∪" and "∩" as the logical "or" and "and" respectively.

We shall be a little more precise about our interpretations in section 3.

In order to construct $(\mathfrak{A}^n, \mathfrak{F})$ we begin with Mereology; add as primitive the term parallelepiped ($\mathcal{P}$); and then define point-class (*DM6*) and between for point-classes (*DM7*) as follows:

DM1 $[AB]:: \mathsf{EXT}(AB) . \equiv :. \mathcal{P}(A) . \mathcal{P}(B) : [D] : \mathcal{P}(D) . \supset . \sim(D \varepsilon (\mathbf{el}(A) \cap \mathbf{el}(B))$

We are defining the relation A is *external* to B iff A and B are parallelepipeds and whenever D is a parallelepiped it is not true that D is contained in both A and B. Figures (A) and (B) show two cases in the plane when the relation holds while figure (C) show a case in the plane when the relation does not hold (we remark that all our figures will be planar).

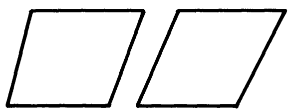

Fig. A

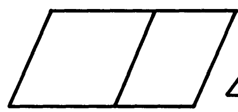

Fig. B

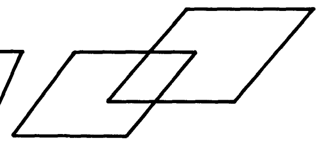

Fig. C

$DM2\ [AB] : \mathrm{ETG}(AB) . \equiv . \mathrm{EXT}(AB) . \mathcal{P}(\mathrm{Kl}(\mathrm{el}(A) \cup \mathrm{el}(B)))$

We are defining the relation A is *externally tangent* to B iff A is external to B and $A \cup B$ is a parallelepiped. Figure (1) below shows a case when the relation holds while figure (2) shows a case when the relation fails because $A \cup B$ is not a parallelepiped.

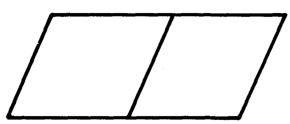

Fig. 1

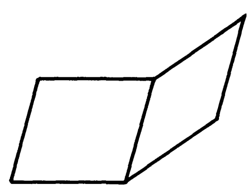

Fig. 2

$DM3\ [AB] :: \mathrm{BIS}(AB) . \equiv : \mathcal{P}(A) . \mathcal{P}(B) . \sim(A \,\varepsilon\, \mathrm{el}(B)) \,.\!\because\!.\, [\exists EF] \,.\!\because\!.\, \mathrm{EXT}(EF) . \mathrm{ETG}(EA) . \mathrm{ETG}(FA) . E \,\varepsilon\, \mathrm{el}(B) . F \,\varepsilon\, \mathrm{el}(B) : [GH] : \mathcal{P}(G) . \mathcal{P}(H) . E \,\varepsilon\, \mathrm{el}(G) . F \,\varepsilon\, \mathrm{el}(G) . H \,\varepsilon\, (\mathrm{el}(A) \cap \mathrm{el}(B)) . \supset . H \,\varepsilon\, \mathrm{el}(G)$

We are defining the relation (not symmetric) A is *bisected* by B iff A and B are parallelepipeds; A is not contained in B; there exist parallelepipeds E and F such that E is external to F, each is externally tangent to A and each is contained in B; and for all parallelepipeds G and H such that E and F are contained in G and if H is in $A \cap B$ then H is contained in G. Figure (A) below indicates a case in which the relation holds while in figure (B) the relation does not hold since H is not in G.

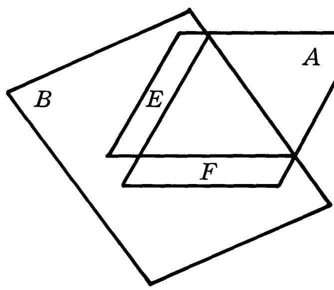

Fig. A

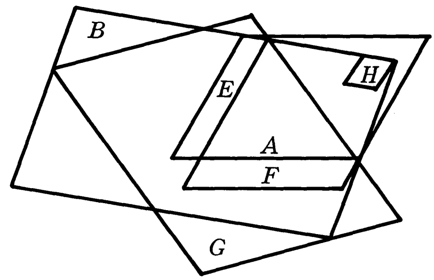

Fig. B

DM4 $[AB]::\ \mathrm{CON}(AB)\,.\!\equiv\therefore \mathcal{P}(B)\,.\,\mathcal{P}(A):[D]:\ \mathrm{BIS}(AD)\,.\!\supset\!.\ \mathrm{BIS}(BD)$

We are defining the relation (not symmetric) B is *concentric* to A iff A and B are parallelepipeds and every bisector of A is a bisector of B. In figure (1) below the relation holds while in figure (2) the relation fails since D bisects A but not B.

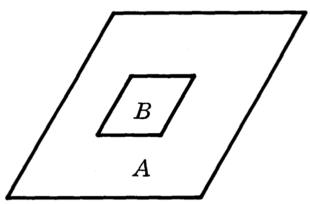

Fig. 1

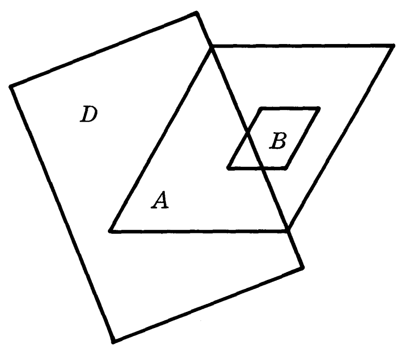

Fig. 2

DM5 $[AB]::\ \mathrm{EQV}(AB)\,.\!\equiv\therefore \mathcal{P}(A)\,.\,\mathcal{P}(B):[EF]:\mathrm{CON}(AE)\,.\,\mathrm{CON}(BF)\,.\!\supset\!.$
$\sim(\mathrm{EXT}(FE))$

We are defining the relation A is *equivalent* to B iff A and B are parallelepipeds and whenever E and F are such that E is concentric to A and F is concentric to B then E and F are not external. This will later mean that the diagonals of A and B are all meeting in the same point.

DM6 $[a]::\ \mathrm{PNT}(a)\,.\!\equiv\therefore [\exists A]\therefore \mathcal{P}(A):[B]:a(B)\,.\!\equiv\!.\ \mathrm{EQV}(BA)$

We are defining a is a *point-class* iff a applies to all parallelepipeds B which are equivalent to a given parallelepiped A. Later this will mean a applies to all parallelepipeds which have the same center as A.

DM7 $[abd]::\ \mathsf{BNT}(abd) \,.\!\equiv\therefore \mathsf{PNT}(a) \,.\, \mathsf{PNT}(b) \,.\, \mathsf{PNT}(d) \,.\!\sim\!(a=b).$
$\sim\!(b=d) \,.\!\sim\!(d=a) : [ABDH] : \mathcal{P}(H) \,.\, a(A) \,.\, b(B) \,.\, d(D) \,.$
$A\ \varepsilon\,\mathsf{el}(H) \,.\, D\ \varepsilon\,\mathsf{el}(H) \,.\supset.\!\sim\!(\mathsf{EXT}(BH))$

We are defining point-class b is *between* point-class a and point-class d iff whenever a parallelepiped H contains a parallelepiped A of a and D of d then H is not external to any parallelepiped B of b. Figure (A) illustrates what happens when the relation fails to hold.

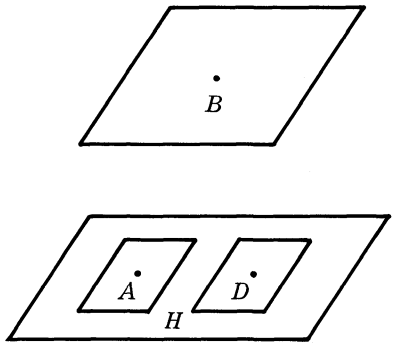

Fig. A

The next step to construct $(\mathfrak{A}^n, \mathfrak{F})$ is to give a set of axioms based on point and between as primitives which are sufficiently strong to give us the class of affine geometries $(\mathfrak{A}^n, \mathfrak{F})$ where $n \geqq 2$, $\mathfrak{A}^n$ is a model for the given axioms, and $\mathfrak{A}^n$ has a vector space structure isomorphic to $\mathfrak{F}^n$ where $\mathfrak{F}$ is an ordered field (in section 2 we prove such a structure for $\mathfrak{A}^n$ exists). In the axioms below $\mathsf{pt}(a)$ is to be interpreted as a is a point while $\mathsf{bet}(abd)$ is to be interpreted as b lies between a and d. We shall present the axioms formally first and then give an informal description of them.

A1 $[abd] : \mathsf{bet}(abd) \,.\supset.\, \mathsf{pt}(a) \,.\, \mathsf{pt}(b) \,.\, \mathsf{pt}(d) \,.\, a \neq b \,.\, b \neq d.\ d \neq a$

A2 $[abd] : \mathsf{bet}(abd) \,.\supset.\!\sim\!(\mathsf{bet}(bda))$

A3 $[ab] : \mathsf{pt}(a) \,.\, \mathsf{pt}(b) \,.\, a \neq b \,.\supset.\, [\exists d] \,.\, \mathsf{bet}(abd)$

DA1 $[abl] :\cdot: \mathbf{S}^1(ab)(l) \,.\!\equiv:: \mathsf{pt}(a) \,.\, \mathsf{pt}(b) \,.\, a \neq b \therefore [d] \therefore d \in l.$
$\equiv : d = a \,.\, \mathsf{v} \,.\, d = b \,.\, \mathsf{v} \,.\, \mathsf{bet}(dab) \,.\, \mathsf{v} \,.\, \mathsf{bet}(adb) \,.\, \mathsf{v} \,.\, \mathsf{bet}(adb)$

A4 $[abefl_1l_2] : \mathbf{S}^1(ab)(l_1) \,.\, \mathbf{S}^1(ef)(l_2) \,.\, e \in l_1 \,.\, f \in l_1 \,.\supset.\, a \in l_2$

A5 $[abefgl_1l_2] : \mathbf{S}^1(be)(l_1) \,.\, \mathbf{S}^1(fg)(l_2) \,.\!\sim\!(a \in l_1) \,.\, \mathsf{pt}(a) \,.$
$\mathsf{bet}(bef) \,.\, \mathsf{bet}(ega) \,.\supset.\, [\exists d] \,.\, d \in l_2 \,.\, \mathsf{bet}(adb)$

DA2 $[mp_0, \ldots, p_mS] \vdots\vdots\ \mathbf{S}^m(p_0, \ldots, p_m)(S) \,.\!\equiv:\cdot: m \in \{2,3,\ldots\}.$
$\mathsf{pt}(p_m) \,.\, [\exists T] \,.\, \mathbf{S}^{m-1}(p_0, \ldots, p_{m-1})(T) \,.\!\sim\!(p_m \in T) :: [p] ::$
$p \in S \,.\!\equiv\therefore [\exists q_0a_0, \ldots, q_ma_mQPl] \therefore \{q_1, \ldots, q_m\} \subset$
$\{p_0, \ldots, p_m\} \,.\, \{a_1, \ldots, a_m\} \subset \{p_0, \ldots, p_m\}.$
$\mathbf{S}^{m-1}(q_1, \ldots, q_m)(Q) \,.\, \mathbf{S}^{m-1}(a_1, \ldots, a_m)(P) \,.\, \{q_0a_0\} \subset$
$(P \cup Q) \,.\, \mathbf{S}^1(q_0a_0)(l) \,.\, p \in l \,.\, q_0 \neq a_0$

A6 $[p_0,\ldots,p_{k+1}akPQR]:k\in\{1,\ldots,n-1\}.\mathsf{S}^k(p_0,\ldots,p_k)(R).$
$\mathsf{S}^{k+1}(p_0,\ldots,p_{k+1})(Q).\mathsf{S}^2(p_0p_{k+1}a)(P).\supset.[\exists bl].$
$\mathsf{S}^1(p_0b)(l).l=P\cap R$

A7 $[abdl_1P]::\mathsf{S}^2(abd)(P).\mathsf{S}^1(ab)(l_1).\supset\therefore[\exists el]\therefore e\in P.$
$\mathsf{S}^1(ed)(l).l\cap l_1=\phi:[fl_2]:f\in P.\mathsf{S}^1(df)(l_2).$
$l_2\cap l_1=\phi.\supset.l_2=l$

DA3 $[abd]:\mathrm{col}(abd).\equiv.[\exists efl].\mathsf{S}^1(ef)(l).a\in l.b\in l.d\in l$

DA4 $[abef]::ab\|ef.\equiv\therefore[\exists gPl_1l_2]\therefore\mathsf{S}^2(abg)(P).\mathsf{S}^1(ab)(l_1).$
$\mathsf{S}^1(ef)(l_2).l_2\subset P:l_1\cap l_2=\phi.\vee.l_1=l_2$

A8 $[abea'b'e']:\mathrm{col}(abe).\mathrm{col}(a'b'e').eb'\|be'.ea'\|ae'.$
$\supset.ba'\|ab'$

A9 $[aa'bb'ee'p]:\mathrm{col}(aa'p).\mathrm{col}(bb'p).\mathrm{col}(ee'p).a\neq a'.$
$b\neq b'.e\neq e'.ab\|a'b'.be\|b'e'.\supset.ae\|a'e'$

A10 $[\exists p_0,\ldots,p_nS]\therefore\mathsf{S}^n(p_0,\ldots,p_n)(S):[a]:\mathrm{pt}(a).\supset.a\in S$

We give now an informal description of our axiom system. Axioms *A1* through *A5* and *A10* where $n=3$ are the usual axioms of order for affine geometry of space given in Forder [3] pp. 44, 45, 48, 60, and 65. In definition *DA1* we are defining l as a line determined by two distinct points a and b. In definition *DA2* we are defining S as an m dimensional space iff there are $m+1$ points $p_0,\ldots,p_m$ which determine an m-simplex (cf. the second, third, and fourth conjuncts of the definiens) and a point belongs to S iff the point lies on a line l determined by two distinct points q_0 and a_0 which lie on two sides P and Q of the m-simplex (P may possibly equal Q). Axiom *A6* is a statement of the fact that if we are given a k-space R, a $(k+1)$-space Q, and a plane P such that $P\neq R$ and $R\subset Q$ then the intersection of P and R is a line l. Axiom *A7* is just the parallel axiom for affine geometry as given in Forder [3] p. 139 ax. *P*. In definition *DA3* we are defining colinear for three not necessarily distinct points a,b, and d while in definition *DA4* we are saying that the line l_1 determined by a and b is parallel to the line l_2 determined by e and f, axiom *A8* is Pappus's axiom which ensures that the points on a line form a field and not just a division ring. Axiom *A9* is necessary only in the case $n=2$. It is Desargues theorem and in the case $n\geqq 3$ it is provable from the other axioms (see Forder [3] pp. 155-7).

We conclude the construction of our system $(\mathfrak{A}^n,\mathfrak{F})$ by adding to Mereology the above axioms and definitions as well as the definitions *DM1* to *DM7* where point is replaced by point-class and between for points is replaced by between for point-classes.

§2. As we mentioned in section 1 we wish to show that if $\mathfrak{A}^n$ satisfies axioms *A1* through *A10* then it is possible to give $\mathfrak{A}^n$ a vector space structure over an ordered field F. We proceed as follows: Axiom *A10* gives us $n+1$ points $p_0,\ldots,p_n$ which generate $\mathfrak{A}^n$. We shall assume $n=2$ for now and then later we shall show how to extend our results inductively to the case $n>2$. Paragraph and page number references in this section refer to Forder [3].

G1 $[\exists P_2l_1l_2p_0p_1p_2].\mathsf{S}^2(p_0p_1p_2)(P_2).\mathsf{S}^1(p_0p_1)(l_1).$
$\mathsf{S}^1(p_0p_2)(l_2)$ [*A10,n/2,DA2,DA1*]

G2 $[pl]:\mathsf{S}^1(p_0p)(l).\supset.l$ forms an ordered field whose additive identity is p_0 and whose multiplicative identity is p [paragraph 23 p. 197]

G3 $[pqll']:\mathsf{S}^1(p_0p)(l).\mathsf{S}^1(p_0q)(l').\supset.l$ is isomorphic to l' [paragraph 13 p. 191]

DG1 $[ll']:l\,||\,l'.\equiv.[\exists abef].\mathsf{S}^1(ab)(l).\mathsf{S}^1(ef)(l').ab\,||\,ef$

G4 $[ab]::\mathrm{pt}(a).\mathrm{pt}(b).\sim(a=b).\supset\therefore[\exists l]\therefore\mathsf{S}^1(ab)(l):$
$[efl']:\mathsf{S}^1(ef)(l').a\in l'.b\in l'.\supset.l'=l$ [paragraph 2 p. 97]

G5 $[ll'l'']:l\,||\,l'.l'\,||\,l''.\supset.l\,||\,l''$ [paragraph 2 p. 140]

G6 $[ll'l'']:l\,||\,l'.\sim(l\,||\,l'').\supset.\sim(l'\,||\,l'')$ [*G5*]

G7 $[ll'abefgQ]:\mathsf{S}^1(ab)(l).\mathsf{S}^1(ef)(l_2).\mathsf{S}^2(abg)(Q).$
$\sim(l\,||\,l').(l\cup l')\subset Q.\supset.[\exists d].(l\cap l')=\{d\}$ [*DG1,DA4,G4*]

G8 $[pqrp'q'r'Q]:\mathsf{S}^2(pqr)(Q).\{p',q',r'\}\subset Q.$
$\sim(\mathrm{col}(p'q'r')).\supset.[\exists Q'].\mathsf{S}^2(p'q'r')(Q').Q'=Q$ [paragraph 14 p. 59]

G9 $[p]:p\in(P_2-(l_1\cup l_2)).\supset.\mathsf{S}^2(p_0p_1p)(P_2).\mathsf{S}^2(p_0p_2p)(P_2)$ [*G8,DA3,G1*]

G10 $\sim(l_1\,||\,l_2).(l_1\cup l_2)\subset P_2$ [*DG1,G1,DA4,G4,DA2,G1*]

G11 $[p]::p\in(P_2-(l_1\cup l_2)).\supset\therefore[\exists\bar{q}\tilde{q}\bar{l}\tilde{l}]\therefore\bar{q}\in P_2.$
$\mathsf{S}^1(p\bar{q})(\bar{l}).(\bar{l}\cap l_2)=\phi:[el]:e\in P_2.\mathsf{S}^1(ep)(l).$
$(l\cap l_2)=\phi.\supset.l=\bar{l}:q\in P_2.\mathsf{S}^1(p\tilde{q})(\tilde{l}).(\tilde{l}\cap l_1)=$
$\phi:[el]:e\in P_2.\mathsf{S}^1(ep)(l).(l\cap l_1)=\phi.\supset.l=\tilde{l}$ [*A7,G9,G1*]

G12 $[p]::p\in(P_2-(l_1\cup l_2)).\supset\therefore[\exists\tilde{l}\bar{l}]::p\in\bar{l}.l_2\,||\,\bar{l}.$
$p\in\tilde{l}.l_1\,||\,\tilde{l}.(\tilde{l}\cup\bar{l})\subset P_2:[l]:p\in l.l\,||\,l_2.\supset.l=\bar{l}:$
$p\in l.l\,||\,l_1.\supset.l=\tilde{l}$ [*G13,DG1,DA4,G4,G9,G1*]

G13 $[ll']:l\,||\,l_2.l'\,||\,l_1.\supset.\sim(l\,||\,l_1).\sim(l'\,||\,l_2)$ [*G6,G12,G10*]

G14 $[p]:p\in(P_2-(l_1\cup l_2)).\supset.[\exists xy\bar{l}\tilde{l}].p\in\bar{l}.l_2\,||\,\bar{l}.$
$\bar{l}\cap l_1=\{x\}.p\in\tilde{l}.l_1\,||\,\tilde{l}.(\tilde{l}\cap l_2)=\{y\}.x\neq p_0.y\neq p_0$ [*G7,G13,G12,G11,G1*]

G15 $[p]::p\in(P_2-(l_1\cup l_2)).\supset\therefore[\exists xy]\therefore x\in l_1.y\in l_2.$
$px\,||\,yp_0.py\,||\,xp_0:[x'y']:x'\in l_1.y'\in l_2.px'\,||\,y'p_0.$
$py'\,||\,x'p_0.\supset.x'=x.y'=y$ [*G14,DA4,DG1,G4,G11,G1*]

With lemma *G15* we are able to define a map α from P to the cartesian product $l_1\times l_2$ as follows (see figure 1 below):

DG2 $$\alpha(p)=\begin{cases}(p,p_0) \text{ if } p\in l_1\\ (p_0,p) \text{ if } p\in l_2\\ (x,y) \text{ if } p\in(P_2-(l_1\cup l_2)) \text{ where } (x,y) \text{ is as given in lemma } G15\end{cases}$$

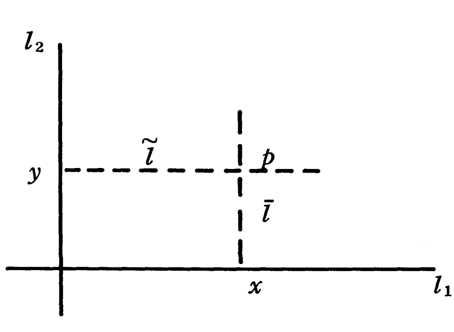

Fig. 1

Our next step is to show that α is a bijective (one-one and onto) map. We do this by proving that there is a map β from $l_1 \times l_2$ to P_2 such that for all $(x,y) \in l_1 \times l_2$ we have $\alpha(\beta(x,y)) = (x,y)$.

G16 $[xy] : x \in l_1 . y \in l_2 . x \neq p_0 . y \neq p_0 . \supset . \sim (x \in l_2) . \sim (y \in l_1)$ [G7,G1,G10]

G17 $[xy] : x \in l_1 . y \in l_2 . x \neq p_0 . y \neq p_0 . \supset . \mathbf{S}^2(p_0 p y)(P_2) . \mathbf{S}^2(p_0 p_2 x)(P_2)$ [G8,DA3,G16,G1]

G18 $[xy] :: x \in l_1 . y \in l_2 . x \neq p_0 . y \neq p_0 . \supset \therefore [\exists \bar{q} \tilde{q} \bar{l} \tilde{l}] \therefore \bar{q} \in P . \mathbf{S}^1(\bar{q}x)(\bar{l}) . (\bar{l} \cap l_2) = \phi : [el] : e \in P_2 . \mathbf{S}^1(ex)(l) . l \cap l_2 = \phi . \supset . l = \bar{l} : \tilde{q} \in P_2 . \mathbf{S}^1(\tilde{q}y)(\tilde{l}) . \tilde{l} \cap l_1 = \phi : [el] : e \in P_2 . \mathbf{S}^1(ey)(l) . l \cap l_1 = \phi . \supset . l = \tilde{l}$ [A7,G17,G1]

G19 $[xy] :\cdot: x \in l_1 . y \in l_2 . x \neq p_0 . y \neq p_0 . \supset :: [\exists \bar{l} \tilde{l}] :: x \in \bar{l} . l_2 \| \bar{l} . y \in \tilde{l} . l_1 \| \tilde{l} . (\tilde{l} \cup \bar{l}) \subset P_2 \therefore [l] \therefore x \in l . l \| l_2 . \supset . l = \bar{l} : y \in l . l \| l_1 . \supset . l = \tilde{l}$ [G18,DG1,DA4,G4,G17,G1]

G20 $[\bar{l} \tilde{l}] : \bar{l} \| l_1 . \tilde{l} \| l_2 . \tilde{l} \| \bar{l} . \supset l_1 \| l_2$ [G5,G1]

G21 $[\bar{l} \tilde{l}] : \bar{l} \| l_1 . \tilde{l} \| l_2 . \supset . \sim (\tilde{l} \| \bar{l})$ [G20,G10,G1]

G22 $[xy] : x \in l_1 . y \in l_2 . x \neq p_0 . y \neq p_0 . \supset . [\exists p \bar{l} \tilde{l}] . x \in \bar{l} . l_2 \| \bar{l} . y \in \tilde{l} . l_1 \| \tilde{l} . (\tilde{l} \cap \bar{l}) = \{p\} . p \in (P_2 - (l_1 \cup l_2))$ [G7,G21,G19,G18,G1]

G23 $[xy] : x \in l_1 . y \in l_2 . x \neq p_0 . y \neq p_0 . \supset \therefore [\exists p] \therefore p \in (P_2 - (l_1 \cup l_2)) . px \| yp_0 . py \| xp_0 : [p'] : p'x \| yp_0 . p'y \| xp_0 . \supset . p' = p$ [G22,DG1,DA4,G4,G18]

With lemma *G23* we may define our map β as follows (see figure 1 above):

DG3 $$\beta(x,y) \begin{cases} x \text{ if } y = p_0 \\ y \text{ if } x = p_0 \\ p \text{ if } x \neq p_0 . y \neq p_0 \text{ where } p \text{ is as given in lemma } G23 \end{cases}$$

It is clear from lemmas *G23* and *G15* that α and β are defined such that $\alpha(\beta(x,y)) = (x,y)$ for all (x,y) in $l_1 \times l_2$.

Now, having shown that α is a bijective map, we may introduce a vector space structure on $\mathfrak{A}^2$ as follows: Let $\mathfrak{F} = l_1$ and let γ be the isomorphism from l_2 to $\mathfrak{F}$ which exists by lemmas *G3* and *G2*. Then let σ be defined on P_2 by $\sigma(p) = (x,\gamma(y))$ where $(x,y) = \alpha(p)$. We have that σ is a bijective map from P onto $\mathfrak{F} \times \mathfrak{F}$. Addition and scalar multiplication can now be defined in P by $p + q = \sigma^{-1}(\sigma(p) \oplus \sigma(q))$ and $tp = t \cdot \sigma(p)$ where $\oplus$ is coordinate-wise addition in $\mathfrak{F} \times \mathfrak{F}$ and $t \cdot (x,y) = (tx,ty)$ for $t,x,y \in \mathfrak{F}$. The remainder of the vector space properties for P_2 follow from the vector space properties of $\mathfrak{F} \times \mathfrak{F}$ over $\mathfrak{F}$ and our map σ. For example, the additive identity in P_2 is p_0 since $\sigma^{-1}(p_0,p_0) = p_0$ and p_0 is the additive identity in $\mathfrak{F}$ by lemma *G2*.

We now show how to extend our results to the case $n > 2$. Without loss of generality we can assume $n = 3$ since by merely repeating the method described below $n - 2$ times one obtains a vector space structure for $\mathfrak{A}^n$ where $n > 3$. Figure 2 illustrates our method.

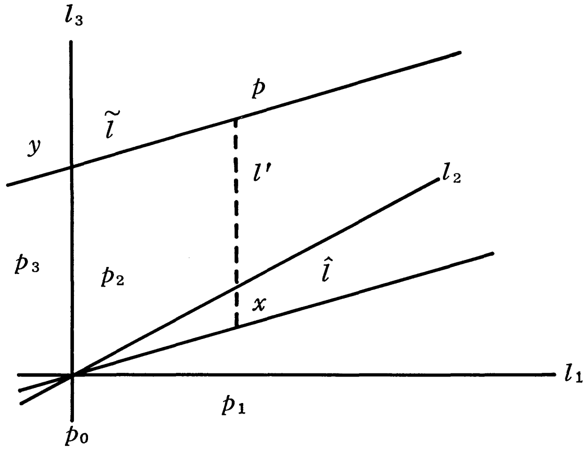

Fig. 2

G24 $[\exists P_3 l_3] \,.\, \mathsf{S}^3(p_0 p_1 p_2 p_3)(P_3) \,.\, \mathsf{S}^1(p_0 p_3)(l_3)$ $[A7, n/3, DA2]$

G25 $[p] : p \in (P_3 - (P_2 \cup l_3)) \,.\supset.\, [\exists Q] \,.\, \mathsf{S}^2(p_0 p_3 p)(Q)$ $[DA2, G24, G1]$

G26 $[p] : p \in (P_3 - (P_2 \cup l_3)) \,.\supset.\, [\exists Q \hat{l} b] \,.\, \mathsf{S}^2(p_0 p_3 p)(Q) \,.$
$\mathsf{S}^1(p_0 b)(\hat{l}) \,.\, \hat{l} = (Q \cap P_2) \,.\, p \in Q = (\hat{l} \cup l_3)$ $[G25, A6, P/Q, Q/P_3, R/P_2, G24, G1]$

G27 $[pQ\hat{l}b] :: p \in (P_3 - (P_2 \cup l_3)) \,.\, \mathsf{S}^2(p_0 p_3 p)(Q) \,.\, \mathsf{S}^1(p_0 b)(\hat{l}) \,.$
$\hat{l} = Q \cap P_2 \,.\, p \in (Q - (\hat{l} \cup l_3)) \,.\supset\therefore [\exists xy] \therefore x \in \hat{l} \,.\, y \in l_3 \,.$
$px \| yp_0 \,.\, py \| xp_0 : [x'y'] : x' \in \hat{l} \,.\, y' \in l_3 \,.\, px' \| y'p_0 \,.$
$py' \| x'p_0 \,.\supset.\, x' = x \,.\, y' = y$ $[G15, P_2/Q, l_1/\hat{l}, l_2/l_3, G26, G8, G24, G4, G1]$

G28 $[x] : x \in (P_2 - l_3) \,.\supset.\, [\exists Q\hat{l}] \,.\, \mathsf{S}^2(p_0 p_3 x)(Q) \,.\, \mathsf{S}^1(p_0 x)(\hat{l}) \,.$
$\hat{l} = Q \cap P_2$ $[DA2, G1, DA1, A6, P/Q, Q/P_3, R/P_2, G8, G4, G24]$

G29 $[xyQ\hat{l}] : x \in (P_2 - l_3) \,.\, y \in l_3 \,.\, y \neq p_0 \,.\, \mathsf{S}^2(p_0 p_3 x)(Q) \,.$
$\mathsf{S}^1(p_0 x)(\hat{l}) \,.\, \hat{l} = Q \cap P_2 \,.\supset.\, x \in \hat{l} \,.\, x \neq p_0$ $[DA1]$

G30 $[xyQ\hat{l}] :: x \in (P_2 - l_3) \,.\, y \in l_3 \,.\, y \neq p_0 \,.\, \mathsf{S}^2(p_0 p_3 x)(Q) \,.$
$\mathsf{S}^1(p_0 x)(\hat{l}) \,.\, \hat{l} = Q \cap P_2 \,.\supset\therefore [\exists p] \therefore p \in (Q - (\hat{l} \cup l_3)) =$
$P_3 - (P_2 \cup l_3) \,.\, px \| yp_0 \,.\, py \| xp_0 : [p'] : p' \in (P_3 - (P_2 \cup l_3)) \,.$
$p'x \| yp_0 \,.\, p'y \| xp_0 \,.\supset.\, p' = p$
$[G23, P_2/Q, l_1/\hat{l}, l_2/l_3, G29, G29, G8, G24, G4, G1]$

Lemmas *G27* and *G30* allow us to define maps $\overline{\alpha}$ from P_3 to $l_1 \times l_2 \times l_3$ and $\overline{\beta}$ from $l_1 \times l_2 \times l_3$ to P_3 respectively such that:

DG4 $$\overline{\alpha}(p) = \begin{cases} (\alpha(p_0), p) & \text{if } p \in l_3 \\ (\alpha(p), p_0) & \text{if } p \in P_2 \\ (\alpha(x), y) & \text{if } p \in (P_3 - (P_2 \cup l_3)) \text{ where } x \text{ and } y \text{ are as given in lemma } G27 \end{cases}$$

DG5 $$\overline{\beta}(x_1, x_2, y) = \begin{cases} \beta(x_1, x_2) & \text{if } y = p_0 \\ y & \text{if } x_1 = p_0 \text{ and } x_2 = p_0 \\ p & \text{if } x_1 \neq p_0 \text{ or } x_2 \neq p_0 \text{ where } p \text{ is determined by lemma } G30 \text{ applied to } \beta(x_1, x_2) \text{ and } y \end{cases}$$

Definitions *DG5, DG4, DG3,* and *DG2* show that $\overline{\alpha}$ is bijective since for all $(x_1,x_2,y) \in l_1 \times l_2 \times l_3$ we have $\overline{\alpha}(\overline{\beta}(x_1,x_2,y)) = (x_1,x_2,y)$. We now introduce a vector space structure on P_3 similar to the way it was done in the case $n = 2$. We let σ be the bijective map from P_2 to $\mathfrak{F} \times \mathfrak{F}$ which was defined above and let γ_3 be the isomorphism from l_3 to $\mathfrak{F}$ which exists by lemmas *G3* and *G2*. Next we define a bijective map $\overline{\sigma}$ from P_3 to $\mathfrak{F} \times \mathfrak{F} \times \mathfrak{F}$ by $\overline{\sigma}(p) = (\sigma(x_1,x_2),\gamma_3(y))$ where $(x_1,x_2,y) = (\alpha(x),y) = \overline{\alpha}(p)$. Now we can define addition and scalar multiplication as before where $\oplus$ is now coordinate-wise addition on $\mathfrak{F} \times \mathfrak{F} \times \mathfrak{F}$ and $t\cdot(x_1,x_2,y) = (tx_1,tx_2,ty)$. The rest of the vector space structure goes through as before except we now use the vector space structure on $\mathfrak{F} \times \mathfrak{F} \times \mathfrak{F}$ and the map $\overline{\sigma}$. In this way we can extend our results to all $n > 2$ where P_k, l_k is replaced at each stage by P_{k+1}, l_{k+1} in lemma *G24* and P_{k-1} is replaced by P_k in lemma *G25* where $3 \leqq k \leqq n - 1$. This then concludes our section 2.

§3. If we look at the construction of our system $(\mathbf{\mathfrak{A}}^n,\mathfrak{F})$ it is clear that it is certainly a model for $(\mathfrak{A}^n,\mathfrak{F})$ since the axioms of $(\mathfrak{A}^n,\mathfrak{F})$ were used to obtain those of $(\mathbf{\mathfrak{A}}^n,\mathfrak{F})$. Therefore, the burden of our proof is to show that $(\mathfrak{A}^n,\mathfrak{F})$ with the vector space structure obtained in section 2 is a model for our system $(\mathbf{\mathfrak{A}}^n,\mathfrak{F})$. In this section we give the preliminaries for our proof:

(A) We use the following notations:

(1) n is a fixed natural number such that $n \geqq 2$.

(2) 0 denotes either the additive identity of $(\mathfrak{A}^n,\mathfrak{F})$ or the additive identity of $\mathfrak{F}$.

(3) 1 denotes the multiplicative identity of $\mathfrak{F}$.

(4) $P,Q,P',Q',P_1,Q_1,P_1',Q_1',\ldots$ denote parallelepipeds as defined in definition *DVP* below.

(5) $\mathbf{a}_0,\mathbf{b}_0,\mathbf{p},\mathbf{q},a',b',p',q',\mathbf{a}_1,\mathbf{b}_1,\mathbf{p}_1,\mathbf{q}_1,a_1',b_1',p_1'q_1',\ldots$ denote points of $\mathfrak{A}^n$.

(6) $S,S_1,S_2,\ldots$ denote sides of parallelepipeds as defined in definition *DVS* below.

(7) $R,R_1,R_2,\ldots$ denote perimeters of parallelepipeds as defined in definition *DVR* below.

(8) $M,M_1,M_2,\ldots$ denote n-simplices as defined in definition *DVM* below.

(9) $H,H_1,H_2,\ldots$ denote halves of parallelepipeds as defined in definition *DVH* below.

(10) $r,s,t,u,v,r',s',u',v',r_1,s_1,t_1,u_1,v_1,r_1',s_1',t_1',u_1',v_1',\ldots$ denote elements of the field $\mathfrak{F}$.

(11) $i,j,k,l,i_1,j_1,k_1,l_1,\ldots$ denote natural numbers.

(12) $<$ and $\leqq$ denote "less than" and "less than or equal" respectively for elements of $\mathfrak{F}$ or for natural numbers.

(13) $0 < t_1,\ldots,t_n < 1$ denotes that each t_i (for $1 \leqq i \leqq n$) is such that $0 < t_i < 1$.

(14) $f{:}A \to B$ denotes f is a function whose domain is A and whose range is B.

(B) We shall use the following definitions:

DV$\bar{P}$ $[\mathbf{a}_0\mathbf{a}_1,\ldots,\mathbf{a}_nP]:\mathsf{P}^n(\mathbf{a}_0,\ldots,\mathbf{a}_n)(P).\equiv.\mathbf{a}_1,\ldots,\mathbf{a}_n$ are linearly independent. $P=\{\mathbf{a}_0+t_1\mathbf{a}_1+\ldots+t_n\mathbf{a}_n \mid 0\leqq t_1,\ldots,t_n\leqq 1\}$.

DVP $[P]:\bar{\mathsf{P}}^n(P).\equiv.[\exists\mathbf{a}_0,\ldots,\mathbf{a}_n].\mathsf{P}^n(\mathbf{a}_0,\ldots,\mathbf{a}_n)(P)$.

DVR $[\mathbf{a}_0\mathbf{a}_1,\ldots,\mathbf{a}_nPR]:\mathsf{R}^n(\mathbf{a}_0,\ldots,\mathbf{a}_nP)(R).\equiv.\mathsf{P}^n(\mathbf{a}_0,\ldots,\mathbf{a}_n)(P).R=\{\mathbf{a}_0+t_1\mathbf{a}_1+\ldots+t_n\mathbf{a}_n \mid [\exists j]:.1\leqq j\leqq n.t_j\in\{0,1\}:[i]:1\leqq i\leqq n.i\neq j.\supset.0\leqq t_i\leqq 1\}$

DV$\bar{S}$ $[\mathbf{a}_0,\ldots,\mathbf{a}_nS_j^iP]:\mathsf{S}^n(\mathbf{a}_0,\ldots,\mathbf{a}_nP)(S_j^i).\equiv.\mathsf{P}^n(\mathbf{a}_0,\ldots,\mathbf{a}_n)(P).i\in\{0,1\}.1\leqq j\leqq n.S_j^i=\{\mathbf{a}_0+t_1\mathbf{a}_1+\ldots+t_n\mathbf{a}_n \mid t_j=i:[k]:1\leqq k\leqq n.k\neq j.\supset.0\leqq t_k\leqq 1\}$

In definition *DV$\bar{S}$* we are saying S_j^i is the j^{th} 0-side (or j^{th} 1-side) of the parallelepiped P determined by $\mathbf{a}_0,\ldots,\mathbf{a}_n$. Figure 3 shows all the S_j^i sides (in the case $n=2$).

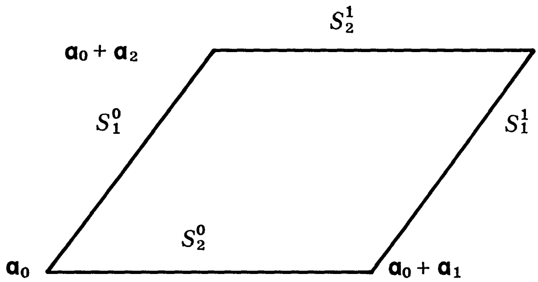

Fig. 3

DVS $[SP]:\bar{\mathsf{S}}^n\{P\}(S).\equiv.[\exists\mathbf{a}_0,\ldots,\mathbf{a}_nS_j^iP].\mathsf{S}^n(\mathbf{a}_0,\ldots,\mathbf{a}_nP)(S_j^i).S=S_j^i$

DVM $[\mathbf{a}_0,\ldots,\mathbf{a}_nM]:\mathsf{M}^n(\mathbf{a}_0,\ldots,\mathbf{a}_n)(M).\equiv.\mathbf{a}_1,\ldots,\mathbf{a}_n$ are linearly independent. $M=\{\mathbf{a}_0+t_1\mathbf{a}_1+\ldots+t_n\mathbf{a}_n \mid 0\leqq t_1,\ldots,t_n\leqq 1.t_1+\ldots+t_n\leqq 1\}$

DVH $[\mathbf{a}_0,\ldots,\mathbf{a}_nH]:\mathsf{H}^n(\mathbf{a}_0,\ldots,\mathbf{a}_n)(H).\equiv.\mathbf{a}_1,\ldots,\mathbf{a}_n$ are linearly independent. $H=\{\mathbf{a}_0+t_1\mathbf{a}_1+\ldots+t_n\mathbf{a}_n \mid 0\leqq t_1,\ldots,t_n\leqq 1.t_1+t_n\leqq 1\}$

(C) The system $(\mathfrak{M}^n,\mathfrak{F})$ shall be interpreted into the system $(\mathfrak{A}^n,\mathfrak{F})$ as follows:

(1) interpret $\mathcal{P}$ as $\bar{\mathsf{P}}^n$ in definition *DVP*
(2) interpret $A\in\mathsf{el}(B)$ as $\bar{\mathsf{P}}^n(A).\bar{\mathsf{P}}^n(B).A\subset B$
(3) interpret $\mathsf{Kl}(a)$ as the union of sets $\bar{\mathsf{P}}^n(A)$ where $A\,\varepsilon\,a$ and we have the implication $[A]:A\,\varepsilon\,a.\supset.\mathcal{P}(A)$

(D) If we apply the definitions given in (C) to our definitions *DM1* to *DM7* we arrive at the following statements:

DV1 $[AB]::\mathsf{EXT}(AB).\equiv.\!\cdot.\bar{\mathsf{P}}^n(A).\bar{\mathsf{P}}^n(B):[D]:\bar{\mathsf{P}}^n(D).\supset.\sim(D\subset(A\cap B))$

DV2 $[AB]:\mathsf{ETG}(AB).\equiv.\mathsf{EXT}(AB).\bar{\mathsf{P}}^n(A\cup B)$

DV3 $[AB]:\cdot:\mathsf{BIS}(AB).\equiv::\bar{\mathsf{P}}^n(A).\bar{\mathsf{P}}^n(B).\sim(A\subset B).\!\cdot.[\exists EF].\!\cdot.\mathsf{EXT}(EF).\mathsf{ETG}(EA).\mathsf{ETG}(FA).E\subset B.F\subset B:[GH]:\bar{\mathsf{P}}^n(G).\bar{\mathsf{P}}^n(H).E\subset G.F\subset G.H\subset(A\cap B).\supset.H\subset G$

DV4 $[AB]::\mathrm{CON}(AB).\equiv.\because.\bar{\mathrm{P}}^n(B).\bar{\mathrm{P}}^n(A):[D]:\mathrm{BIS}(AD).\supset.\mathrm{BIS}(BD)$
DV5 $[AB]::\mathrm{EQV}(AB).\equiv.\because.\bar{\mathrm{P}}^n(A).\bar{\mathrm{P}}^n(B):[EF]:\mathrm{CON}(AE).\mathrm{CON}(BF).$
$\supset.\sim(\mathrm{EXT}(FD))$
DV6 $[a]::\mathrm{PNT}(a).\equiv.\because.[\exists A].\therefore.\bar{\mathrm{P}}^n(A):[B]:a(B).\equiv.\mathrm{EQV}(BA)$
DV7 $[abd]::\mathrm{BNT}(abd).\equiv.\because.\mathrm{PNT}(a).\mathrm{PNT}(b).\mathrm{PNT}(d).\sim(a=b).\sim(b=d).$
$\sim(d=a):[ABDH]:\bar{\mathrm{P}}^n(H).a(A).b(B).d(D).A\subset H.D\subset H.\supset.\sim(\mathrm{EXT}(BH))$

(E) We conclude this section with the following remarks:

(1) In section four we show there exists a 1-1 and onto correspondence σ from the set $\{a|\mathrm{PNT}(a)\}$ onto the points of $(\mathfrak{A}^n,\mathfrak{F})$. Then in section 5 our correspondence preserves the relation of Between (i.e. $\mathrm{BNT}(abd).\equiv.[\exists t].0<t<1.\sigma(b)=\sigma(a)+t(\sigma(d)-\sigma(a))$). Finally in section 6 we outline the proof that the axioms of $(\mathfrak{M}^n,\mathfrak{F})$ are provable in $(\mathfrak{A}^n,\mathfrak{F})$ using the interpretation given in (C).

(2) To facilitate the reading of the proofs many lemmas are preceded by an informal statement of their content and a figure illustrating the lemma in the case $n=2$.

(3) We shall, where practical, omit the definitions *DV$\bar{P}$, DVP, DVR, DV$\bar{S}$, DVS, DVM,* and *DVH* as reasons in our proof. Also we assume an elementary acquaintance with the theory of vector spaces.

§4. The main result of this section will be the following: $[\exists\sigma].\sigma:\{a|\mathrm{PNT}(a)\}\to\mathfrak{A}^n.\sigma$ is bijective.

Using lemmas *L1* to *L4* we shall show in lemmas *L5* and *L6* that if we have $\mathrm{P}^n(\mathfrak{a}_0,\ldots,\mathfrak{a}_n)(P)$ then P is essentially uniquely determined by $\mathfrak{a}_0,\ldots,\mathfrak{a}_n$ where $\mathfrak{a}_0$ is a vertex of P and all other vertices are obtained by adding any number of the other vectors (with coefficient 1) onto $\mathfrak{a}_0$. We show this in figure 3′.

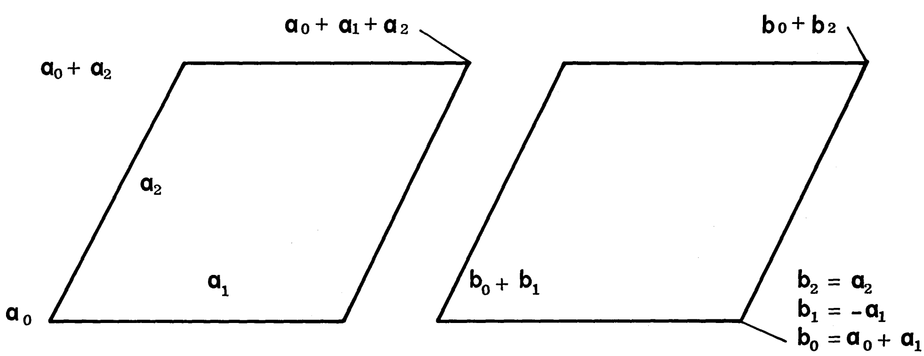

Fig. 3′

L1 $[\mathfrak{a}_0\mathfrak{b}_0,\ldots,\mathfrak{a}_n\mathfrak{b}_nPt_1,\ldots,t_n]:\mathrm{P}^n(\mathfrak{a}_0,\ldots,\mathfrak{a}_n)(P).$
$\mathrm{P}^n(\mathfrak{b}_0,\ldots,\mathfrak{b}_n)(P).0<t_1<1.0\leqq t_2\leqq 1,\ldots,0\leqq t_n$
$\leqq 1.\mathfrak{b}_0=\mathfrak{a}_0+t_1\mathfrak{a}_1+\ldots+t_n\mathfrak{a}_n.\supset.t_1=0$

PF $[\mathfrak{a}_0\mathfrak{b}_0,\ldots,\mathfrak{a}_n\mathfrak{b}_nPt_1,\ldots,t_n]:\mathrm{Hp}(5).\supset.[\exists r_1,\ldots,r_ns_1,\ldots,s_n].$

6) $0 \leqq r_1 \leqq 1, \ldots, 0 \leqq r_n \leqq 1.$
7) $0 \leqq s_1 \leqq 1, \ldots, 0 \leqq s_n \leqq 1.$ [1,2,3,4]
8) $\mathbf{a}_0 + t_2\mathbf{a}_2 + \ldots + t_n\mathbf{a}_n = \mathbf{b}_0 + r_1\mathbf{b}_1 + \ldots + r_n\mathbf{b}_n.$
9) $\mathbf{a}_0 + \mathbf{a}_1 + t_2\mathbf{a}_2 + \ldots + t_n\mathbf{a}_n = \mathbf{b}_0 + s_1\mathbf{b}_1 + \ldots + s_n\mathbf{b}_n.$ [1,2,3,4]
10) $\mathbf{a}_1 = (s_1 - r_1)\mathbf{b}_1 + \ldots + (s_n - r_n)\mathbf{b}_n.$ [8,9]
11) $\mathbf{a}_0 + t_1\mathbf{a}_1 + \ldots + t_n\mathbf{a}_n = t_1[(s_1 - r_1)\mathbf{b}_1 + \ldots + (s_n - r_n)\mathbf{b}_n] + \mathbf{b}_0 + r_1\mathbf{b}_1 + \ldots + r_n\mathbf{b}_n.$ [10,8]
12) $0 = [r_1 + t_1(s_1 - r_1)]\mathbf{b}_1 + \ldots + [r_n + t_1(s_n - r_n)]\mathbf{b}_n.$ [11,5]
13) $r_1 + t_1(s_1 - r_1) = 0, \ldots, r_n + t_1(s_n = r_n) = 0.$ [12,3,6,7]
14) $r_1 + t_1(s_1 - r_1) = (1 - t_1)r_1 + t_1 s_1, \ldots, r_n + t_1(s_n - r_n) = (1 - t_1)r_n + t_1 s_n.$ [3,6,7]
15) $(1 - t_1)(r_1 + \ldots + r_n) + t_1(s_1 + \ldots + s_n) = 0.$ [14]
16) $r_1 + \ldots + r_n = 0.$ [15,3,6,7]
17) $r_1 = 0, \ldots, r_n = 0.$ [16,6]
18) $\mathbf{b}_0 = \mathbf{a}_0 + t_2\mathbf{a}_2 + \ldots + t_n\mathbf{a}_n.$ [8,17]
$t_1 = 0$ [5,8]

L2 $[\mathbf{a}_0, \ldots, \mathbf{a}_n\mathbf{b}_0, \ldots, \mathbf{b}_n P t_1, \ldots, t_n] : \mathbf{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n)(P) . \mathbf{P}^n(\mathbf{b}_0, \ldots, \mathbf{b}_n)(P) . 0 < t_1 < 1 . 0 \leqq t_2 \leqq 1, \ldots, 0 \leqq t_n \leqq 1 . \mathbf{b}_0 + \mathbf{b}_1 = \mathbf{a}_0 + t_1\mathbf{a}_1 + \ldots + t_n\mathbf{a}_n . \supset . t_1 = 0$

PF $[\mathbf{a}_0, \ldots, \mathbf{a}_n\mathbf{b}_0, \ldots, \mathbf{b}_n P t_1, \ldots, t_n] : \mathrm{Hp}(5) . \supset . [\exists r_1, \ldots, r_n s_1, \ldots, s_n].$

6) $0 \leqq r_1 \leqq 1, \ldots, 0 \leqq r_n \leqq 1 . 0 \leqq s_1 \leqq 1, \ldots, 0 \leqq s_n \leqq 1.$
7) $\mathbf{a}_0 + t_2\mathbf{a}_2 + \ldots + t_n\mathbf{a}_n = \mathbf{b}_0 + (1 - r_1)\mathbf{b}_1 + r_2\mathbf{b}_2 + \ldots + r_n\mathbf{b}_n.$
8) $\mathbf{a}_0 + \mathbf{a}_1 + t_2\mathbf{a}_2 + \ldots + t_n\mathbf{a}_n = \mathbf{b}_0 + (1 - s_1)\mathbf{b}_1 + s_2\mathbf{b}_2 + \ldots + s_n\mathbf{b}_n.$ [1,2,3,4]
9) $\mathbf{a}_1 = (r_1 - s_1)\mathbf{b}_1 + (s_2 - r_2)\mathbf{b}_2 + \ldots + (s_n - r_n)\mathbf{b}_n.$ [7,8]
10) $\mathbf{a}_0 + t_1\mathbf{a}_1 + \ldots + t_n\mathbf{a}_n = t_1[(r_1 - s_1)\mathbf{b}_1 + (s_2 - r_2)\mathbf{b}_2 + \ldots + (s_n - r_n)\mathbf{b}_n] + \mathbf{b}_0 + (1 - r_1)\mathbf{b}_1 + r_2\mathbf{b}_2 + \ldots + r_n\mathbf{b}_n.$ [9,7]
11) $0 = [-r_1 + t_1(r_1 - s_1)]\mathbf{b}_1 + [r_2 + t_1(s_2 - r_2)]\mathbf{b}_2 + \ldots + [r_n + t_1(s_n - r_n)]\mathbf{b}_n.$ [10,5]
12) $[-r_1 + t_1(r_1 - s_1)] = 0 . [r_2 + t_1(s_2 - r_2)] = 0, \ldots, [r_n + t_1(s_n - r_n)] = 0.$ [11,6,3]
13) $[-r_1 + t_1(r_1 - s_1)] = [r_1 + t_1(s_1 - r_1)]$ [12]
14) $(1 - t_1)[r_1 + r_2 + \ldots + r_n] + t_1[s_1 + \ldots + s_n] = 0.$ [12,13]
15) $r_1 + \ldots + r_n = 0.$ [14,6,3]
16) $r_1 = 0, \ldots, r_n = 0.$ [15,6]
17) $\mathbf{b}_0 + \mathbf{b}_1 = \mathbf{a}_0 + t_2\mathbf{a}_2 + \ldots + t_n\mathbf{a}_n.$ [16,7]
$t_1 = 0$ [17,5]

L3 $[\mathbf{a}_0\mathbf{b}_0, \ldots, \mathbf{a}_n\mathbf{b}_n P] : \mathbf{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n)(P) . \mathbf{P}^n(\mathbf{b}_0, \ldots, \mathbf{b}_n)(P) . \supset . [\exists t_{01}, \ldots, t_{0n}, \ldots, t_{nn}] . \{t_{11}, \ldots, t_{1n}, \ldots, t_{nn}\} \subset \{0,1\} . \mathbf{b}_0 = \mathbf{a}_0 + t_{01}\mathbf{a}_1 + \ldots + t_{0n}\mathbf{a}_n . \mathbf{b}_0 + \mathbf{b}_1 = \mathbf{a}_0 + t_{11}\mathbf{a}_1 + \ldots + t_{1n}\mathbf{a}_n, \ldots, \mathbf{b}_0 + \mathbf{b}_n = \mathbf{a}_0 + t_{n1}\mathbf{a}_1 + \ldots + t_{nn}\mathbf{a}_n$ [*L1* (n times), *L2*(n^2 times),1,2,$\rightarrow\leftarrow$]

L4 $[\mathfrak{a}_0\mathfrak{b}_0, \ldots, \mathfrak{a}_n\mathfrak{b}_n P t_{01}, \ldots, t_{0n}] : \mathrm{P}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n)(P) .$
$\mathrm{P}^n(\mathfrak{b}_0, \ldots, \mathfrak{b}_n)(P) . \{t_{01}, \ldots, t_{0n}\} \subset \{0,1\} . \mathfrak{b}_0 = \mathfrak{a}_0$
$+ t_{01}\mathfrak{a}_1 + \ldots + t_{0n}\mathfrak{a}_n . \supset . \{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\}$
$= \{(1 - 2t_{01})\mathfrak{a}_1, \ldots, (1 - 2t_{0n})\mathfrak{a}_n\}$

PF $[\mathfrak{a}_0\mathfrak{b}_0, \ldots, \mathfrak{a}_n\mathfrak{b}_n P t_{01}, \ldots, t_{0n}] :: \mathrm{Hp}(4) . \supset :\cdot:$
$[\exists t_{11}, \ldots, t_{1n}, \ldots, t_{n1}, \ldots, t_{nn}] :\cdot:$

5) $\{t_{11}, \ldots, t_{1n}, \ldots, t_{n1}, \ldots, t_{nn}\} \subset \{0,1\} .$
6) $\mathfrak{b}_0 + \mathfrak{b}_1 = \mathfrak{a}_0 + t_{11}\mathfrak{a}_1 + \ldots + t_{1n}\mathfrak{a}_n .$
$\vdots$
$\mathfrak{b}_0 + \mathfrak{b}_n = t_{n1}\mathfrak{a}_1 + \ldots + t_{nn}\mathfrak{a}_n + \mathfrak{a}_0 .$ [*L3*,1,2]

7) $\mathfrak{b}_1 = (t_{11} - t_{01})\mathfrak{a}_1 + \ldots + (t_{1n} - t_{0n})\mathfrak{a}_n .$
$\vdots$
$\mathfrak{b}_n = (t_{n1} - t_{01})\mathfrak{a}_1 + \ldots + (t_{nn} - t_{0n})\mathfrak{a}_n ::$ [6,4]
$[\exists j_1, \ldots, j_n] ::$

8) $1 \leqq j_1 \leqq n, \ldots, 1 \leqq j_n \leqq n.$
9) $(t_{1j_1} - t_{0j_1}) \neq 0, \ldots, (t_{nj_n} - t_{0j_n}) \neq 0 .$ [2]
10) $(t_{1j_1} - t_{0j_1}) = \pm 1, \ldots, (t_{nj_n} - t_{0j_n}) = \pm 1:$ [9,5,3]
11) $(t_{2j_1} - t_{0j_1}) \neq 0 . \supset . (t_{2j_1} - t_{0j_1}) = (t_{1j_1} - t_{0j_1}):$
$\vdots$
$(t_{nj_1} - t_{0j_1}) \neq 0 . \supset . (t_{nj_1} - t_{0j_1}) = (t_{1j_1} - t_{0j_1}) .\cdot.$ [5,3]
$[\exists k] .\cdot.$

12) k is the number of non-zero coefficients of $\mathfrak{a}_{j_1}$ in step 7. [7]
13) $1 \leqq k \leqq n.$ [12,9]
14) $\mathfrak{b}_0 + \mathfrak{b}_1 + \ldots + \mathfrak{b}_n = \mathfrak{a}_0 +$
$[t_{0j_1} + k(t_{1j_1} - t_{0j_1})]\mathfrak{a}_{j_1} +$ [etc]: [7,11,12]
15) $k \geqq 2 . \supset . [t_{0j_1} + k(t_{1j_1} - t_{0j_1})]$
$\notin \{t | 0 \leqq t \leqq 1\}:$ [5,3]
16) $k = 1 :$ [15,14,1,2,→←]
17) $(t_{2j_1} - t_{0j_1}) = 0, \ldots, (t_{nj_1} - t_{0j_1}) = 0.$ [16,12]
18) $j_2 \neq j_1, \ldots, j_n \neq j_1 .$ [17,9]
19) $j_1, \ldots, j_n$ are all distinct. [12-17j_1/j_i i=1, …, n]
20) $1j_1, \ldots, nj_n$ appear in (all) n distinct columns as subscripts in step 7: [19,7]
21) $[ij] : 1 \leqq i \leqq n . 1 \leqq j \leqq n . j \neq j_i . \supset .$
$(t_{ij} - t_{0j}) = 0:$ [20,12-17j_1/j_i i=1, …, n]
22) $\mathfrak{b}_1 = (t_{ij_1} - t_{0j_1})\mathfrak{a}_{j_1} .$
$\vdots$
$\mathfrak{b}_n = (t_{1j_n} - t_{0j_n})\mathfrak{a}_{j_n} .$ [21]
23) $\mathfrak{b}_1 = (1 - 2t_{0j_1})\mathfrak{a}_{j_1}, \ldots, \mathfrak{b}_n =$
$(1 - 2t_{0j_n})\mathfrak{a}_{j_n} .$ [22,9,5,3]
24) $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ are distinct:·: [2]

$\{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\} = \{(1 - 2t_{01})\mathfrak{a}_1, \ldots, (1 - 2t_{0n})\mathfrak{a}_n\}$ [24,23,19]

L5 $[\mathfrak{a}_0\mathfrak{b}_0, \ldots, \mathfrak{a}_n\mathfrak{b}_n P] : \mathrm{P}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n)(P) . \mathrm{P}^n(\mathfrak{b}_0, \ldots, \mathfrak{b}_n)(P) .$
$\supset . [\exists t_1, \ldots, t_n] . \{t_1, \ldots, t_n\} \subset \{0,1\} . \mathfrak{b}_0 = \mathfrak{a}_0 + t_1\mathfrak{a}_1$
$+ \ldots + t_n\mathfrak{a}_n . \{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\} = \{(1 - 2t_1)\mathfrak{a}_1, \ldots, (1 - 2t_n)\mathfrak{a}_n\}$ [*L4*,*L3*]

L6 $[\mathbf{a}_0\mathbf{b}_0, \ldots, \mathbf{a}_n\mathbf{b}_n P t_1, \ldots, t_n] : \mathsf{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n)(P)$.
$\{t_1, \ldots, t_n\} \subset \{0,1\}$. $\mathbf{b}_0 = \mathbf{a}_0 + t_1\mathbf{a}_1 + \ldots + t_n\mathbf{a}_n$.
$\mathbf{b}_1 = (1 - 2t_1)\mathbf{a}_1, \ldots, \mathbf{b}_n = (1 - 2t_n)\mathbf{a}_n$. $\supset$.
$\mathsf{P}^n(\mathbf{b}_0, \ldots, \mathbf{b}_n)(P)$

PF $[\mathbf{a}_0\mathbf{b}_0, \ldots, \mathbf{a}_n\mathbf{b}_n P t_1, \ldots, t_n]$: : Hp(4) . $\supset$.·.
5) $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are linearly independent. [1,2,4]
6) $[s_1, \ldots, s_n]$. $\mathbf{b}_0 + s_1\mathbf{b}_1 + \ldots + s_n\mathbf{b}_n = \mathbf{a}_0 +$
$[t_1 + s_1(1 - 2t_1)]\mathbf{a}_1 + \ldots + [t_n + s_n(1 - 2t_n)]\mathbf{a}_n$: [3,4]
7) $[s_1, \ldots, s_n] : 0 \leqq s_1 \leqq 1, \ldots, 0 \leqq s_n \leqq 1$. ≡.
$0 \leqq t_1 + s_1(1 - 2t_1) \leqq 1, \ldots, 0 \leqq t_n + s_n(1 - 2t_n) \leqq 1$: [2]
8) $[s_1, \ldots, s_n] : 0 \leqq s_1 \leqq 1, \ldots, 0 \leqq s_n \leqq 1$. ≡ .
$\mathbf{b}_0 + s_1\mathbf{b}_1 + \ldots + s_n\mathbf{b}_n \in P$: [1,6,7]
9) $P = \{\mathbf{b}_0 + s_1\mathbf{b}_1 + \ldots + s_n\mathbf{b}_n \mid 0 \leqq s_1 \leqq 1, \ldots, 0 \leqq s_n \leqq 1\}$.·. [8]
$\mathsf{P}^n(\mathbf{b}_0, \ldots, \mathbf{b}_n)(P)$ [5,9]

With lemmas *L7* - *L17* we will show that the relation EXT(PQ) defined by *DV1* holds iff there is no point in the interior of P which belongs to Q. Lemma *L7* shows that if M is an n-simplex all of whose vertices lie in a parallelepiped P then M also is contained in P (see figure 4).

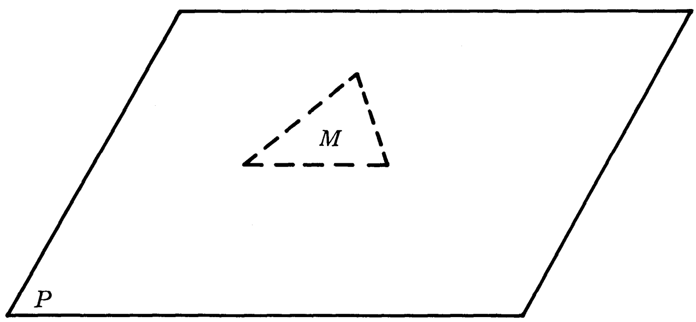

Fig. 4

L7 $[\mathbf{a}_0\mathbf{b}_0, \ldots, \mathbf{a}_n\mathbf{b}_n \mathbf{p} P M] : \mathsf{M}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n)(M)$. $\mathsf{P}^n(\mathbf{b}_0, \ldots, \mathbf{b}_n)(P)$.
$\mathbf{a}_0 \in P$. $\mathbf{a}_0 + \mathbf{a}_1 \in P, \ldots, \mathbf{a}_0 + \mathbf{a}_n \in P$. $\mathbf{p} \in M$. $\supset$. $\mathbf{p} \in P$

PF $[\mathbf{a}_0\mathbf{b}_0, \ldots, \mathbf{a}_n\mathbf{b}_n \mathbf{p} P M]$: : Hp(5) . $\supset$.·.
$[\exists t_{01}, \ldots, t_{0n}, \ldots, t_{n1}, \ldots, t_{nn}]$.·.
6) $0 \leqq t_{01} \leqq 1, \ldots, 0 \leqq t_{0n} \leqq 1, \ldots, 0 \leqq t_{n1} \leqq 1, \ldots, 0 \leqq t_{nn} \leqq 1$.
7) $\mathbf{a}_0 = \mathbf{b}_0 + t_{01}\mathbf{b}_1 + \ldots + t_{0n}\mathbf{b}_n$.
$\mathbf{a}_0 + \mathbf{a}_1 = \mathbf{b}_0 + t_{11}\mathbf{b}_1 + \ldots + t_{1n}\mathbf{b}_n$.
$\vdots$
$\mathbf{a}_0 + \mathbf{a}_n = \mathbf{b}_0 + t_{n1}\mathbf{b}_1 + \ldots + t_{nn}\mathbf{b}_n$: [2,3,4]
$[\exists r_1, \ldots, r_n]$:

8) $0 \leqq r_1, \ldots, 0 \leqq r_n . r_1 + \ldots + r_n \leqq 1.$

9) $\mathbf{p} = \mathbf{a}_0 + r_1\mathbf{a}_1 + \ldots + r_n\mathbf{a}_n.$ [1,5]

10) $\mathbf{p} = \mathbf{b}_0 + t_{01}\mathbf{b}_1 + \ldots + t_{0n}\mathbf{b}_n + r_1[(\mathbf{b}_0 - \mathbf{a}_0) + t_{11}\mathbf{b}_1 + \ldots + t_{1n}\mathbf{b}_n] + \ldots + r_n[(\mathbf{b}_0 - \mathbf{a}_0) + t_{n1}\mathbf{b}_1 + \ldots + t_{nn}\mathbf{b}_n].$ [9,7]

11) $\mathbf{p} = \mathbf{b}_0 + t_{01}\mathbf{b}_1 + \ldots + t_{0n}\mathbf{b}_n + r_1[(-t_{01}\mathbf{b}_1 - \ldots - t_{0n}\mathbf{b}_n) + t_{11}\mathbf{b}_1 + \ldots + t_{1n}\mathbf{b}_n] + \ldots + r_n[(-t_{01}\mathbf{b}_1 - \ldots - t_{0n}\mathbf{b}_n) + t_{n1}\mathbf{b}_1 + \ldots + t_{nn}\mathbf{b}_n].$ [10,7]

12) $\mathbf{p} = \mathbf{b}_0 + (t_{01} + r_1 t_{11} + \ldots + r_n t_{n1} - r_1 t_{01} - \ldots - r_n t_{01})\mathbf{b}_1 + \ldots + (t_{0n} + r_1 t_{1n} + \ldots + r_n t_{nn} - r_1 t_{0n} - \ldots - r_n t_{0n})\mathbf{b}_n.$ [11]

13) $\mathbf{p} = \mathbf{b}_0 + [t_{01} + r_1 t_{11} + \ldots + r_n t_{n1} - (r_1 + \ldots + r_n)\, t_{01}]\mathbf{b}_1 + \ldots + [t_{0n} + r_1 t_{1n} + \ldots + r_n t_{nn} - (r_1 + \ldots + r_n) t_{0n}]\mathbf{b}_n.$ [12]

14) $\mathbf{p} = \mathbf{b}_0 + [t_{01}(1 - (r_1 + \ldots + r_n)) + r_1 t_{11} + \ldots + r_n t_{n1}]\mathbf{b}_1 + \ldots + [t_{0n}(1 - (r_1 + \ldots + r_n)) + r_1 t_{1n} + \ldots + r_n t_{nn}]\mathbf{b}_n.$ [13]

15) $0 \leqq t_{01}(1 - (r_1 + \ldots + r_n)) + r_1 t_{11} + \ldots + r_n t_{n1}\ 0 \leqq t_{0n}(1 - (r_1 + \ldots + r_n)) + r_1 t_{1n} + \ldots + r_n t_{nn}.$ [8,6]

16) $t_{01}(1 - (r_1 + \ldots + r_n)) + r_1 t_{11} + \ldots + r_n t_{n1} \leqq t_{01}(1 - r_1 + \ldots + r_n) + r_1 + \ldots + r_n \leqq 1, \ldots, t_{0n}(1 - r_1 + \ldots + r_n) + r_1 t_{1n} + \ldots + r_n t_{nn} \leqq t_{0n}(1 - r_1 + \ldots + r_n) + r_1 + \ldots + r_n \leqq 1 \therefore$ [8,6]

$\mathbf{p} \in P$ [12,14,15,16]

In lemma *L8* we show that each n-simplex M contains as a subset an n parallelepiped (Figure 5).

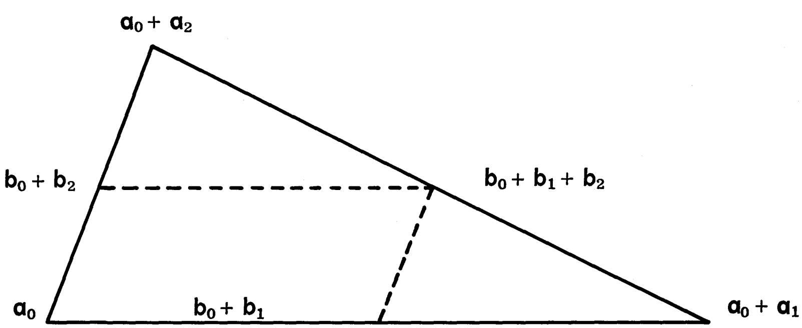

Fig. 5

L8 $[\mathsf{a}_0, \ldots, \mathsf{a}_n M \mathsf{p}] :: \mathsf{M}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(M) . \supset .\therefore$
$[\exists \mathsf{b}_1, \ldots, \mathsf{b}_n P] .\therefore \mathsf{P}^n(\mathsf{a}_0 \mathsf{b}_1, \ldots, \mathsf{b}_n)(P) : \mathsf{p} \in P . \supset . \mathsf{p} \in M$

PF $[\mathsf{a}_0, \ldots, \mathsf{a}_n M \mathsf{p}] :: \mathrm{Hp}(1) . \supset .\therefore$
$[\exists \mathsf{b}_0, \ldots, \mathsf{b}_n P] .\therefore$

2) $\mathsf{b}_0 = \mathsf{a}_0 . \mathsf{b}_1 = \frac{1}{n}\mathsf{a}_1, \ldots, \mathsf{b}_n = \frac{1}{n}\mathsf{a}_n .$ [1]

3) $\mathsf{b}_1, \ldots, \mathsf{b}_n$ are linearly independent. [2,1]

4) $\mathsf{P}^n(\mathsf{b}_0, \ldots, \mathsf{b}_n)(P) . \mathsf{a}_0 \in P$: [2,3]

5) $[\mathsf{p} t_1, \ldots, t_n] : 0 \leqq t_1 \leqq 1, \ldots, 0 \leqq t_n \leqq 1.$
$\mathsf{p} = \mathsf{b}_0 + t_1 \mathsf{b}_1 + \ldots + t_n \mathsf{b}_n . \supset . \mathsf{p} = \mathsf{a}_0 +$
$\frac{t_1}{n}\mathsf{a}_1 + \ldots + \frac{t_n}{n}\mathsf{a}_n . 0 \leqq \frac{t_1}{n}, \ldots, 0 \leqq \frac{t_n}{n}.$
$\frac{t_1}{n} + \ldots + \frac{t_n}{n} \leqq \frac{1}{n} + \ldots + \frac{1}{n} = 1:$ [2]

6) $\mathsf{p} \in P . \supset . \mathsf{p} \in M .\therefore$ [5,1]

$[\exists \mathsf{b}_0, \ldots, \mathsf{b}_n P] .\therefore \mathsf{P}^n(\mathsf{b}_0, \ldots, \mathsf{b}_n)(P) : \mathsf{p} \in P . \supset . \mathsf{p} \in M$ [4,6]

Using *L7* and *L8* we formulate (*L9*, *L10*, *L11*) an equivalent condition to $\mathsf{EXT}(P_1 P_2)$ which instead of requiring a parallelepiped Q to be contained in $P_1 \cap P_2$ requires an n-simplex M to be contained in $P_1 \cap P_2$. Figure 6 is to be used with *L9* and figure 7 is to be used with *L10*.

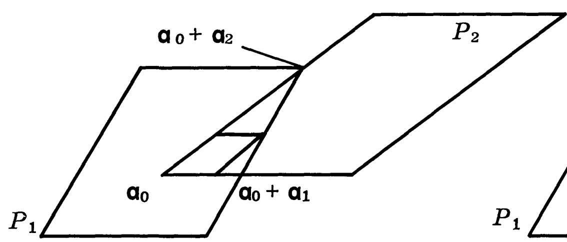

Fig. 6

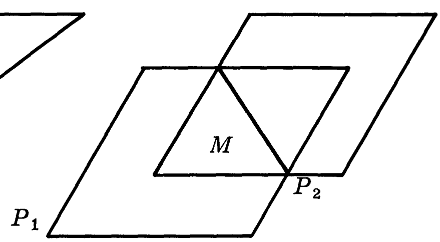

Fig.7

L9 $[P_1 P_2] : \mathsf{EXT}(P_1 P_2) . \supset . \sim[\exists \mathsf{a}_0, \ldots, \mathsf{a}_n M] . \mathsf{M}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)$
$(M) . \mathsf{a}_0 \in P_1 \cap P_2 . \mathsf{a}_0 + \mathsf{a}_1 \in P_1 \cap P_2, \ldots, \mathsf{a}_0 + \mathsf{a}_n \in$
$P_1 \cap P_2$ [*DV1*, *L8*, $\rightarrow\leftarrow$]

L10 $[P_1 P_2] : \bar{\mathsf{P}}^n(P_1) . \bar{\mathsf{P}}^n(P_2) . \sim \mathsf{EXT}(P_1 P_2) . \supset . [\exists \mathsf{a}_0, \ldots, \mathsf{a}_n M].$
$\mathsf{M}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(M) . M \subset (P_1 \cap P_2)$

PF $[P_1 P_2] .\therefore \mathrm{Hp}(3) . \supset :$
$[\exists Q \mathsf{a}_0, \ldots, \mathsf{a}_n] :$

4) $\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(Q) .$

5) $Q \subset P_1 \cap P_2 .$ [1,2,3]

6) $\mathsf{a}_0 \in Q . \mathsf{a}_0 + \mathsf{a}_1 \in Q, \ldots, \mathsf{a}_0 + \mathsf{a}_n \in Q.$ [4]
$[\exists M].$

7) $\mathsf{M}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(M).$ [4]

8) $M \subset Q.$ [*L7*,4,6]

9) $M \subset P_1 \cap P_2$: [8,5]

$[\exists \mathfrak{a}_0, \ldots, \mathfrak{a}_n M] . \mathsf{M}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n)(M) . M \subset P_1 \cap P_2$ [9,7]

L11 $[P_1 P_2] .\therefore \bar{\mathsf{P}}^n(P_1) . \bar{\mathsf{P}}^n(P_2) . \supset : \mathsf{EXT}(P_1 P_2) . \equiv .$
$\sim([\exists \mathfrak{a}_0, \ldots, \mathfrak{a}_n M] . \mathsf{M}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n)(M) . M \subset P_1 \cap P_2)$ [*L9*,*L7*;*L10*]

In *L12* we are saying if P is a parallelepiped, R is the perimeter of P, $\mathfrak{p}$ belongs to the interior of P (i.e. P-R), and $\mathfrak{q}$ does not belong to P then there is a point $\mathfrak{a}$ which lies between $\mathfrak{p}$ and $\mathfrak{q}$ and $\mathfrak{a} \in R$ (see figure 8).

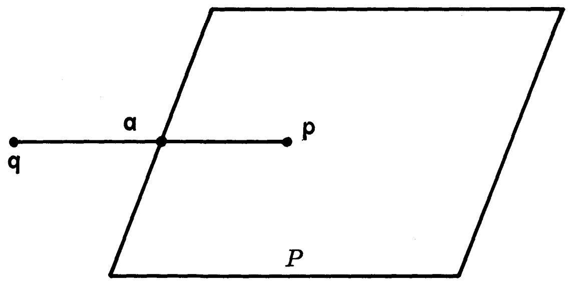

Fig. 8

L12 $[\mathfrak{a}_0, \ldots, \mathfrak{a}_n \mathfrak{p}\mathfrak{q} PR] : \mathsf{R}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n P)(R) . \mathfrak{p} \in (P - R).$
$\sim(\mathfrak{q} \in P) . \supset . [\exists \mathfrak{a} t] . 0 < t < 1 . \mathfrak{a} = \mathfrak{p} + t(\mathfrak{q} - \mathfrak{p}) . \mathfrak{a} \in R$

PF $[\mathfrak{a}_0, \ldots, \mathfrak{a}_n \mathfrak{p}\mathfrak{q} PR] :: \mathrm{Hp}(3) . \supset :\cdot:$

$[\exists s_1, \ldots, s_n t_1, \ldots, t_n] :\cdot:$

4) $0 < s_1 < 1, \ldots, 0 < s_n < 1.$
5) $\mathfrak{p} = \mathfrak{a}_0 + s_1 \mathfrak{a}_1 + \ldots + s_n \mathfrak{a}_n .$ [1,2]

6) $\mathfrak{q} = \mathfrak{a}_0 + t_1 \mathfrak{a}_1 + \ldots + t_n \mathfrak{a}_n .$
7) $\sim (0 \leqq t_1 \leqq 1, \ldots, 0 \leqq t_n \leqq 1) ::$ [1,3]

$[\exists k_1, \ldots, k_n] ::$

8)
$$k_1 = \begin{cases} 1 & \text{if } 0 \leqq t_1 \leqq 1 \\ \dfrac{1 - s_1}{t_1 - s_1} & \text{if } t_1 > 1 \\ \dfrac{s_1}{s_1 - t_1} & \text{if } t_1 < 0 \end{cases}$$
$$\vdots \qquad \vdots$$
$$k_n = \begin{cases} 1 & \text{if } 0 \leqq t_n \leqq 1 \\ \dfrac{1 - s_n}{t_n - s_n} & \text{if } t_n > 1 \\ \dfrac{s_n}{s_n - t_n} & \text{if } t_n < 0 :: \end{cases}$$ [5,6]

$[\exists t] .\therefore$

9) $t = \min \{k_1, \ldots, k_n\}$ [8]

10) $0 < t < 1.$ [9,8,4]

11) $0 \leqq s_1 + t(t_1 - s_1) \leqq 1, \ldots, 0$
$\leqq s_n + t(t_n - s_n) \leqq 1:$ [8,4]

$[\exists l]:$

12) $1 \leqq l \leqq n.$ [8,7,4]
13) $s_l + t(t_l - s_l) \epsilon \{0,1\}.$

$[\exists \mathsf{a}]$.

14) $\mathsf{a} = \mathsf{p} + t(\mathsf{q} - \mathsf{p})$. [2,9]
15) $\mathsf{a} \epsilon R$:·: [13,14,5,6]

$[\exists \mathsf{a} t] . 0 < t < 1 . \mathsf{a} = \mathsf{p} + t(\mathsf{q} - \mathsf{p}) . \mathsf{a} \epsilon R$ [10,14,15]

With *L13* we achieve half of the goal which was announced just prior to *L7*; namely, we show that if P_1 is a parallelepiped with perimeter R, P_2 is a parallelepiped, and p is a point in $P_2 \cap (P_1 - R)$ then P_1 and P_2 are not external. See figure 9.

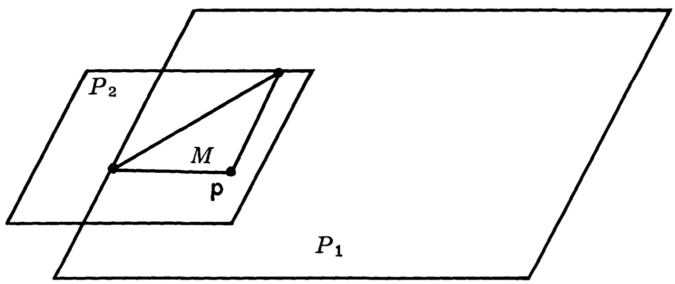

Fig. 9

L13 $[\mathsf{a}_0\mathsf{b}_0, \ldots, \mathsf{a}_n\mathsf{b}_n P_1 P_2 R \mathsf{p}] : \mathsf{R}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n P_1)(R)$.
$\mathsf{P}^n(\mathsf{b}_0, \ldots, \mathsf{b}_n)(P_2) . \mathsf{p} \epsilon P_2 \cap (P_1 - R) . \supset . \sim(\mathsf{EXT}(P_1 P_2))$

PF $[\mathsf{a}_0\mathsf{b}_0, \ldots, \mathsf{a}_n\mathsf{b}_n P_1 P_2 R \mathsf{p}]$ ⁝ Hp(3) . ⊃ ⁖ $[\exists t_1 s_1 r_1, \ldots, t_n s_n r_n]$ ⁖

4) $\mathsf{a}_0 = t_1 \mathsf{a}_1 + \ldots + t_n \mathsf{a}_n$. [1]
5) $\mathsf{p} = \mathsf{a}_0 + s_1 \mathsf{a}_1 + \ldots + s_n \mathsf{a}_n$. [1,3]
6) $0 < s_1 < 1, \ldots, 0 < s_n < 1$.
7) $\mathsf{p} = \mathsf{b}_0 + r_1 \mathsf{b}_1 + \ldots + r_n \mathsf{b}_n$. [2,3]
8) $0 \leqq r_1 \leqq 1, \ldots, 0 \leqq r_n \leqq 1$ ⁝

$[\exists t_{i1}, \ldots, t_{in}]$ ⁝

9) $1 \leqq i \leqq n$. [1,2]
10) $\mathsf{b}_i - \mathsf{a}_0 = t_{i1} \mathsf{a}_1 + \ldots + t_{in} \mathsf{a}_n$.
11) $\mathsf{b}_i =$

$(t_1 + t_{i1})\mathsf{a}_1 + \ldots + (t_n + t_{in})\mathsf{a}_n$ ⁖ [10]

$[\exists u_i t'_{ij} u_{ij}]$ ⁖

12) $1 \leqq j \leqq n . 1 \leqq i \leqq n$. [7]

13) $$u_i = \begin{cases} \frac{1}{2}(1 - r_i) & \text{if } r_i < 1. \\ -\frac{1}{2} & \text{if } r_i = 1. \end{cases}$$

14) $$t'_{ij} = \begin{cases} \frac{1 - s_j}{t_{ij} + t_j} & \text{if } t_{ij} + t_j > 0. \\ 1 & \text{if } t_{ij} + t_j = 0. \\ \frac{-s_j}{t_{ij} + t_j} & \text{if } t_{ij} + t_j < 0. \end{cases}$$ [11,5]

15) $u_{ij} = \begin{cases} \dfrac{-s_j}{t_{ij}+t_j} & \text{if } t_{ij}+t_j > 0. \\ -1 & \text{if } t_{ij}+t_j = 0. \\ \dfrac{1-s_j}{t_{ij}+t_j} & \text{if } t_{ij}+t_j < 0. \end{cases}$ [11,5]

16) $u_{ij} \leqq 0 \leqq t'_{ij}$. [14,15,6]

17) $0 \leqq s_j + t'_{ij}(t_{ij}+t_j) \leqq 1$ [14,6]

18) $0 \leqq s_j + u_{ij}(t_{ij}+t_j) \leqq 1$: : [15,6]

$[\exists t'_i u'_i]$: :

19) $1 \leqq i \leqq n$.

20) $t'_i = \min\{t'_{i1}, \ldots, t'_{in}\}$. [14,15]

21) $u'_i = \min\{u_{i1}, \ldots, u_{in}\}$.

22) $0 \leqq s_j + t'_i(t_{ij} + \mathbf{a}_j) \leqq 1$. [20,14,6]

23) $0 \leqq s_j + u'_i(t_{ij} + \mathbf{a}_j) \leqq 1$. [21,15,6]

24) $\mathbf{p} + t'_i\mathbf{b}_i = \mathbf{a}_0 + (s_1 + t'_i(t_i + t_{i1}))\mathbf{a}_1 + \ldots + (s_n + t'_i(t_i + t_{in}))\mathbf{a}_n$. [5,11]

25) $\mathbf{p} + u'_i\mathbf{b}_i = \mathbf{a}_0 + (s_1 + u'_i(t_i + t_{i1}))\mathbf{a}_1 + \ldots + (s_n + t'_i(t_i + t_{in}))\mathbf{a}_n$. [5,11]

26) $\mathbf{p} + t'_i\mathbf{b}_i \epsilon P_1$. [24,22]

27) $\mathbf{p} + u'_i\mathbf{b}_i \epsilon P_1$.·. [25,23]

$[\exists r'_i]$.·.

28) $1 \leqq i \leqq n$.

29) $r'_i = \begin{cases} \min\{t'_i, u_i\} & \text{if } r_i < 1. \\ \max\{u'_i, u_i\} & \text{if } r_i = 1. \end{cases}$ [20,21,13,8]

30) $0 \leqq r_i + r'_i \leqq 1$.

31) $0 \leqq s_j + r'_i(t_{ij} + t_j) \leqq 1$. [29,20,21,14,15,6]

32) $r'_i \neq 0$.

33) $r'_1\mathbf{b}_1, \ldots, r'_n\mathbf{b}_n$ are linearly independent. [29,2]

34) $\mathbf{p} + r'_i\mathbf{b}_i \epsilon (P_1 \cap P_2)$: [30,31,1,2,5,7]

$[\exists M]$:

35) $\mathbf{M}^n(\mathbf{p}, r'_1\mathbf{b}_1, \ldots, r'_n\mathbf{b}_n)(M)$. [33,5,2]

$M \subset P_1 \cap P_2$. [L7,35,34,3]

$[\exists Q]$.

36) $\overline{\mathbf{P}}^n(Q)$.

37) $Q \subset P_1 \cap P_2$:·: [L6,35]

$\sim(\text{EXT}(P_1 P_2))$ [37,36]

In *L14* we are saying if two points $\mathbf{p}_1$ and $\mathbf{p}_2$ belong to a parallelepiped P then every point $\mathbf{q}$ between $\mathbf{p}_1$ and $\mathbf{p}_2$ also belongs to P.

L14 $[\mathbf{a}_0, \ldots, \mathbf{a}_n \mathbf{p}_1 \mathbf{p}_2 \mathbf{q} t P] : \mathbf{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n)(P) . 0 \leqq t \leqq 1.$
$\mathbf{p}_1 \epsilon P . \mathbf{p}_2 \epsilon P . \mathbf{q} = \mathbf{p}_1 + t(\mathbf{p}_2 - \mathbf{p}_1) . \supset . \mathbf{q} \epsilon P$

PF $[\mathbf{a}_0, \ldots, \mathbf{a}_n \mathbf{p}_1 \mathbf{p}_2 \mathbf{q} t P] : \text{Hp}(5) . \supset .$
$[\exists s_1 t_1, \ldots, s_n t_n]$.

6) $0 \leqq s_1 \leqq 1 . 0 \leqq t_1 \leqq 1, \ldots, 0 \leqq s_n \leqq 1.$
$0 \leqq t_n \leqq 1.$
7) $p_1 = a_0 + s_1 a_1 + \ldots + s_n a_n .$ [1,3,4]
8) $p_2 = a_0 + t_1 a_1 + \ldots + t_n a_n .$
9) $q = a_0 + (s_1 + t(t_1 - s_1)) a_1 + \ldots +$
$(s_n + t(t_n - s_n)) a_n .$ [5,7,8]
10) $0 \leqq s_1 - ts_1 \leqq s_1 - ts_1 + tt_1 = s_1 +$
$t(t_1 - s_1), \ldots, 0 \leqq s_n - ts_n \leqq s_n - ts_n +$
$tt_n = s_n + t(t_n - s_n).$ [2,6]
11) $s_1 + t(t_1 - s_1) = s_1(1-t) + tt_1 \leqq (1-t) +$
$t = 1, \ldots, s_n + t(t_n - s_n) = s_n(1-t) + tt_n \leqq$
$(1-t) + t = 1.$ [2,6]
$q \in P$ [1,9,10,11]

In our next lemma we refine *L14* a little to say that if p_1 and p_2 besides being in the parallelepiped P also do not lie on the same side then if q is between p_1 and p_2 we have that q belongs to the interior of P (i.e., $q \in P - R$).

L15 $[a_0, \ldots, a_n p_1 p_2 q t P R] : R^n(a_0, \ldots, a_n P)(R) . p_1 \in P .$
$p_2 \in P . 0 < t < 1 . q = p_1 + t(p_2 - p_1) . \sim [\exists S_j^i] .$
$S^n(a_0, \ldots, a_n P)(S_j^i) . p_1 \in S_j^i . p_2 \in S_j^i) . \supset . q \in (P - R)$

PF $[a_0, \ldots, a_n p_1 p_2 q t P R] :: \mathrm{Hp}(6) . \supset .\because$
$[\exists s_1 t_1, \ldots, s_n t_n] .\because$
7) $0 \leqq s_1 \leqq 1 . 0 \leqq t_1 \leqq 1, \ldots, 0 \leqq s_n \leqq 1.$
$0 \leqq t_n \leqq 1.$ [1,2,3]
8) $p_1 = a_0 + s_1 a_1 + \ldots + s_n a_n .$
9) $p_2 = a_0 + t_1 a_1 + \ldots + t_n a_n .$
10) $q = a_0 + (s_1 + t(t_1 - s_1)) a_1 + \ldots +$
$(s_n + t(t_n - s_n)) a_n .$ [5,8,9]
11) $q \in P.$ [*L14*,1,2,3,4,5]
12) $0 \leqq s_1 + t(t_1 - s_1) \leqq 1, \ldots, 0 \leqq s_n +$
$t(t_n - s_n) \leqq 1:$ [11,10,1]
13) $s_1 + t(t_1 - s_1) = 0 . \supset . s_1(1-t) + tt_1 = 0:$ [7,4]
14) $s_1 + t(t_1 - s_1) = 0 . \supset . s_1(1-t) = 0 . tt_1 = 0:$ [13,7,4]
15) $s_1 + t(t_1 - s_1) = 0 . \supset . s_1 = 0 . t_1 = 0:$ [14,7,4]
16) $s_1 + t(t_1 - s_1) = 0 . \supset . p_1 \in S_1^\circ . p_2 \in S_1^\circ :$ [15,6]
17) $\sim (s_1 + t(t_1 - s_1) = 0), \ldots, \sim (s_n + t(t_n - s_n) = 0):$
[16,6, $\rightarrow\leftarrow$(n times)]
18) $s_1 + t(t_1 - s_1) = 1 . \supset . s_1(1-t) + tt_1 = 1:$ [7,4]
19) $s_1 + t(t_1 - s_1) = 1 . \supset . s_1 = 1 . t_1 = 1:$ [18,7,4]
20) $s_1 + t(t_1 - s_1) = 1 . \supset . p_1 \in S_1^1 . p_2 \in S_1^1 :$ [19,6]
21) $\sim (s_1 + t(t_1 - s_1) = 1), \ldots, \sim (s_n + t(t_n - s_n) = 1) .$
[20,6, $\rightarrow\leftarrow$(n times)]
22) $0 < s_1 + t(t_1 - s_1) < 1, \ldots, 0 < s_n +$
$t(t_n - s_n) < 1 .\because$ [12,17,21]
$q \in P - R$ [22,11,1]

Our next three lemmas show by induction that if all the vertices of a parallelepiped P_1 belong to a parallelepiped P_2 then P_1 is contained in P_2.

L16 $[P_1 P_2 \mathsf{a}_0, \ldots, \mathsf{a}_n t_1, \ldots, t_n \mathsf{b}_0 \mathsf{b}_1 Q] : \bar{\mathsf{P}}^n(P_2) . \mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(P_1) .$
$\{t_1, \ldots, t_n\} \subset \{0,1\} . \mathsf{b}_0 = \mathsf{a}_0 + t_1 \mathsf{a}_1 + \ldots +$
$t_n \mathsf{a}_n . \mathsf{b}_1 \epsilon \{(1-2t_1)\mathsf{a}_1, \ldots, (1-2t_n)\mathsf{a}_n\} . \{\mathsf{b}_0, \mathsf{b}_1\} \subset P_2 .$
$\mathsf{P}^1(\mathsf{b}_0 \mathsf{b}_1)(Q) . \supset . Q \subset P_2$ [*L14*]

L17 $[P_1 P_2 \mathsf{a}_0, \ldots, \mathsf{a}_n \mathsf{b}_0, \ldots, \mathsf{b}_k \mathsf{p}_0, \ldots, \mathsf{p}_{k+1} t_1 s_1, \ldots, t_n s_n Q Q_1 \mathsf{p}]$
$\therefore \bar{\mathsf{P}}^n(P_2) . \mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(P_1) . \{s_1, \ldots, s_n\} \subset \{0,1\} .$
$1 \leqq k \leqq n-1 . \mathsf{p}_0 = \mathsf{a}_0 + s_1 \mathsf{a}_1 + \ldots + s_n \mathsf{a}_n .$
$\{\mathsf{p}_1, \ldots, \mathsf{p}_{k+1}\} \subset \{(1-2s_1)\mathsf{a}_1, \ldots, (1-2s_n)\mathsf{a}_n\} .$
$\{\mathsf{p}_0 + s_1' \mathsf{a}_1 + \ldots + s_n' \mathsf{a}_n | \{s_1', \ldots, s_n'\} \subset \{0,1\}\} \subset P_2 .$
$\mathsf{P}^{k+1}(\mathsf{p}_0, \ldots, \mathsf{p}_{k+1})(Q_1) . \mathsf{p} \epsilon Q_1 : \mathsf{P}^k(\mathsf{b}_0, \ldots, \mathsf{b}_k)(Q).$
$\mathsf{b}_0 = \mathsf{a}_0 + t_1 \mathsf{a}_1 + \ldots + t_n \mathsf{a}_n . \{t_1, \ldots, t_n\} \subset \{0,1\} .$
$\{\mathsf{b}_1, \ldots, \mathsf{b}_k\} \subset \{(1-2t_1)\mathsf{a}_1, \ldots, (1-2t_n)\mathsf{a}_n\} . \supset . Q \subset P_2 :$
$\supset . \mathsf{p} \epsilon P_2$

PF $[P_1 P_2 \mathsf{a}_0, \ldots, \mathsf{a}_n \mathsf{b}_0, \ldots, \mathsf{b}_k \mathsf{p}_0, \ldots, \mathsf{p}_{k+1} t_1 s_1, \ldots, t_n s_n Q Q_1 \mathsf{p}]$
$:: \mathrm{Hp}(10) : \supset \therefore$

$[\exists Q_2 Q' \mathsf{b}_0 \mathsf{b}_0' \mathsf{b}_1 \mathsf{b}_1', \ldots, \mathsf{b}_k \mathsf{b}_k'] \therefore$

11) $\mathsf{b}_0 = \mathsf{p}_0, \ldots, \mathsf{b}_k = \mathsf{p}_k .$
12) $\mathsf{P}^k(\mathsf{b}_0, \ldots, \mathsf{b}_k)(Q_2) .$
13) $\mathsf{b}_0' = \mathsf{p}_0 + \mathsf{p}_{k+1} . \mathsf{b}_1' = \mathsf{p}_1, \ldots, \mathsf{b}_k' = \mathsf{p}_k .$
14) $\mathsf{P}^k(\mathsf{b}_0', \ldots, \mathsf{b}_k')(Q') .$ [2,3,4,5,6]
15) $Q_2 \subset P_2 .$ [10,12,11,3,5,6]
16) $Q' \subset P_2 :$ [10,14,13,3,5,6]

$[\exists r_1, \ldots, r_{k+1}] :$

17) $0 \leqq r_1, \ldots, r_{k+1} \leqq 1 .$
18) $\mathsf{p} = \mathsf{p}_0 + r_1 \mathsf{p}_1 + \ldots + r_{k+1} \mathsf{p}_{k+1} .$ [4,8,9]

$[\exists \mathsf{q}_1 \mathsf{q}_2] .$

19) $\mathsf{q}_1 = \mathsf{p}_0 + r_1 \mathsf{p}_1 + \ldots + r_k \mathsf{p}_k .$
20) $\mathsf{q}_2 = \mathsf{p}_0 + \mathsf{p}_{k+1} + r_1 \mathsf{p}_1 + \ldots + r_k \mathsf{p}_k .$ [17,8]
21) $\mathsf{q}_1 \epsilon Q_2 .$ [19,17,12,11]
22) $\mathsf{q}_2 \epsilon Q' .$ [20,17,14,13]
23) $\mathsf{q}_1 \epsilon P_2 . \mathsf{q}_2 \epsilon P_2 .$ [15,16,21,22]
24) $\mathsf{p} = \mathsf{q}_1 + r_{k+1}(\mathsf{q}_2 - \mathsf{q}_1) \therefore$ [18,19,20]

$\mathsf{p} \epsilon P_2$ [*L14*,1,17,23,24]

L18 $[P_1 P_2 \mathsf{a}_0, \ldots, \mathsf{a}_n] : \bar{\mathsf{P}}^n(P_2) . \mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(P_1) .$
$\{\mathsf{a}_0 + t_1 \mathsf{a}_1 + \ldots + t_n \mathsf{a}_n | \{t_1, \ldots, t_n\} \subset \{0,1\}\} \subset P_2 . \supset .$
$P_1 \subset P_2$ [*L16*,*L17*]

We are now ready to prove the other half of what we said we would show just before *L7*. Namely, we now show that if P_1 and P_2 are parallelepipeds which contain a parallelepiped Q in their intersection then there exists a point p belonging to $P_2 \cap (P_1 - R)$ where R is the perimeter of P_1.

L19 $[\mathsf{a}_0 \mathsf{a}_0', \ldots, \mathsf{a}_n \mathsf{a}_n' P_1 P_2 R] : \mathsf{R}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n P_1)(R) . P^n(\mathsf{a}_0', \ldots, \mathsf{a}_n')(P_2) .$
$\sim \mathrm{EXT}(P_1 P_2) . \supset . [\exists \mathsf{q}] . \mathsf{q} \epsilon P_2 \cap (P_1 - R)$

PF $[\mathsf{a}_0\mathsf{a}_0', \ldots, \mathsf{a}_n\mathsf{a}_n' P_1 P_2 R] \therefore \mathrm{Hp}(3) . \supset ::$

$[\exists \mathsf{b}_0, \ldots, \mathsf{b}_n Q] ::$

4) $\mathsf{P}^n(\mathsf{b}_0, \ldots, \mathsf{b}_n)(Q)$

5) $Q \subset P_1 \cap P_2 :.$ [1,2,3]

$[\exists r_1 s_1 t_1, \ldots, r_n s_n t_n] .\because$

6) $0 \leqq r_1, \ldots, r_n \leqq 1.$

7) $0 \leqq s_1, \ldots, s_n \leqq 1.$

8) $0 \leqq t_1, \ldots, t_n \leqq 1.$

9) $\mathsf{b}_0 = \mathsf{a}_0 + r_1\mathsf{a}_1 + \ldots + r_n\mathsf{a}_n .$

10) $\mathsf{b}_0 + \frac{1}{2}\mathsf{b}_1 + \ldots + \frac{1}{2}\mathsf{b}_n = \mathsf{a}_0 + s_1\mathsf{a}_1 + \ldots + s_n\mathsf{a}_n .$ [5,1]

11) $\mathsf{b}_0 + \mathsf{b}_1 + \ldots + \mathsf{b}_n = \mathsf{a}_0 + t_1\mathsf{a}_1 + \ldots + t_n\mathsf{a}_n .$

12) $\frac{1}{2}\mathsf{b}_1 + \ldots + \frac{1}{2}\mathsf{b}_n = (t_1 - s_1)\mathsf{a}_1 + \ldots + (t_n - s_n)\mathsf{a}_n .$ [11,10]

13) $r_1 + (t_1 - s_1) = s_1, \ldots, r_n + (t_n - s_n) = s_n .$ [1,9,12,10]

14) $r_1 + t_1 - 2s_1, \ldots, r_n + t_n = 2s_n :$ [13]

15) $s_1 = 0 . \supset . t_1 = 0, \ldots, s_n = 0 . \supset . t_n = 0 :$ [7,8,14]

16) $s_1 = 0 . \supset . (t_1 - s_1) = 0, \ldots, s_n = 0 . \supset (t_n - s_n) = 0 :$ [15]

17) $\{\mathsf{b}_1, \ldots, \mathsf{b}_n\}$ has dimension n. [1]

18) $\sim(s_1 = 0), \ldots, \sim(s_n = 0) :$ [17,12,16,→←]

19) $s_1 = 1 . \supset . t_1 = 1, \ldots, s_n = 1 . \supset . t_n = 1 :$ [14,6,8]

20) $s_1 = 1 . \supset . (t_1 - s_1) = 0, \ldots, s_n = 1 . \supset . (t_n - s_n) = 0 :$ [19]

21) $\sim(s_1 = 1), \ldots, \sim(s_n = 1) .$ [17,12,20,→←]

$[\exists \mathsf{q}] .$

22) $\mathsf{q} = \mathsf{b}_0 + \frac{1}{2}\mathsf{b}_1 + \ldots + \frac{1}{2}\mathsf{b}_n .$ [4]

23) $\mathsf{q} \in (P_1 - R) ::$ [22,5,1,10,18,21]

$[\exists \mathsf{q}] . \mathsf{q} \in P_2 \cap (P_1 - R)$ [23,22,4,5]

We now state as *L20* the reduction of *DV1* to the statement that two parallelepipeds P_1 and P_2 are external iff there exists a point q in the interior of one and which belongs to the other.

L20 $[\mathsf{a}_0, \ldots, \mathsf{a}_n P_1 P_2 R] \therefore \mathsf{R}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n P_1)(R) . \bar{\mathsf{P}}^n(P_2) . \supset :$
$\mathrm{EXT}(P_1 P_2) . \equiv . \sim([\exists \mathsf{q}] . \mathsf{q} \in P_2 \cap (P_1 - R))$ [*L19*,*L13*]

Our next step will be to show that if two parallelepipeds P_1 and P_2 are externally tangent (*DV2*) then there exist vectors $\mathsf{a}_0, \ldots, \mathsf{a}_n\mathsf{a}_n'$ and $t < 0$ such that we have $\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(P_1)$, $\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_{n-1}\mathsf{a}_n')(P_2)$ and $\mathsf{a}_n' = t\mathsf{a}_n$. We do this using lemmas *L21* - *L29*. In lemma *L21* we show that if p_1 belongs to parallelepiped P and $\mathbf{p}_2$ does not then there exists a point $\mathbf{q}$ between $\mathbf{p}_1$ and $\mathbf{p}_2$ which also does not belong to P (figure 10).

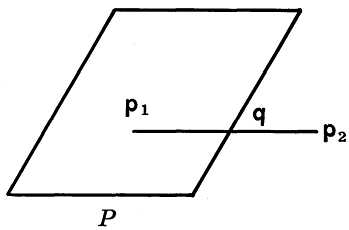

Fig. 10

L21 $[\mathbf{a}_0, \ldots, \mathbf{a}_n \mathbf{p}_1 \mathbf{p}_2 P] : \mathbf{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n)(P) \,.\, \mathbf{p}_1 \epsilon P \,.\, \sim(\mathbf{p}_2 \epsilon P).$
$\supset . [\exists \mathbf{q} t] \,.\, 0 < t < 1 . \mathbf{q} = \mathbf{p}_1 + t(\mathbf{p}_2 - \mathbf{p}_1) \,.\, \sim (\mathbf{q} \epsilon P)$

PF $[\mathbf{a}_0, \ldots, \mathbf{a}_n \mathbf{p}_1 \mathbf{p}_2 P] :: \mathrm{Hp}(3) . \supset :\cdot:$

$[\exists t_1, \ldots, t_{2n}] :\cdot:$

4) $0 \leqq t_1 \leqq 1, 0 \leqq t_3 \leqq 1, \ldots, 0 \leqq t_{2n-1} \leqq 1.$ [1,2]

5) $\mathbf{p}_1 = \mathbf{a}_0 + t_1 \mathbf{a}_1 + t_3 \mathbf{a}_2 + \ldots + t_{2n-1} \mathbf{a}_n .$ [1,2]

6) $\mathbf{p}_2 = \mathbf{a}_0 + t_2 \mathbf{a}_1 + t_4 \mathbf{a}_2 + \ldots + t_{2n} \mathbf{a}_n ::$ [1,2]

$[\exists i j u_1 u_2] ::$

7) $i \epsilon \{2, 4, \ldots, 2n\} .$ [6,1,3]

8) $j = i - 1 :$ [6,1,3]

9) $t_i > 1 . \vee . t_i < 0 :$ [6,1,3]

10) $u_1 = \dfrac{\frac{1}{2}(t_i + 1) - t_j}{t_i - t_j} .$ [5,6]

11) $u_2 = \dfrac{\frac{1}{2} t_i - t_j}{t_i - t_j} \therefore$ [5,6]

$[\exists s_1 s_1', s_3 s_3', \ldots, s_{2n-1} s_{2n-1}' \mathbf{q}_1 \mathbf{q}_2] \therefore$

12) $\mathbf{q}_1 = \mathbf{a}_0 + s_1 \mathbf{a}_1 + s_3 \mathbf{a}_2 + \ldots + s_{2n-1} \mathbf{a}_n .$ [2,10,11,6]

13) $\mathbf{q}_2 = \mathbf{a}_0 + s_1' \mathbf{a}_1 + s_3' \mathbf{a}_2 + \ldots + s_{2n-1}' \mathbf{a}_n .$ [2,10,11,6]

14) $\mathbf{q}_1 = \mathbf{p}_1 + u_1(\mathbf{p}_2 - \mathbf{p}_1).$ [2,10,11,6]

15) $\mathbf{q}_2 = \mathbf{p}_1 + u_2(\mathbf{p}_2 - \mathbf{p}_1).$ [2,10,11,6]

16) $s_j = t_j + u_1(t_i - t_j) .$ [14,12,5,6]

17) $s_j' = t_j + u_2(t_i - t_j) :$ [15,13,5,6]

18) $t_i > 1 . \supset . 0 < u_1 < 1 :$ [10,8,6,4]

19) $t_i > 1 . \supset . s_j > 1 :$ [16,10]

20) $t_i > 1 . \supset . \sim(\mathbf{q}_1 \epsilon P) :$ [19,12,1]

21) $t_i < 0 . \supset . 0 < u_2 < 1 :$ [11,8,6,4]

22) $t_i < 0 . \supset . s_j' < 0$: [17,11]

23) $t_i < 0 . \supset . \sim(q_2 \in P)$:·: [22,13,1]

$[\exists q t] . q = p_1 + t(p_2 - p_1) . \sim(q \in P)$ [9,20,14,23,15]

With *L19* (and *L20*) we show that if P_1 and P_2 are externally tangent parallelepipeds, P_1 is determined by the vectors $a_0, \ldots, a_n$, p belongs to $P_2 - P_1$ and the j^{th} coordinate "t_j" of p with respect to $\{a_0, \ldots, a_n\}$, and $t_j > 1 (t_j < 0)$ then the side $S_j^1 (S_j^0)$ of P_1 is contained in $P_1 \cap P_2$ (figure 11 shows the case $t_1 > 1$).

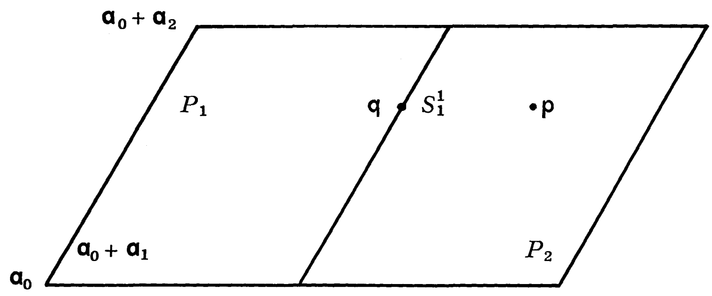

Fig. 11

L22 $[a_0, \ldots, a_n t_1, \ldots, t_n j p_1 p_2 S_j^1 P_1 P_2] : S^n(a_0, \ldots, a_n P_1)(S_j^1)$.
ETG $(P_1 P_2) . p_1 \in (P_2 - P_1) . p_1 = a_0 + t_1 a_1 + \ldots + t_n a_n$.
$t_j > 1 . p_2 \in S_j^1 . \supset . p_2 \in (P_1 \cap P_2)$

PF $[a_0, \ldots, a_n t_1, \ldots, t_n j p_1 p_2 S_j^1 P_1 P_2]$:·: Hp(6). $\supset$: :
$[\exists s_1, \ldots, s_n]$: :

7) $p_2 = a_0 + s_1 a_1 + \ldots + s_n a_n$.

8) $0 \leqq s_1 \leqq 1, \ldots, 0 \leqq s_{j-1} \leqq 1 . 0 \leqq s_{j+1} \leqq 1, \ldots, 0 \leqq s_n \leqq 1$.

9) $s_j = 1$ ∴ [1,6]

$[\exists Q]$ ∴

10) $\bar{P}^n(Q)$.

11) $Q = P_1 \cup P_2$. [2]

12) $p_1 \in Q . p_2 \in Q$: [11,3,6,1]

13) $[t] : 0 < t < 1 . \supset . t_j + t(s_j - t_j) = t_j(1 - t) + t s_j > 1$: [9,5]

14) $[qt] : 0 < t < 1 . q = p_1 + t(p_2 - p_1) . \supset . \sim(q \in P_1)$: [13,3,4,7]

15) $[qt] : 0 < t < 1 . q = p_1 + t(p_2 - p_1) . \supset . q \in Q$: [*L14*,1,12]

16) $[qt] : 0 < t < 1 . q = p_1 + t(p_2 - p_1) . \supset . q \in P_2$: [11,15,14]

17) $p_2 \in P_2$: : [*L21*,1,3,16,→←]

$p_2 \in P_1 \cap P_2$ [1,6,17]

L23 $[\mathfrak{a}_0, \ldots, \mathfrak{a}_n t_1, \ldots, t_n j \mathfrak{p}_1 \mathfrak{p}_2 S_j^0 P_1 P_2] : \mathsf{S}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n P_1)(S_j^0) .$
$\mathrm{ETG}(P_1 P_2) . \mathfrak{p}_1 \in (P_2 - P_1) . \mathfrak{p}_1 = \mathfrak{a}_0 + t_1\mathfrak{a}_1 + \ldots + t_n\mathfrak{a}_n .$
$t_j < 0 . \mathfrak{p}_2 \in S_j^0 . \supset . \mathfrak{p}_2 \in (P_1 \cap P_2)$

PF $[\mathfrak{a}_0, \ldots, \mathfrak{a}_n t_1, \ldots, t_n j \mathfrak{p}_1 \mathfrak{p}_2 S_j^0 P_1 P_2] :\cdot: \mathrm{Hp}(6) . \supset ::$

$[\exists s_1, \ldots, s_n] ::$

7) $\mathfrak{p}_2 = \mathfrak{a}_0 + s_1\mathfrak{a}_1 + \ldots + s_n\mathfrak{a}_n .$
8) $0 \leqq s_1 \leqq 1, \ldots, 0 \leqq s_{j-1} \leqq 1 . 0 \leqq s_{j+1} \leqq 1, \ldots, 0 \leqq s_n \leqq 1.$
9) $s_j = 0 .\because$ [1,6]

$[\exists Q] .\because$

10) $\bar{\mathsf{P}}^n(Q) .$
11) $Q = P_1 \cup P_2 .$
12) $\mathfrak{p}_1 \in Q . \mathfrak{p}_2 \in Q :$ [2]
13) $[t] : 0 < t < 1 . \supset . t_j + t(s_j - t_j) = t_j(1 - t) < 0 :$ [9,5]
14) $[\mathfrak{q}t] : 0 < t < 1 . \mathfrak{q} = \mathfrak{p}_1 + t(\mathfrak{p}_2 - \mathfrak{p}_1) . \supset . \sim(\mathfrak{q} \in P_1) :$ [13,3,7]
15) $[\mathfrak{q}t] : 0 < t < 1 . \mathfrak{q} = \mathfrak{p}_1 + t(\mathfrak{p}_2 - \mathfrak{p}_1) . \supset . \mathfrak{q} \in Q :$ [*L14*,1,12]
16) $[\mathfrak{q}t] : 0 < t < 1 . \mathfrak{q} = \mathfrak{p}_1 + t(\mathfrak{p}_2 - \mathfrak{p}_1) . \supset . \mathfrak{q} \in P_2 :$ [11,15,14]
17) $\mathfrak{p}_2 \in P_2 ::$ [*L21*,1,3,16,→←]

$\mathfrak{p}_2 \in P_1 \cap P_2$ [1,6,17]

We now summarize the result of the last two lemmas in our next lemma by saying if P_1 and P_2 are externally tangent then there exists a side of P_1 contained in $P_1 \cap P_2$.

L24 $[P_1 P_2] : \mathrm{ETG}(P_1 P_2) . \supset . [\exists S] . \bar{\mathsf{S}}^n\{P_1\}(S) . S \subset (P_1 \cap P_2)$

PF $[P_1 P_2] :: \mathrm{Hp}(1) . \supset :\cdot:$

2) $P_2 \subset P_1 . \supset . \sim \mathrm{EXT}(P_1 P_2) :\cdot:$ [1]

$[\exists \mathfrak{p}] :\cdot:$

3) $\mathfrak{p} \in (P_2 - P_1) ::$ [1,2,→←]

$[\exists \mathfrak{a}_0, \ldots, \mathfrak{a}_n t_1, \ldots, t_n j] ::$

4) $\mathsf{P}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n)(P_1) .$ [1]
5) $\mathfrak{p} = \mathfrak{a}_0 + t_1\mathfrak{a}_1 + \ldots + t_n\mathfrak{a}_n$ [3,4]
6) $1 \leqq j \leqq n :$
7) $t_j < 0 . \vee . t_j > 1 .\because$ [3,5,4]

$[\exists S_j^i] .\because$

8) $\mathsf{S}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n P_1)(S_j^i) .$
9) $t_j > 1 . \supset . i = 1 :$
10) $t_j < 0 . \supset . i = 0 :$ [4]
11) $S_j^i \subset P_1 \cap P_2 :\cdot:$ [*L22*,*L23*,8,1,3,5,9,10,7]

$[\exists S] . \bar{\mathsf{S}}^n\{P_1\}(S) . S \subset (P_1 \cap P_2).$ [8,11]

Now we work on the uniqueness of having one side of P_1 contained in $P_1 \cap P_2$ if P_1 and P_2 are externally tangent. Our next lemma shows that if

S is a side of P_1 and p belongs to $P_1 - S$ then if P_1 and P_2 are externally tangent and $S \subset P_1 \cap P_2$ it follows that p doesn't belong to P_2 (since otherwise P_1 and P_2 would not be external).

L25 $[P_1 P_2 S \mathrm{p}]$: $\mathrm{ETG}(P_1 P_2) . \bar{\mathrm{S}}^n \{P_1\} (S) . \mathrm{p} \in (P_1 - S) . S \subset (P_1 \cap P_2) .$
$\supset . \sim (\mathrm{p} \in P_2)$

PF $[P_1 P_2 S \mathrm{p}] :: \mathrm{Hp}(4) . \supset :\cdot:$
$[\exists \mathrm{a}_0, \ldots, \mathrm{a}_n S_j^i] :\cdot:$

5) $\mathrm{S}^n(\mathrm{a}_0, \ldots, \mathrm{a}_n P_1)(S_j^i) .$
6) $S = S_j^i ::$ [2]

$[\exists \mathrm{q} t_1, \ldots, t_n] ::$

7) $\mathrm{q} = \mathrm{a}_0 + t_1 \mathrm{a}_1 + \ldots + t_n \mathrm{a}_n .$
8) $t_j = i .$
9) $t_1 = \frac{1}{2}, \ldots, t_{j-1} = \frac{1}{2} . t_{j+1} = \frac{1}{2}, \ldots, t_n = \frac{1}{2} .$ [6]
10) $\mathrm{q} \in S .\because$

$[\exists R] .\because$

11) $\mathrm{R}^n(\mathrm{a}_0, \ldots, \mathrm{a}_n P_1)(R) :$ [5]
12) $\mathrm{p} \in R . \vee . \mathrm{p} \in (P_1 - R) :$ [3]
13) $\mathrm{p} \in (P_1 - R) . \mathrm{p} \in P_2 . \supset . \sim \mathrm{EXT}(P_1 P_2) :$ [*L13*,11,1]
14) $\mathrm{p} \in R . \supset . \sim ([\exists S_j^i] \mathrm{S}^n(\mathrm{a}_0, \ldots, \mathrm{a}_n P_1)(S_j^i) . \mathrm{p} \in S_j^i . \mathrm{q} \in S_j^i) :$ [7,8,9,3,10]
15) $\mathrm{p} \in R . \mathrm{p} \in P_2 . \supset . \mathrm{p} + \frac{1}{2}(\mathrm{q} - \mathrm{p}) \in (P_1 - R) :$ [*L15*,11,10,14,p₂/q]
16) $\mathrm{p} \in R . \mathrm{p} \in P_2 . \supset . \mathrm{p} + \frac{1}{2}(\mathrm{q} - \mathrm{p}) \in P_2 :$ [*L14*,11,10,6,4]
17) $\mathrm{p} \in R . \mathrm{p} \in P_2 . \supset . \sim \mathrm{EXT}(P_1 P_2) :\cdot:$ [*L13*,11,15,16,1]

$\sim (\mathrm{p} \in P_2)$ [1,12,13,17,→←]

In *L26* we state the uniqueness mentioned above. Notice that *L21* means $P_1 \cap P_2 = S$.

L26 $[P_1 P_2 S S_1] : \mathrm{ETG}(P_1 P_2) . \bar{\mathrm{S}}^n \{P_1\} (S) . \bar{\mathrm{S}}^n \{P_2\} (S_1) . S \subset P_1 \cap P_2 .$
$S_1 \subset P_1 \cap P_2 . \supset . S = S_1$

PF $[P_1 P_2 S S_1] : \mathrm{Hp}(5) . \supset .$

6) $P_1 \cap P_2 = S .$ [*L25*,1,2,4]
7) $P_1 \cap P_2 = S_1 .$ [*L25*,1,3,5]

$S = S_1$ [6,7]

In *L27* we show that if two parallelepipeds P_1 and P_2 are externally tangent then except for one vector they can be generated by the same set of vectors, (see figure 12). We use *L5* and *L6* where n is replaced by $n - 1$.

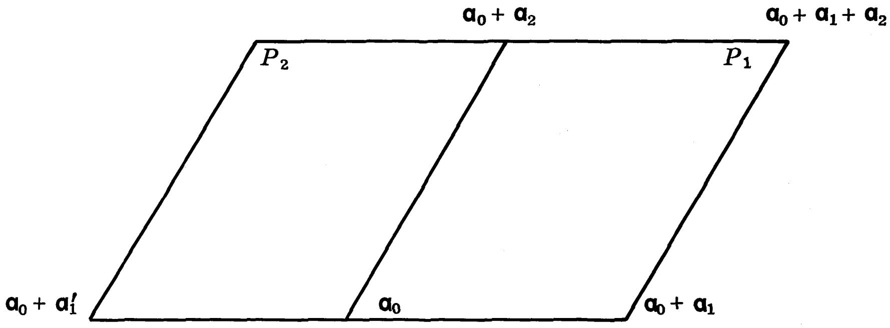

Fig. 12

(we can use $\mathbf{a}_0, \mathbf{a}_1', \mathbf{a}_2$ to generate P_2 where $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ generate P_1)

L27 $[P_1 P_2] : \mathrm{ETG}(P_1 P_2) \,.\supset.\, [\exists \mathbf{a}_0, \ldots, \mathbf{a}_n \mathbf{a}_n'] \,.\, \mathrm{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n)(P_1) \,.$
$\mathrm{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}, \mathbf{a}_n')(P_2)$

PF $[P_1 P_2] :: \mathrm{Hp}(1) \,.\supset :\cdot:$
$[\exists \mathbf{b}_0 \mathbf{b}_0', \ldots, \mathbf{b}_n \mathbf{b}_n' S_j^i S_k^l] :\cdot:$

2) $\mathrm{S}^n(\mathbf{b}_0, \ldots, \mathbf{b}_n P_1)(S_j^i) .$
3) $\mathrm{S}^n(\mathbf{b}_0', \ldots, \mathbf{b}_n' P_2)(S_k^l) .$
4) $P_1 \cap P_2 = S_j^i .$
5) $P_2 \cap P_1 = S_k^l .$ [*L24*,*L25*,1,(2 times)]
6) $S_j^i = S_k^l .$ [4,5]
7) $\mathrm{P}^{n-1}(\mathbf{b}_0 + i\mathbf{b}_j, \mathbf{b}_1, \ldots, \mathbf{b}_{j-1}, \mathbf{b}_{j+1}, \ldots, \mathbf{b}_n)(S_j^i) .$ [*L6*,2]
8) $\mathrm{P}^{n-1}(\mathbf{b}_0' + l\mathbf{b}_k', \mathbf{b}_1', \ldots, \mathbf{b}_{k-1}', \mathbf{b}_{k+1}', \ldots, \mathbf{b}_n')(S_k^l) ::$ [*L6*,3]
$[\exists \mathbf{a}_0, \ldots, \mathbf{a}_n, \mathbf{p}_0, \ldots, \mathbf{p}_n] ::$

9) $\mathbf{a}_0 = \mathbf{b}_0 + i\mathbf{b}_j .$
10) $\mathbf{a}_1 = \mathbf{b}_1, \ldots, \mathbf{a}_{j-1} = \mathbf{b}_{j-1} .$
11) $\mathbf{a}_j = (1 - 2i)\mathbf{b}_j .$ [*L6*,2]
12) $\mathbf{a}_{j+1} = \mathbf{b}_{j+1}, \ldots, \mathbf{a}_n = \mathbf{b}_n .$
13) $\mathrm{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n)(P_1) .$
14) $\mathrm{P}^{n-1}(\mathbf{a}_0, \ldots, \mathbf{a}_{j-1}, \mathbf{a}_{j+1}, \ldots, \mathbf{a}_n)(S_j^i) .$ [7,9,10,11,12]
15) $\mathbf{p}_0 = \mathbf{b}_0' + l\mathbf{b}_k .$
16) $\mathbf{p}_1 = \mathbf{b}_1', \ldots, \mathbf{p}_{k-1} = \mathbf{b}_{k-1}' .$
17) $\mathbf{p}_k = (1 - 2l)\mathbf{b}_k' .$ [*L6*,3]
18) $\mathbf{p}_{k+1} = \mathbf{b}_{k+1}', \ldots, \mathbf{p}_n = \mathbf{b}_n' .$
19) $\mathrm{P}^n(\mathbf{p}_0, \ldots, \mathbf{p}_n)(P_2) .$
20) $\mathrm{P}^{n-1}(\mathbf{p}_0, \ldots, \mathbf{p}_{k-1}, \mathbf{p}_{k+1}, \ldots, \mathbf{p}_n)(S_k^l) \,.\cdot.$ [8,15,16,17,18]
$[\exists t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n] \,.\cdot.$

21) $\{t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n\} \subset \{0,1\} .$
22) $\mathbf{a}_0 = \mathbf{p}_0 + t_1\mathbf{p}_1 + \ldots + t_{k-1}\mathbf{p}_{k-1}$
$+ t_{k+1}\mathbf{p}_{k+1} + \ldots + t_n\mathbf{p}_n .$ [*L5*,14,20,6]
23) $\{\mathbf{a}_1, \ldots, \mathbf{a}_{j-1}, \mathbf{a}_{j+1}, \ldots, \mathbf{a}_n\} =$
$\{(1 - 2t_1)\mathbf{p}_1, \ldots, (1 - 2t_{k-1})\mathbf{p}_{k-1},$
$(1 - 2t_{k+1})\mathbf{p}_{k+1}, \ldots, (1 - 2t_n)(\mathbf{p}_n)\} :$
$[\exists \mathbf{b}_0, \ldots, \mathbf{b}_n] :$

24) $\mathsf{b}_0 = \mathsf{a}_0$.
25) $\mathsf{b}_1 = \mathsf{a}_1, \ldots, \mathsf{b}_{j-1} = \mathsf{a}_{j-1}$.
26) $\mathsf{b}_j = \mathsf{p}_k$. [L6,23,22,19]
27) $\mathsf{b}_{j+1} = \mathsf{a}_{j+1}, \ldots, \mathsf{b}_n = \mathsf{a}_n$.
28) $\mathsf{P}^n(\mathsf{b}_0, \ldots, \mathsf{b}_n)(P_2)$.
29) $\mathsf{P}^n(\mathsf{b}_0, \ldots, \mathsf{b}_{j-1}, \mathsf{b}_{j+1}, \ldots, \mathsf{b}_n, \mathsf{b}_j)(P_2)$ [L6,28]
30) $\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_{j-1}, \mathsf{a}_{j+1}, \ldots, \mathsf{a}_n, \mathsf{b}_j)(P_2)$. [29,24,25,26,27]
31) $\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_{j-1}, \mathsf{a}_{j+1}, \ldots, \mathsf{a}_n, \mathsf{a}_j)(P_1)$:·: [L6,13]

$[\exists \mathsf{a}_0, \ldots, \mathsf{a}_n, \mathsf{a}_n'] . \mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(P_1) . \mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_{n-1}\, \mathsf{a}_n')(P_2)$ [30,31]

We now prove, for later use, a variation of lemma *L27* which says that if $\mathsf{a}_0, \ldots, \mathsf{a}_n$ generate P_1, P_1 and P_2 are externally tangent, and the side S_n^0 of P_1 equals the intersection $P_1 \cap P_2$ then there is a point a_n' such that $\mathsf{a}_0, \ldots, \mathsf{a}_{n-1}, \mathsf{a}_n'$ generate P_2.

L28 $[\mathsf{a}_0, \ldots, \mathsf{a}_n P_1 P_2 S_n^0] : \mathsf{ETG}(P_1 P_2) . \mathsf{S}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n P_1)(S_n^0)$.
$S_n^0 \subset P_2 . \supset . [\exists \mathsf{a}_n'] . \mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_{n-1}, \mathsf{a}_n')(P_2)$

PF $[\mathsf{a}_0, \ldots, \mathsf{a}_n P_1 P_2 S_n^0] :: \mathrm{Hp}(3) . \supset .\cdot.$

4) $S_n^0 = P_1 \cap P_2$. [1,2,3,*L21*]
5) $\mathsf{P}^{n-1}(\mathsf{a}_0, \ldots, \mathsf{a}_{n-1})(S_n^0)$: [2]

$[\exists \mathsf{b}_0, \ldots, \mathsf{b}_n, S_j^i]$:

6) $\mathsf{S}^n(\mathsf{b}_0, \ldots, \mathsf{b}_n P_2)(S_j^i)$. [1]
7) $S_j^i \subset P_1 \cap P_2$.
8) $S_j^i = P_1 \cap P_2$. [*L21*,7]
9) $S_j^i = S_n^0$. [*L26*,1,2,3,7]
10) $\mathsf{P}^{n-1}(\mathsf{b}_0 + i\mathsf{b}_j, \mathsf{b}_1, \ldots, \mathsf{b}_{j-1}, \mathsf{b}_{j+1}, \ldots, \mathsf{b}_n)(S_j^i)$. [*L6*,6]

$[\exists t_1, \ldots, t_{j-1} t_{j+1}, \ldots, t_n]$.

11) $\mathsf{a}_0 = \mathsf{b}_0 + i\mathsf{b}_j + t_1 \mathsf{b}_1 + \ldots + t_{j-1}\mathsf{b}_{j-1} + t_{j+1}\mathsf{b}_{j+1} + \ldots + t_n \mathsf{b}_n$ [*L5*,10,5,9]
12) $\{\mathsf{a}_1, \ldots, \mathsf{a}_{n-1}\} = \{(1-2t_1)\mathsf{b}_1, \ldots, (1-2t_{j-1})\mathsf{b}_{j-1}, (1-2t_{j+1})\mathsf{b}_{j+1}, \ldots, (1-2t_n)\mathsf{b}_n\}$.
13) $\mathsf{P}^{n-1}(\mathsf{a}_0, \ldots, \mathsf{a}_{n-1})(S_j^i)$. [*L6*,10,11,12]
14) $\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_{n-1}(1-2i)\mathsf{b}_j)(P_2)$:. [*L6*,6,11,12]

$[\exists \mathsf{a}_n'] . \mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_{n-1}, \mathsf{a}_n')(P_2)$ [14]

We are now ready for the last stage in obtaining the characterization for *DV2*, mentioned just before lemma *L21*. Namely, we show in lemma *L29* that if P_1 and P_2 are externally tangent and we have vectors $\mathsf{a}_0, \ldots, \mathsf{a}_n \mathsf{a}_n'$ such that $\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(P_1)$ holds and $\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_{n-1}\mathsf{a}_n')(P_2)$ holds (such vectors exist by *L27*) then there is a "t" such that $\mathsf{a}_n' = t\,\mathsf{a}_n$ and $t < 0$. Conversely, in lemma *L30*, we show that if $\mathsf{a}_n' = t\mathsf{a}_n$ and $t < 0$ then $\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_{n-1}\mathsf{a}_n')(P_2)$ implies that P_2 is externally tangent to P_1.

L29 $[\mathsf{a}_0, \ldots, \mathsf{a}_n \mathsf{a}_n' P_1 P_2] : \mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(P_1)$.
$\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_{n-1}, \mathsf{a}_n')(P_2)$.
$\mathsf{ETG}(P_1 P_2) . \supset : [\exists t] . \mathsf{a}_n' = t\mathsf{a}_n . t < 0$

PF $[\mathsf{a}_0, \ldots, \mathsf{a}_n \mathsf{a}_n' P_1 P_2] :: \mathrm{Hp}(3) . \supset :\cdot:$

$[\exists t_1, \ldots, t_n] :\cdot:$

4) $\mathfrak{a}_n' = t_1\mathfrak{a}_1 + \ldots + t_n\mathfrak{a}_n.$ [1,2]

5) $\mathfrak{a}_0 + \ldots + \mathfrak{a}_{n-1} + \mathfrak{a}_n' = \mathfrak{a}_0 + (1+t_1)\mathfrak{a}_1 + \ldots$
$+ (1 + t_{n-1})\mathfrak{a}_{n-1} + t_n\mathfrak{a}_n.$ [4]

6) $\mathfrak{a}_0 + \mathfrak{a}_n' = \mathfrak{a}_0 + t_1\mathfrak{a}_1 + \ldots + t_n\mathfrak{a}_n:$ [4]

7) $t_1 > 0 . \vee ., \ldots, . \vee . t_{n-1} > 0 . \supset .$
$\sim[(\mathfrak{a}_0 + \ldots + \mathfrak{a}_{n-1} + \mathfrak{a}_n') \epsilon P_2]:$ [2,5]

8) $\sim(t_1 > 0) ., \ldots, . \sim(t_{n-1} > 0):$ [2,7]

9) $t_1 < 0 . \vee ., \ldots, . \vee . t_{n-1} < 0 . \supset . \sim[(\mathfrak{a}_0 + \mathfrak{a}_n') \epsilon P_2]:$ [2,6]

10) $t_1 = 0 ., \ldots, t_{n-1} = 0:$ [9,2,8]

11) $t_n = 0 . \supset . \mathfrak{a}_n' = 0:$ [10,6]

12) $\sim(t_n = 0)::$ [11,2]

$[\exists t]::$

13) $\sim(t = 0).$

14) $\mathfrak{a}_n' = t\mathfrak{a}_n .\!\cdot\!.$ [6,10,12]

$[\exists s S_n^0] .\!\cdot\!.$

15) $s = \min\{t, 1\}.$ [14]

16) $\mathsf{S}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n P_1)(S_n^0).$

17) $S_n^0 \subset P_1 \cap P_2.$ [1,2]

18) $S_n^0 = P_1 \cap P_2:$ [*L25*,3,16,17]

19) $t > 0 . \supset . \mathfrak{a}_0 + s\mathfrak{a}_n \in P_1:$ [1,15]

20) $t > 0 . \supset . \mathfrak{a}_0 + s\mathfrak{a}_n \in P_2:$ [2,15,14]

21) $t > 0 . \supset . \mathfrak{a}_0 + s\mathfrak{a}_n \in P_1 \cap P_2:$ [20,19]

22) $t > 0 . \supset . \sim(\mathfrak{a}_0 + s\mathfrak{a}_n) \epsilon S_n^0:$ [16,15]

23) $\sim(t > 0).$ [18,21,22]

24) $t < 0 :\cdot:$ [23,13]

$[\exists t] . \mathfrak{a}_n' = t\mathfrak{a}_n . t < 0$ [14,24]

L30 $[P_1 P_2 \mathfrak{a}_0, \ldots, \mathfrak{a}_n \mathfrak{b}\, t] : \mathsf{P}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n)(P_1) . t < 0 . \mathfrak{b} = t\mathfrak{a}_n.$
$\mathsf{P}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_{n-1}\mathfrak{b})(P_2) . \supset . \mathrm{ETG}(P_1 P_2)$

PF $[P_1 P_2 \mathfrak{a}_0, \ldots, \mathfrak{a}_n \mathfrak{b} t] :\cdot: \mathrm{Hp}(4) . \supset ::$

$[\exists R]::$

5) $\mathsf{R}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n P_1)(R):$ [1]

6) $[s_1 t_1, \ldots, s_n t_n] : 0 \leqq s_n \leqq 1 . 0 \leqq t_n \leqq 1.$
$\mathfrak{a}_0 + t_1\mathfrak{a}_1 + \ldots + t_n\mathfrak{a}_n = \mathfrak{a}_0 + s_1\mathfrak{a}_1 + \ldots$
$+ s_{n-1}\mathfrak{a}_{n-1} + s_n\mathfrak{b} . \supset . s_1 = t_1, \ldots, s_{n-1} =$
$t_{n-1} . t_n = 0.$ [1,2,3]

7) $P_1 \cap P_2 \subset R.$ [5,6]

8) $\sim([\exists \mathfrak{q}] . \mathfrak{q} \in (P_1 - R) \cap P_2).$ [7]

9) $\mathrm{EXT}(P_1 P_2) .\!\cdot\!.$ [*L19*,5,4,8,→←]

$[\exists \mathfrak{p}_0, \ldots, \mathfrak{p}_n Q] .\!\cdot\!.$

10) $\mathfrak{p}_0 = \mathfrak{a}_0 + t\mathfrak{a}_n.$
$\mathfrak{p}_1 = \mathfrak{a}_1, \ldots, \mathfrak{p}_{n-1} = \mathfrak{a}_{n-1}.$
$\mathfrak{p}_n = (1 - t)\mathfrak{a}_n.$

11) $\mathfrak{p}_0 = \mathfrak{a}_0 + \mathfrak{b}.$

12) $\mathfrak{p}_n = \dfrac{(1-t)}{t}\,\mathfrak{b}.$

13) $\mathsf{P}^n(\mathfrak{p}_0, \ldots, \mathfrak{p}_n)(Q):$ [1,2,3]

14) $[t_n] : 0 \leqq t_n \leqq \dfrac{t}{t-1} . \supset . -1 \leqq t_n\left(\dfrac{1-t}{t}\right) \leqq 0:$ [2]

15) $[t_n] : 0 \leqq t_n \leqq \frac{t}{t-1} . \supset . 0 \leqq 1 + t_n\left(\frac{1-t}{t}\right) \leqq 1 :$ [14]

16) $[t_n] . \mathfrak{p}_0 + t_n\mathfrak{p}_n = \mathfrak{a}_0 + \left[1 + t_n\left(\frac{1-t}{t}\right)\right]\mathfrak{b}.$ [10,3,12]

17) $[t_n] : \frac{t}{t-1} \leqq t_n \leqq 1 . \supset . t \leqq t_n(1-t) \leqq 1-t :$ [2]

18) $[t_n] : \frac{t}{t-1} \leqq t_n \leqq 1 . \supset . 0 \leqq t + t_n(1-t) \leqq 1 :$ [17]

19) $[t_n] . \mathfrak{p}_0 + t_n\mathfrak{p}_n = \mathfrak{a}_0 + [t + t_n(1-t)]\mathfrak{a}_n.$ [10,3,12]

20) $Q \subset P_1 \cup P_2 :$ [13,10,1,4,16,15,19,18]

21) $[u] : 0 \leqq u \leqq 1 . \supset . \frac{t}{t-1} \leqq \frac{u-t}{(1-t)} \leqq 1.$ [2]

22) $[u_1, \ldots, u_n] . \mathfrak{a}_0 + u_1\mathfrak{a}_1 + \ldots + u_n\mathfrak{a}_n =$

$\mathfrak{p}_0 = u_1\mathfrak{p}_1 + \ldots + u_{n-1}\mathfrak{p}_{n-1} + \left(\frac{u_n - t}{1-t}\right)\mathfrak{p}_n.$ [19,10]

23) $P_1 \subset Q :$ [1,13,22,21,2]

24) $[r] : 0 \leqq r \leqq 1 . \supset . 0 \leqq \frac{t(r-1)}{1-t} \leqq \frac{t}{t-1} :$ [2]

25) $[r_1, \ldots, r_n] . \mathfrak{a}_0 + r_1\mathfrak{a}_1 + \ldots + r_n\mathfrak{a}_n =$

$\mathfrak{p}_0 + r_1\mathfrak{p}_1 + \ldots + r_{n-1}\mathfrak{p}_{n-1} + \left(\frac{t(r_n - 1)}{1-t}\right)\mathfrak{p}_n.$ [16,10]

26) $P_2 \subset Q.$ [4,13,25,24,2]

27) $Q = P_1 \cup P_2 : :$ [20,23,26]

$\mathrm{ETG}(P_1 P_2)$ [9,27]

Having given a relationship between the generating vectors of two externally tangent parallelepipeds, we turn our attention to a characterization of the definition of bisector (*DV3*). We want to show that P_2 bisects P_1 iff there is an H such that we have $\mathrm{H}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n)(H)$ and $H = P_1 \cap P_2$ for a suitable choice of generating vectors $\mathfrak{a}_0, \ldots, \mathfrak{a}_n$ of P_1. This will be the object of lemmas *L31* - *L38*. In lemma *L31* we show that if two parallelepipeds Q_1 and Q_2 are externally tangent to the same side of parallelepiped P then Q_1 and Q_2 are not external (figure 13).

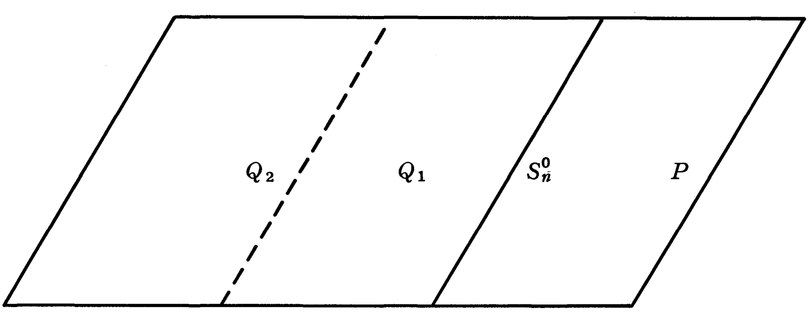

Fig. 13

L31 $[\mathfrak{a}_0, \ldots, \mathfrak{a}_n P Q_1 Q_2 S_n^0] : \mathsf{S}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n P)(S_n^0) . \mathsf{ETG}(PQ_1) .$
$\mathsf{ETG}(PQ_2) . P \cap Q_1 = S_n^0 . P \cap Q_2 = S_n^0 . \supset . \sim(\mathsf{EXT}(Q_1 Q_2))$

PF $[\mathfrak{a}_0, \ldots, \mathfrak{a}_n P Q_1 Q_2 S_n^0] .\therefore \mathrm{Hp}(5) . \supset :$
$[\exists \mathfrak{b}_n \mathfrak{b}_n' t t' R] :$

6) $\mathsf{P}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_{n-1} \mathfrak{b}_n)(Q_1).$ [*L28*,1,2,4]
7) $\mathfrak{b}_n = t\mathfrak{a}_n .$
8) $t < 0.$ [*L29*,6,1,2]
9) $\mathsf{P}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_{n-1} \mathfrak{b}_n')(Q_2).$ [*L28*,1,3,5]
10) $\mathfrak{b}_n' = t'\mathfrak{a}_n .$
11) $t' < 0 .$ [*L29*,9,1,3]
12) $\mathsf{R}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_{n-1} \mathfrak{b}_n Q_1)(R).$ [6]
$[\exists s \mathfrak{p}] .$
13) $s = \frac{1}{2} \max\{t, t'\} .$
14) $\mathfrak{p} = \mathfrak{a}_0 + \frac{1}{2}\mathfrak{a}_1 + \ldots + \frac{1}{2}\mathfrak{a}_{n-1} + \frac{s}{t}\mathfrak{b}_n .$ [8,11,6]
15) $\mathfrak{p} = \mathfrak{a}_0 + \frac{1}{2}\mathfrak{a}_1 + \ldots + \frac{1}{2}\mathfrak{a}_{n-1} + \frac{s}{t'}\mathfrak{b}_n' .$ [14,13,10,7]
16) $0 < \frac{s}{t} < 1 . 0 < \frac{s}{t'} < 1.$ [13,11,8]
17) $\mathfrak{p} \in Q_1 - R.$ [14,16,12,6]
18) $\mathfrak{p} \in Q_2 :$ [15,16,9]
$\sim(\mathsf{EXT}(Q_1, Q_2))$ [*L13*,12,9,17,18]

Our next lemma says that under suitable hypotheses the four parallelepipeds $P_1 P_2 Q_1 Q_2$ have relative positions such as is shown in figure 14. Note that the hypotheses form part of the conditions necessary for P_2 to bisect P_1 (*DV3*) and as the figure indicates these hypotheses are enough to guarantee that P_2 contains at least "half" of P_1 but they are not sufficient to force P_2 to contain exactly "half" of P_1.

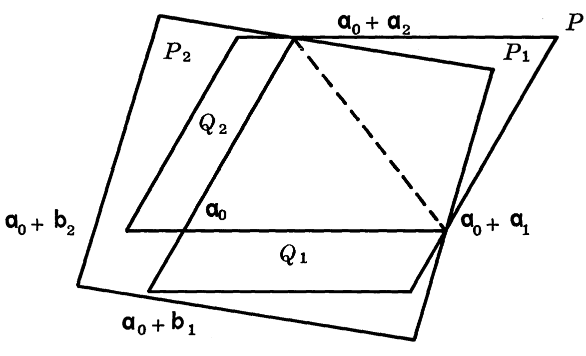

Fig. 14

L32 $[P_1 P_2 Q_1 Q_2]$: $\mathsf{ETG}(P_1 Q_1) . \mathsf{ETG}(P_1 Q_2) . \mathsf{EXT}(Q_1 Q_2) . \bar{\mathsf{P}}^n(P_2)$.
$Q_1 \subset P_2 . Q_2 \subset P_2 . \sim(P_1 \subset P_2) . \supset . [\exists \mathsf{a}_0, \ldots, \mathsf{a}_n \mathsf{b}_1 \mathsf{b}_2]$.
$\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(P_1) . \{\mathsf{a}_0 + t_1 \mathsf{a}_1 + \ldots + t_n \mathsf{a}_n \mid \{t_1, \ldots, t_n\}$
$\subset \{0,1\} . \sim(t_1 = 1 . t_n = 1)\} \subset P_2 . \mathsf{P}^n(\mathsf{a}_0, \mathsf{a}_1, \ldots, \mathsf{a}_{n-1}, \mathsf{b}_1)(Q_1)$.
$\mathsf{P}^n(\mathsf{a}_0, \mathsf{b}_2, \mathsf{a}_2, \ldots, \mathsf{a}_n)(Q_2)$

PF $[P_1 P_2 Q_1 Q_2]$: : Hp (7) . $\supset$.˙.
$[\exists \mathsf{p}_0, \ldots, \mathsf{p}_n S_1 S_2 S_j^i S_k^l S_n^0 \bar{S}_n^0]$.˙.

8) $\mathsf{P}^n(\mathsf{p}_0, \ldots, \mathsf{p}_n)(P_1)$. [1]
9) $\bar{\mathsf{S}}^n\{P_1\}(S_1) . \mathsf{S}^n(\mathsf{p}_0, \ldots, \mathsf{p}_n P_1)(S_j^i)$.
10) $S_1 = S_j^i$. [*L21*, *L20*, 1]
11) $S_1 = P_1 \cap Q_1$.
12) $S_1 \subset P_2$. [11,5]
13) $\mathsf{P}^n(\mathsf{p}_0 + i\mathsf{p}_j, \mathsf{p}_1, \ldots, \mathsf{p}_{j-1}, \mathsf{p}_{j+1}, \ldots, \mathsf{p}_n,$
$(1-2i)\mathsf{p}_j)(P_1)$. [*L6*,8,9]
14) $\mathsf{S}^n(\mathsf{p}_0 + i\mathsf{p}_j, \mathsf{p}_1, \ldots, \mathsf{p}_{j-1}, \mathsf{p}_{j+1}, \ldots, \mathsf{p}_n,$
$(1-2i)\mathsf{p}_j P_1)(S_n^0)$. [13]
15) $S_1 = S_n^0$. [10,9,14]
16) $\bar{\mathsf{S}}^n\{P_1\}(S_2)$.
17) $\mathsf{S}^n(\mathsf{p}_0, \ldots, \mathsf{p}_n P_1)(S_k^l)$. [*L21*, *L29*, 2]
18) $S_2 = S_k^l$.
19) $S_2 = P_1 \cap Q_2$.
20) $\mathsf{P}^n(\mathsf{p}_0 + l\mathsf{p}_k, \mathsf{p}_1, \ldots, \mathsf{p}_{k-1}, \ldots, \mathsf{p}_n (1-2l)\mathsf{p}_k)$
(P_1). [*L6*,8,17]
21) $\mathsf{S}^n(\mathsf{p}_0 + l\mathsf{p}_k, \mathsf{p}_1, \ldots, \mathsf{p}_{k-1}, \mathsf{p}_{k+1}, \ldots, \mathsf{p}_n,$
$(1-2l)\mathsf{p}_k P_1)(\bar{S}_n^0)$. [20]
22) $S_2 = \bar{S}_n^0$. [18,17,21]
23) $S_2 \subset P_2$: [19,6]
24) $S_2 = S_1 . \supset . \sim(\mathsf{EXT}(Q_1 Q_2))$: [*L26*,14,1,2,11,15,19]
25) $S_j^i = S_k^{l-1} . \supset . P_1 \subset P_2$: [*L18*,4,8,18,10,17,9]
26) $j \neq k$. [25,24,18,10,3,7]
27) $\mathsf{P}^n(\mathsf{p}_0 + i\mathsf{p}_j + l\mathsf{p}_k, \mathsf{p}_1, \ldots, \mathsf{p}_{j-1}, \mathsf{p}_{j+1}, \ldots, \mathsf{p}_{k-1},$
$\mathsf{p}_{k+1}, \ldots, \mathsf{p}_n, (1-2i)\mathsf{p}_j, (1-2l)\mathsf{p}_k)(P_1)$. [*L6*,8,9,17]
28) $\mathsf{P}^n(\mathsf{p}_0 + i\mathsf{p}_j + l\mathsf{p}_k, \mathsf{p}_1, \ldots, \mathsf{p}_{j-1}, \mathsf{p}_{j+1}, \ldots, \mathsf{p}_{k-1},$
$\mathsf{p}_{k+1}, \ldots, \mathsf{p}_n, (1-2l)\mathsf{p}_k, (1-2i)\mathsf{p}_j)(P_1)$: [*L6*,27]
$[\exists \mathsf{a}_0, \ldots, \mathsf{a}_n \mathsf{b}_1 \mathsf{b}_2]$:
29) $\mathsf{a}_0 = \mathsf{p}_0 + i\mathsf{p}_j + l\mathsf{p}_k . \mathsf{a}_1 = (1-2l)\mathsf{p}_k$.
$\mathsf{a}_2 = \mathsf{p}_1 ., \ldots, . \mathsf{a}_{j-1} = \mathsf{p}_{j-2} . \mathsf{a}_j = \mathsf{p}_{j-1}$.
$\mathsf{a}_{j+1} = \mathsf{p}_{j+1} ., \ldots, . \mathsf{a}_{k-1} = \mathsf{p}_{k-1} . \mathsf{a}_k =$
$\mathsf{p}_{k+1}, \ldots, \mathsf{a}_{n-1} = \mathsf{p}_n . \mathsf{a}_n = (1-2i)\mathsf{p}_j$. [8,9,17]
30) $\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(P_1)$. [*L6*,8,9,17,29]
31) $\mathsf{S}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n P_1)(S_n^0)$. [29,14]
32) $\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_{n-1}, \mathsf{b}_1)(Q_1)$. [*L28*,1,31,15,12]
33) $\mathsf{P}^n(\mathsf{a}_0, \mathsf{a}_2, \ldots, \mathsf{a}_n, \mathsf{a}_1)(P_1)$. [*L6*,30]
34) $\mathsf{S}^n(\mathsf{a}_0, \mathsf{a}_2, \ldots, \mathsf{a}_n, \mathsf{a}_1 P_1)(\bar{S}_n^0)$. [29,21]
35) $\mathsf{P}^n(\mathsf{a}_0, \mathsf{a}_2, \ldots, \mathsf{a}_n, \mathsf{b}_2)(Q_2)$. [*L28*,2,34,22,23]
36) $\mathsf{P}^n(\mathsf{a}_0, \mathsf{b}_2, \mathsf{a}_2, \ldots, \mathsf{a}_n)(Q_2)$ [*DV*$\bar{P}$,35]

37) $\{a_0 + t_1 a_1 + \ldots + t_n a_n | \{t_1, \ldots, t_n\} \subset \{0,1\}. \sim(t_1 = 1. t_n = 1)\} \subset P_2 .\because$ [35,32,5,6]
$[\exists a_0, \ldots, a_n b_1 b_2] . P^n(a_0, \ldots, a_n)(P_1) .$
$\{a_0 + t_1 a_1 + \ldots + t_n a_n | \{t_1, \ldots, t_n\} \subset \{0,1\}.$
$\sim(t_1 = 1. t_n = 1)\} \subset P_2 . P^n(a_0, \ldots, a_{n-1}, b_1)(Q_1).$
$P^n(a_0, b_2, a_2, \ldots, a_n)(Q_2)$ [30,36,33,35]

In lemma *L33* we show that if parallelepipeds Q_1 and Q_2 are externally tangent to "adjacent" sides of parallelepiped P_1 then there exists a parallelepiped P_2 containing Q_1, Q_2, and exactly "half" of P_1. Figure 15 illustrates *L33*.

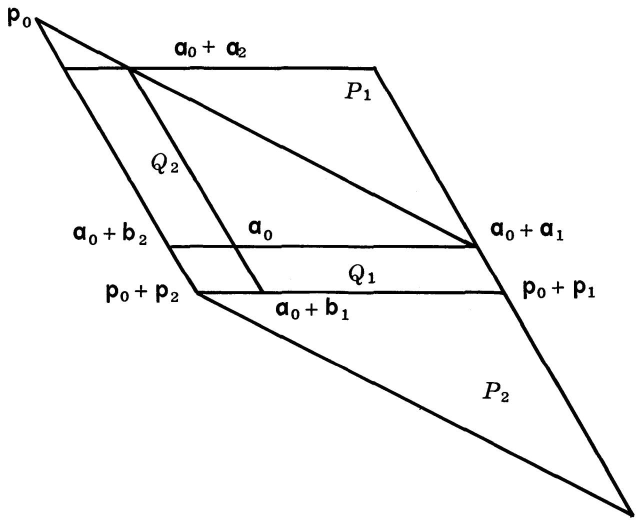

Fig. 15

L33 $[P_1 Q_1 Q_2 a_0, \ldots, a_n b_1 b_2] : P^n(a_0, \ldots, a_n)(P_1).$
$P^n(a_0, \ldots, a_{n-1}, b_1)(Q_1) . P^n(a_0, b_2, a_2, \ldots, a_n)(Q_2) .$
$ETG(P_1 Q_1) . ETG(P_1 Q_2) . \supset . [\exists p_0, \ldots, p_n P_2 H].$
$P^n(p_0, \ldots, p_n)(P_2) . Q_1 \subset P_2 . Q_2 \subset P_2 . H^n(a_0, \ldots, a_n P_1)$
$(H) . P_1 \cap P_2 = H$

PF $[P_1 Q_1 Q_2 a_0, \ldots, a_n b_1 b_2] :\cdot: Hp(5) . \supset ::$
$[\exists us] ::$
6) $s < 0.$
7) $b_1 = s a_n .$
8) $u < 0.$
9) $b_2 = u a_1 :\cdot:$ [*L29*,1,2,3,4,5]
$[\exists p_0, \ldots, p_n] :\cdot:$

10) $\mathbf{p}_0 = \mathbf{a}_0 + u\mathbf{a}_1 + (1-u)\mathbf{a}_n.$

11) $\mathbf{p}_0 + \mathbf{p}_1 = \mathbf{a}_0 + (1-s)\mathbf{a}_1 + s\mathbf{a}_n.$

12) $\mathbf{p}_0 + \mathbf{p}_n = \mathbf{a}_0 + u\mathbf{a}_1 + s\mathbf{a}_n.$

13) $\mathbf{p}_2 = \mathbf{a}_2, \ldots, \mathbf{p}_{n-1} = \mathbf{a}_{n-1}.$ [7,9,1]

14) $\mathbf{p}_1 = (1-s-u)\mathbf{a}_1 + (s+u-1)\mathbf{a}_n.$ [10,11]

15) $\mathbf{p}_n = (s+u-1)\mathbf{a}_n : :$ [10,12]

$[\exists P_2] : :$

16) $\mathbf{P}^n(\mathbf{p}_0, \ldots, \mathbf{p}_n)(P_2) :$ [10,12,13,15,1]

17) $[r_1 t_1, \ldots, r_n t_n] : \mathbf{a}_0 + r_1\mathbf{a}_1 + \ldots + r_{n-1}\mathbf{a}_{n-1} + r_n u\mathbf{a}_n = \mathbf{p}_0 + t_1\mathbf{p}_1 + \ldots + t_n\mathbf{p}_n . \supset .$
$r_1 = u + t_1(1-s-u) . r_n s = (1-u) + t_1 (s+u-1) + t_n(s+u-1) :$ [16,1,7,9,10,12,15]

18) $[r_1 t_1, \ldots, r_n t_n] : \mathbf{a}_0 + r_1\mathbf{a}_1 + \ldots + r_{n-1}\mathbf{a}_{n-1} + r_n u\mathbf{a}_n = \mathbf{p}_0 + t_1\mathbf{p}_1 + \ldots + t_n\mathbf{p}_n . \supset . t_1 = \dfrac{r_1 - u}{1-s-u} . t_n = \dfrac{r_n s + u - 1 - t_1(s+u-1)}{s+u-1} :$ [17]

19) $[r_1 t_1, \ldots, r_n t_n] : \mathbf{a}_0 + r_1\mathbf{a}_1 + \ldots + r_{n-1}\mathbf{a}_{n-1} + r_n s\mathbf{a}_n = \mathbf{p}_0 + t_1\mathbf{p}_1 + \ldots + t_n\mathbf{p}_n . 0 \leqq r_1, \ldots, r_n \leqq 1 . \supset . 0 \leqq t_1, \ldots, t_n \leqq 1 :$ [17,18,7,1,16]

20) $Q_1 \subset P_2 :$ [19,18,16,2]

21) $[r_1 t_1, \ldots, r_n t_n] : \mathbf{a}_0 + r_1 u\mathbf{a}_1 + r_2\mathbf{a}_2 + \ldots + r_n\mathbf{a}_n = \mathbf{p}_0 + t_1\mathbf{p}_1 + \ldots + t_n\mathbf{p}_n . \supset . r_1 u = u + t_1(1-s-u) . r_n = (1-u) + t_1(s+u-1) + t_n(s+u-1) :$ [16,1,7,9,10,14,15]

22) $[r_1 t_1, \ldots, r_n t_n] : \mathbf{a}_0 + r_1 u\mathbf{a}_1 + r_2\mathbf{a}_2 + \ldots + r_n\mathbf{a}_n = \mathbf{p}_0 + t_1\mathbf{p}_1 + \ldots + t_n\mathbf{p}_n . \supset . t_1 = \dfrac{r_1 u - u}{(1-s-u)} . t_n = \dfrac{r_n + (u-1) - t_1(s+u-1)}{s+u-1} = \dfrac{r_n + ur_1 - 1}{s+u-1} :$ [21]

23) $[r_1 t_1, \ldots, r_n t_n] : \mathbf{a}_0 + r_1 u\mathbf{a}_1 + r_2\mathbf{a}_2 + \ldots + r_n\mathbf{a}_n = \mathbf{p}_0 + t_1\mathbf{p}_1 + \ldots + t_n\mathbf{p}_n . 0 \leqq r_1, \ldots, r_n \leqq 1 . \supset . 0 \leqq t_1, \ldots, t_n \leqq 1 :$ [22,6,8,1,16]

24) $Q_2 \subset P_2 :$ [23,22,16,3]

25) $[r_1 t_1, \ldots, r_n t_n] : \mathbf{a}_0 + r_1\mathbf{a}_1 + \ldots + r_n\mathbf{a}_n = \mathbf{p}_0 + t_1\mathbf{p}_1 + \ldots + t_n\mathbf{p}_n . \supset . r_1 = u + t_1(1-s-u) . r_2 = t_2, \ldots, r_{n-1} = t_{n-1} . r_n = (1-u) + (t_1 + t_n)(s+u-1) :$ [1,16,7,9,10,14,15]

26) $[r_1 t_1, \ldots, r_n t_n] : \mathbf{q}_0 + r_1\mathbf{a}_1 + \ldots + r_n\mathbf{a}_n = \mathbf{p}_0 + t_1\mathbf{p}_1 + \ldots + t_n\mathbf{p}_n . \supset . r_1 + r_n = 1 + t_n(s+u-1) \leqq 1 .\because.$ [25,6,8,1,16]

$[\exists H] .\because.$

27) $\mathbf{H}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n P_1)(H).$ [1]

28) $P_1 \cap P_2 \subset H.$ [1,16,26]

29) $\mathbf{a}_0 + \mathbf{a}_1 = \mathbf{p}_0 + \left(\dfrac{1-u}{1-s-u}\right)\mathbf{p}_1.$ [10,14,15]

30) $\mathbf{a}_0 + \mathbf{a}_n = \mathbf{p}_0 + \left(\frac{u}{s+u-1}\right)\mathbf{p}_1.$

31) $\mathbf{a}_0 + \mathbf{a}_1 \epsilon (P_1 \cap P_2).$ [1,16,29]

32) $\mathbf{a}_0 + \mathbf{a}_n \epsilon (P_1 \cap P_2):$ [1,16,30]

33) $[r_2, \ldots, r_{n-1}]: 0 \leqq r_2, \ldots, r_{n-1} \leqq 1.$
$\supset . \mathbf{a}_0 + r_2\mathbf{a}_2 + \ldots + r_{n-1}\mathbf{a}_{n-1} =$
$\mathbf{p}_0 + \frac{u}{(u+s-1)} \mathbf{p}_1 + r_2\mathbf{p}_2 + \ldots +$
$r_{n-1}\mathbf{p}_{n-1} + \frac{1}{(1-s-u)} \mathbf{p}_n:$ [10,15,12,13,1]

34) $[r_2, \ldots, r_{n-1}]: 0 \leqq r_2, \ldots, r_{n-1} \leqq 1.$
$\supset . \mathbf{a}_0 + r_2\mathbf{a}_2 + \ldots + r_{n-1}\mathbf{a}_{n-1} \epsilon$
$(P_1 \cap P_2):$ [1,16,33,6,8]

35) $[r_2, \ldots, r_{n-1}]: 0 \leqq r_2, \ldots, r_{n-1} \leqq 1.$
$\supset . [\exists \mathbf{b}_0 M] . \mathbf{b}_0 = \mathbf{a}_0 + r_2\mathbf{a}_2 + \ldots +$
$r_{n-1}\mathbf{a}_{n-1} . \mathbf{M}^2(\mathbf{b}_0\mathbf{a}_1\mathbf{a}_2)(M) . M \subset (P_1 \cap P_2):$
[L7,34,16,1,32,31]

36) $H \subset (P_1 \cap P_2).$ [35]

37) $P_1 \cap P_2 = H ::$ [35,28]

$[\exists \mathbf{p}_0, \ldots, \mathbf{p}_n P_2 H] . \mathbf{P}^n(\mathbf{p}_0, \ldots, \mathbf{p}_n)(P_2) . Q_1 \subset P_2 .$
$Q_2 \subset P_2 . \mathbf{H}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n P_1)(H) . P_1 \cap P_2 = H$ [16,20,24,27,37]

Our next lemma says that if parallelepiped P_2 contains one point $\mathbf{p}$ more than "half" of parallelepiped P_1 then it contains a parallelepiped Q more than half (see figure 16).

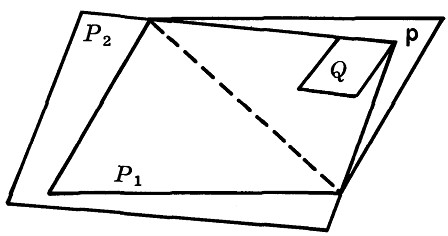

Fig. 16

L34 $[\mathbf{a}_0, \ldots, \mathbf{a}_n \mathbf{b}_1 \mathbf{b}_2 \mathbf{p} P_1 P_2 Q_1 Q_2 H] : \overline{\mathbf{P}}^n(P_2) . \mathbf{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n)(P_1) .$
$\mathbf{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}, \mathbf{b}_1)(Q_1) . \mathbf{P}^n(\mathbf{a}_0, \mathbf{b}_2, \mathbf{a}_2, \ldots, \mathbf{a}_n)(Q_2) .$
$Q_1 \subset P_2 . Q_2 \subset P_2 . \mathbf{H}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n P_1)(H) . \mathbf{p} \epsilon ((P_1 \cap P_2) - H)$
$. \supset . [\exists Q] . \mathbf{p} \epsilon Q . \overline{\mathbf{P}}^n(Q) . Q \subset ((P_2 \cap P_1) - H)$

PF $[\mathbf{a}_0, \ldots, \mathbf{a}_n \mathbf{b}_1 \mathbf{b}_2 \mathbf{p} P_1 P_2 Q_1 Q_2 H] :: \mathrm{Hp}(8) . \supset :\cdot:$
$[\exists r_1, \ldots, r_n] :\cdot:$

9) $\mathbf{p} = \mathbf{a}_0 + r_1\mathbf{a}_1 + \ldots + r_n\mathbf{a}_n.$

10) $0 \leqq r_1, \ldots, r_n \leqq 1.$ [2,8]

11) $r_1 + r_n > 1 ::$

$[\exists \mathbf{p}_1, \ldots, \mathbf{p}_n] ::$

12) $\mathbf{p}_1 = \mathbf{a}_0 + \mathbf{a}_1$.
13) $\mathbf{p}_2 = \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2, \dots, \mathbf{p}_{n-1} = \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_{n-1}$.
14) $\mathbf{p}_n = \mathbf{a}_0 + \mathbf{a}_n \therefore$ [2]

$[\exists \mathbf{q}_1, \dots, \mathbf{a}_n] \therefore$

15) $\mathbf{q}_1 = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}), \dots, \mathbf{q}_n = \frac{1}{2}(\mathbf{p}_n - \mathbf{p})$: [12,13,14,9]

16) $[s_1, \dots, s_n] : s_1\mathbf{q}_1 + \dots + s_n\mathbf{q}_n = 0 . \supset . s_1(\mathbf{p} - \mathbf{p}_1) + \dots + s_n(\mathbf{p} - \mathbf{p}_n) = 0$: [15]

17) $[s_1, \dots, s_n]: s_1\mathbf{q}_1 + \dots + s_n\mathbf{q}_n = 0 . \supset .$
$s_1(r_1 - 1)\mathbf{a}_1 + s_1 r_2 \mathbf{a}_2 + \dots + s_1 r_n \mathbf{a}_n +$
$s_2(r_1 - 1)\mathbf{a}_1 + s_2(r_2 - 1)\mathbf{a}_2 + s_2 r_3 \mathbf{a}_3 + \dots +$
$s_2 r_n \mathbf{a}_n + s_3(r_1 - 1)\mathbf{a}_1 + s_3 r_2 \mathbf{a}_2 +$
$s_3(r_3 - 1)\mathbf{a}_3 + s_3 r_4 \mathbf{a}_4 + \dots + s_3 r_n \mathbf{a}_n +$

$\vdots$

$s_n r_1 \mathbf{a}_1 + s_n r_2 \mathbf{a}_2 + \dots + s_n r_{n-1} \mathbf{a}_{n-1} +$
$s_n(r_n - 1)\mathbf{a}_n = 0$: [14,12,13,14,9]

18) $\mathbf{a}_1, \dots, \mathbf{a}_n$ are linearly independent: [2]

19) $[s_1, \dots, s_n] : s_1\mathbf{q}_1 + \dots + s_n\mathbf{q}_n = 0 . \supset .$
$(s_1 + \dots + s_n)r_1 = s_1 + \dots + s_{n-1}$.
$(s_1 + \dots + s_n)r_2 = s_2, \dots, (s_1 + \dots + s_n)r_n = s_n$: [18,17]

20) $[s_1, \dots, s_n] : s_1\mathbf{q}_1 + \dots + s_n\mathbf{q}_n = 0 . \supset .$
$(s_1 + \dots + s_n)(r_1 + r_n) = s_1 + \dots + s_n$: [19]

21) $[s_1, \dots, s_n] : s_1\mathbf{q}_1 + \dots + s_n\mathbf{q}_n = 0 . \supset .$
$s_1 + \dots + s_n = 0$: [20,10]

22) $[s_1, \dots, s_n] : s_1\mathbf{q}_1 + \dots + s_n\mathbf{q}_n = 0 . \supset .$
$s_1 = 0, \dots, s_n = 0$: [21,19]

23) $\mathbf{q}_1, \dots, \mathbf{q}_n$ are linearly independent: [22,15]

24) $[s_1, \dots, s_n] : 0 \leqq s_1, \dots, s_n \leqq 1.$
$s_1 + \dots + s_n \leqq 1 . \supset . \mathbf{p} + s_1\mathbf{q}_1 + \dots +$
$s_n\mathbf{q}_n = \mathbf{a}_0 + \left[r_1 + \frac{1}{2}\{(s_1 + \dots + s_{n-1}) - (s_1 + \dots + s_n)r_1\}\right]\mathbf{a}_1 + \left[r_2 + \frac{1}{2}\{s_2 - (s_1 + \dots + s_n)r_2\}\right]\mathbf{a}_2 + \dots + [r_n + \frac{1}{2}\{s_n - (s_1 + \dots + s_n)r_n\}]\mathbf{a}_n$: [15,9,12,13,14]

25) $[s_1, \dots, s_n] : 0 \leqq s_1, \dots, s_n \leqq 1.$
$s_1 + \dots + s_n \leqq 1 . \supset . 0 < r_1 + \frac{1}{2}\{(s_1 + \dots + s_{n-1}) - (s_1 + \dots + s_n)r_1\} \leqq 1.$
$0 < r_n + \frac{1}{2}\{s_n - (s_1 + \dots + s_n)r_n\} \leqq 1 . 1 < r_1 + r_n + \frac{1}{2}\{(s_1 + \dots + s_n) - (s_1 + \dots + s_n)(r_1 + r_n)\}$
$\leqq 2 . 0 \leqq r_2 + \frac{1}{2}\{s_2 - (s_1 + \dots + s_n)r_2\}, \dots, r_{n-1}$
$+ \frac{1}{2}\{s_{n-1} - (s_1 + \dots + s_n)r_{n-1}\} \leqq 1$: [10,11]

26) $[s_1, \ldots, s_n] : 0 \leqq s_1, \ldots, s_n \leqq 1 . s_1 + \ldots +$
$s_n \leqq 1 . \supset . \mathsf{p} + s_1\mathsf{q}_1 + \ldots + s_n\mathsf{q}_n \in (P_1 - H)$: [24,25,2,7]
27) $\mathsf{p}_1 \in P_2, \ldots, \mathsf{p}_n \in P_2 . \mathsf{p} \in P_2$. [12,13,14,3,4,5,6]
28) $\mathsf{p} + \mathsf{q}_1 \in P_2, \ldots, \mathsf{p} + \mathsf{q}_n \in P_2 . \mathsf{p} \in P_2$.·. [$L14$,1,15,27]
$[\exists M]$:
29) $\mathsf{M}^n(\mathsf{p}, \mathsf{q}_1, \ldots, \mathsf{q}_n)(M)$. [$DVM$,8,23,15,12,13,14,2]
30) $M \subset P_2$. [$L7$,29,1,28]
31) $M \subset ((P_2 \cap P_1) - H)$. [30,26]
$[\exists Q]$.
32) $\bar{\mathsf{P}}^n(Q)$.
33) $Q \subset M$. [$L8$,29]
34) $\mathsf{p} \in Q$.
35) $Q \subset ((P_2 \cap P_1) - H)$:·: [33,31]
$[\exists Q] . \mathsf{p} \in Q . \bar{\mathsf{P}}^n(Q) . Q \subset ((P_1 \cap P_2) - H)$ [35,34,32]

Next we use the previous lemma to show that if a parallelepiped Q contains more than "half" of parallelepiped P_1 and contains two parallelepipeds Q_1 and Q_2 which are externally tangent to adjacent sides of P_1 then Q does not satisfy the last conjunct of bisector (see $DV3$ and figure 17).

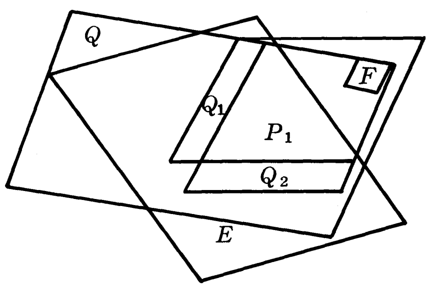

Fig. 17

$L35$ $[P_1 Q Q_1 Q_2 H \mathsf{p} \mathsf{a}_0, \ldots, \mathsf{a}_n \mathsf{b}_1 \mathsf{b}_2] : \mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(P_1)$.
$\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_{n-1}\mathsf{b}_1)(Q_1) . \mathsf{P}^n(\mathsf{a}_0, \mathsf{b}_2, \mathsf{a}_2, \ldots, \mathsf{a}_n)(Q_2)$.
$\bar{\mathsf{P}}^n(Q) . \mathrm{ETG}(Q_1 P_1) . \mathrm{ETG}(Q_2 P_1) . Q_1 \subset Q . Q_2 \subset Q$.
$\mathsf{H}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n P_1)(H) . \mathsf{p} \in ((Q \cap P_1) - H) . \supset . [\exists EF] \bar{\mathsf{P}}^n(E)$.
$\bar{\mathsf{P}}^n(F) . Q_1 \subset E . Q_2 \subset E . F \subset P_1 \cap Q . \sim(F \subset E)$

PF $[P_1 Q Q_1 Q_2 H \mathsf{p} \mathsf{a}_0, \ldots, \mathsf{a}_n \mathsf{b}_1 \mathsf{b}_2]$: : Hp(10) . ⊃ .·.
$[\exists E]$.·.
11) $\bar{\mathsf{P}}^n(E)$.
12) $Q_1 \subset E$.
13) $Q_2 \subset E$. [$L33$,1,2,3,5,6]
14) $P_1 \cap E = H$:
$[\exists F]$:
15) $\bar{\mathsf{P}}^n(F)$.
16) $F \subset ((P_1 \cap Q) - H)$.

17) $\mathsf{p} \in F$. [$L34$,11,1,2,3,10,12,13]
18) $\sim(\mathsf{p} \in E)$. [10,14]
19) $\sim(F \subset E)$.·. [17,18]
$[\exists EF]$. $\bar{\mathsf{P}}^n(E)$. $\bar{\mathsf{P}}^n(F)$. $Q_1 \subset E$. $Q_2 \subset E$. $F \subset P_1 \cap Q$.
$\sim(F \subset E)$

Our final lemma, before obtaining our geometric characterization of the definition of bisector ($DV3$), says that if Q_1 and Q_2 are externally tangent to adjacent sides of P_1; and P_2 contains Q_1 and Q_2 then P_2 contains at least half of P_1.

L36 $[P_1 Q_1 Q_2 P_2 H \mathsf{a}_0, \ldots, \mathsf{a}_n \mathsf{b}_0 \mathsf{b}_1]$: $\bar{\mathsf{P}}^n(P_2)$. $\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(P_1)$.
$\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n \mathsf{b}_1)(Q_1)$. $\mathsf{P}^n(\mathsf{a}_0, \mathsf{b}_2, \mathsf{a}_2, \ldots, \mathsf{a}_n)(Q_2)$.
$Q_1 \subset P_2$. $Q_2 \subset P_2$. $\mathsf{H}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n P_1)(H)$. $\supset$. $H \subset P_1 \cap P_2$

PF $[P_1 Q_1 Q_2 P_2 H \mathsf{a}_0, \ldots, \mathsf{a}_n \mathsf{b}_0 \mathsf{b}_1]$: : Hp(7) . $\supset$.·.
8) $[t_2, \ldots, t_{n-1}]$: $0 \leqq t_2, \ldots, t_{n-1} \leqq 1$. $\supset$.
$[\exists \mathsf{b}_0 \mathsf{b}_1 \mathsf{b}_2 M]$. $\mathsf{b}_0 = \mathsf{a}_0 + t_2 \mathsf{a}_2 + \ldots + t_{n-1} \mathsf{a}_{n-1}$.
$\mathsf{b}_1 = \mathsf{a}_0 + \mathsf{a}_1$. $\mathsf{b}_2 = \mathsf{a}_0 + \mathsf{a}_n$. $\mathsf{M}^2(\mathsf{b}_0, \mathsf{b}_1, \mathsf{b}_2)(M)$.
$M \subset (P_1 \cap P_2)$: [2,$L7$,2,$L7$,1,3,4,5,6]
9) $[t_1, \ldots, t_n]$: $0 \leqq t_1, \ldots, t_n \leqq 1 . t_1 + t_n \leqq 1$. $\supset$.
$\mathsf{a}_0 + t_1 \mathsf{a}_1 + \ldots + t_n \mathsf{a}_n \in (P_1 \cap P_2)$.·. [8]
$H \subset P_1 \cap P_2$ [9]

We now prove that P_2 bisects P_1 iff P_2 contains exactly half (in the sense of DVH) of P_1. Each implication is given as a separate lemma.

L37 $[P_1 P_2]$: $\mathsf{BIS}(P_1 P_2)$. $\supset$. $[\exists \mathsf{a}_0, \ldots, \mathsf{a}_n \mathsf{b}_1 \mathsf{b}_2 Q_1 Q_2 H]$.
$\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(P_1)$. $\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_{n-1}, \mathsf{b}_1)(Q_1)$.
$\mathsf{P}^n(\mathsf{a}_0, \mathsf{b}_2, \mathsf{a}_2, \ldots, \mathsf{a}_n)(Q_2)$. $\mathsf{ETG}(Q_1 P_1)$. $\mathsf{ETG}(Q_2 P_1)$.
$Q_1 \subset P_2$. $Q_2 \subset P_2$. $\mathsf{H}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n P_1)(H)$. $P_1 \cap P_2 = H$

PF $[P_1 P_2]$: : Hp(1) . $\supset$:·:
$[\exists Q_1 Q_2]$:·:
2) $\mathsf{EXT}(Q_1 Q_2)$.
3) $\mathsf{EXT}(Q_1 P_1)$.
4) $\mathsf{ETG}(Q_2 P_1)$. [$DV3$,1]
5) $Q_1 \subset P_2$.
6) $Q_2 \subset P_2$: :
$[\exists \mathsf{a}_0, \ldots, \mathsf{a}_n \mathsf{b}_1 \mathsf{b}_2]$: :
7) $\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(P_1)$.
8) $\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_{n-1}, \mathsf{b}_1)(Q_1)$. [$L32$,1,2,3,4,5,6]
9) $\mathsf{P}^n(\mathsf{a}_0, \mathsf{b}_2, \mathsf{a}_2, \ldots, \mathsf{a}_n)(Q_2)$.·.
$[\exists H]$.·.
10) $\mathsf{H}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n P_1)(H)$. [7]
11) $H \subset (P_1 \cap P_2)$: [$L36$,1,7,8,9,5,6]
12) $[\mathsf{p}]$: $\mathsf{p} \in ((P_2 \cap P_1) - H)$. $\supset$.
$\sim(\mathsf{BIS}(P_1 P_2))$: [$L35$,7,8,9,1,5,6,3,4]

13) $P_1 \cap P_2 = H$:·: [12,11,1]
$[\exists \mathbf{a}_0, \ldots, \mathbf{a}_n \mathbf{b}_1 \mathbf{b}_2 Q_1 Q_2 H] . \mathsf{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n)(P_1)$.
$\mathsf{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}\mathbf{b}_1)(Q_1) . \mathsf{P}^n(\mathbf{a}_0, \mathbf{b}_2, \mathbf{a}_2, \ldots, \mathbf{a}_n)(Q_2)$.
$\mathsf{ETG}(Q_1 P_1) . \mathsf{ETG}(Q_2 P_1) . Q_1 \subset P_2 . Q_2 \subset P_2$.
$\mathsf{H}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n P_1)(H) . P_1 \cap P_2 = H$ [7,8,9,3,4,5,6,10,13]

L38 $[P_1 P_2 Q_1 Q_2 H \mathbf{a}_0, \ldots, \mathbf{a}_n \mathbf{b}_1 \mathbf{b}_2] : \bar{\mathsf{P}}^n(P_2) . \mathsf{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n)(P_1)$.
$\mathsf{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}\mathbf{b}_1)(Q_1) . \mathsf{P}^n(\mathbf{a}_0, \mathbf{b}_2, \mathbf{a}_2, \ldots, \mathbf{a}_n)(Q_2)$.
$\mathsf{ETG}(Q_1 P_1) . \mathsf{ETG}(Q_2 P_1) . Q_1 \subset P_2 . Q_2 \subset P_2$.
$\mathsf{H}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n P_1)(H) . P_1 \cap P_2 = H . \supset . \mathsf{BIS}(P_1 P_2)$

PF $[P_1 P_2 Q_1 Q_2 H \mathbf{a}_0, \ldots, \mathbf{a}_n \mathbf{b}_1 \mathbf{b}_2]$:·: Hp(10) . ⊃ : :
11) $\sim(P_1 \subset P_2)$. [1,2,10,9]
12) $\mathsf{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n, \mathbf{b}_2)(Q_2)$. [$DV\bar{P}$,4]
13) $\mathsf{P}^n(\mathbf{a}_0, \mathbf{a}_2, \ldots, \mathbf{a}_n, \mathbf{a}_1)(P_1)$: : [$DV\bar{P}$,2]
$[\exists st]$: :
14) $s < 0$.
15) $\mathbf{b}_1 = s\mathbf{a}_n$.
16) $t < 0$.
17) $\mathbf{b}_2 = t\mathbf{a}_1$: [*L29*,2,3,13,12]
18) $[t_1 t'_1, \ldots, t_n t'_n] : \mathbf{a}_0 + t_1 \mathbf{a}_1 + \ldots + t_{n-1} + \mathbf{a}_{n-1}$
$t_n \mathbf{b}_1 = \mathbf{a}_0 + t'_1 \mathbf{b}_2 + t'_1 \mathbf{a}_2 + \ldots + t'_n \mathbf{a}_n$. ⊃.
$t_n s \mathbf{a}_n = t'_n . t'_1 t \mathbf{a}_1 = t_1$: [2,15,17]
19) $[t_1 t'_1, \ldots, t_n t'_n] : 0 \leqq t_1, t'_1, \ldots, t_n, t'_n \leqq 1$.
$\mathbf{a}_0 + t_1 \mathbf{a}_1 + \ldots + t_{n-1} \mathbf{a}_{n-1} + t_n \mathbf{b}_1 =$
$\mathbf{a}_0 + t'_1 \mathbf{b}_2 + t'_2 \mathbf{a}_2 + \ldots + t'_n \mathbf{a}_n . \supset . t_n = 0$.
$t'_n = 0 . t_1 = 0 . t'_1 = 0$.·. [18,14,16]
$[\exists R_1 R_2]$.·.
20) $\mathsf{R}^n(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}, \mathbf{b}_1 Q_1)(R_1)$. [3]
21) $\mathsf{R}^n(\mathbf{a}_0, \mathbf{b}_2, \mathbf{a}_2, \ldots, \mathbf{a}_n Q_2)(R_2)$: [4]
22) $[\mathbf{p}] : \mathbf{p} \in (Q_1 \cap Q_2) . \supset . \mathbf{p} \in (R_1 \cap R_2)$: [4,3,19,21,20]
23) $\sim([\exists \mathbf{q}] . \mathbf{q} \in (Q_2 \cap (Q_1 - R_1)))$.·. [22]
24) $\mathsf{EXT}(Q_1 Q_2)$: [*L19*,23,20,4]
25) $[EF] : Q_1 \subset E . Q_2 \subset E . F \subset P_1 \cap P_2 . \bar{\mathsf{P}}^n(E) . \bar{\mathsf{P}}^n(F)$.
$\supset . F \subset H \subset E$: : [10,9,*L36*,P_2/E,2,3,4,5,6]
$\mathsf{BIS}(P_1 P_2)$ [1,2,3,4,7,8,11,24,25]

We can combine lemmas *L33* and *L38* to obtain:

L39 $[P_1 Q_1 Q_2 \mathbf{a}_0, \ldots, \mathbf{a}_n, \mathbf{b}_1 \mathbf{b}_2] : \mathsf{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n)(P_1)$.
$\mathsf{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}, \mathbf{b}_1)(Q_1) . \mathsf{P}^n(\mathbf{a}_0, \mathbf{b}_2, \mathbf{a}_2, \ldots, \mathbf{a}_n)(Q_2)$.
$\mathsf{ETG}(Q_1 P_1) . \mathsf{ETG}(Q_2 P_1) . \supset . [\exists H P_2] . \mathsf{BIS}(P_1 P_2)$.
$\mathsf{H}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n P_1)(H) . P_1 \cap P_2 = H$

Next we give a characterization of the definition of concentric (*DV4*). Here we want to show that if two parallelepipeds are concentric then they have the same center. In lemma *L40* we show that any bisector of P_1 contains the center of P_1. We then go on to show that only the center of P_1 is contained in the intersection of all the bisectors of P_1.

L40 $[P_1P_2\mathbf{a}_0,\ldots,\mathbf{a}_n]:\mathrm{BIS}(P_1P_2).\mathrm{P}^n(\mathbf{a}_0,\ldots,\mathbf{a}_n)(P_1).\supset.$
$\mathbf{a}_0+\frac{1}{2}\mathbf{a}_1+\ldots+\frac{1}{2}\mathbf{a}_n\in P_2$

PF $[P_1P_2\mathbf{a}_0,\ldots,\mathbf{a}_n]\therefore \mathrm{Hp}(2).\supset:$
$[\exists H\mathbf{a}_0',\ldots,\mathbf{a}_n']:$

3) $\mathrm{H}^n(\mathbf{a}_0',\ldots,\mathbf{a}_n'P_1)(H).$
4) $H=P_1\cap P_2.$ [*L37*,1]
5) $\mathrm{P}^n(\mathbf{a}_0',\ldots,\mathbf{a}_n')(P_1).$
6) $\mathbf{a}_0'+\frac{1}{2}\mathbf{a}_1'+\ldots+\frac{1}{2}\mathbf{a}_n'\in H.$ [*DVH*,3]
7) $\mathbf{a}_0'+\frac{1}{2}\mathbf{a}_1'+\ldots+\frac{1}{2}\mathbf{a}_n'\in P_2.$ [6,4]

$[\exists s_1,\ldots,s_n].$

8) $\{s_1,\ldots,s_n\}\subset\{0,1\}.$
9) $\mathbf{a}_0'=\mathbf{a}_0+s_1\mathbf{a}_1+\ldots+s_n\mathbf{a}_n.$ [*L5*,2,5]
10) $\{\mathbf{a}_1',\ldots,\mathbf{a}_n'\}=\{(1-2s_1)\mathbf{a}_1,\ldots,(1-2s_n)\mathbf{a}_n\}.$
11) $\mathbf{a}_0+\frac{1}{2}\mathbf{a}_1+\ldots+\frac{1}{2}\mathbf{a}_n=\mathbf{a}_0'+\left[s_1+\frac{1}{2}(1-2s_1)\right]\mathbf{a}_1+\ldots+\left[s_n+\frac{1}{2}(1-2s_n)\right]\mathbf{a}_n'=\mathbf{a}_0+\frac{1}{2}\mathbf{a}_1+\ldots+\frac{1}{2}\mathbf{a}_n:$ [8,9,10]

$\mathbf{a}_0+\frac{1}{2}\mathbf{a}_1+\ldots+\frac{1}{2}\mathbf{a}_n\in P_2$ [11,7]

In lemmas *L41*, *L42*, *L43*, *L44*, and *L45* we examine five cases where a point $\mathbf{p}$ of parallelepiped P_1 does not lie in the "center" of P_1 and show in each case that there is a bisector P_2 of P_1 such that $\sim(\mathbf{p}\in P_2)$. Then in lemma *L46* we show that if two parallelepipeds are concentric then their centers are equal. In the following "$\mathbf{b}_1,\ldots,\hat{\mathbf{b}}_i,\ldots,\mathbf{b}_n$" denotes that all objects $\mathbf{b}_1,\ldots,\mathbf{b}_n$ are to be considered except $\mathbf{b}_i$.

L41 $[P_1\mathbf{p}\mathbf{a}_0,\ldots,\mathbf{a}_nt_1,\ldots,t_nij]:1\leq i\leq n.1\leq j\leq n.i\neq j.$
$t_i+t_j>1.\mathrm{P}^n(\mathbf{a}_0,\ldots,\mathbf{a}_n)(P_1).\mathbf{p}=\mathbf{a}_0+t_1\mathbf{a}_1+\ldots+$
$t_n\mathbf{a}_n.\mathbf{p}\in P_1.\supset.[\exists P_2].\mathrm{BIS}(P_1P_2).\sim(\mathbf{p}\in P_2)$

PF $[P_1\mathbf{p}\mathbf{a}_0,\ldots,\mathbf{a}_nt_1,\ldots,t_nij]\therefore\mathrm{Hp}(7).\supset:$

8) $\mathrm{P}^n(\mathbf{a}_0,\ldots,\hat{\mathbf{a}}_i,\ldots,\mathbf{a}_n\mathbf{a}_i)(P_1).$
9) $\mathrm{P}^n(\mathbf{a}_0,\ldots,\hat{\mathbf{a}}_j,\ldots,\mathbf{a}_n\mathbf{a}_j)(P_1).$ [*L6*,5]
10) $\mathrm{P}^n(\mathbf{a}_0,\mathbf{a}_i,\ldots,\hat{\mathbf{a}}_i,\hat{\mathbf{a}}_j,\ldots,\mathbf{a}_n,\mathbf{a}_j)(P_1):$

$[\exists Q_1Q_2]:$

11) $\mathrm{P}^n(\mathbf{a}_0,\ldots,\hat{\mathbf{a}}_j,\ldots,\mathbf{a}_n,-\mathbf{a}_j)(Q_1).$ [5]
12) $\mathrm{ETG}(P_1Q_1).$ [*L30*,8,11]
13) $\mathrm{P}^n(\mathbf{a}_0,\ldots,\hat{\mathbf{a}}_i,\ldots,\mathbf{a}_n,-\mathbf{a}_i)(Q_2).$ [5]
14) $\mathrm{ETG}(P_1Q_2).$ [*L30*,9,13]
15) $\mathrm{P}^n(\mathbf{a}_0-\mathbf{a}_i,\mathbf{a}_1,\ldots,\hat{\mathbf{a}}_i,\ldots,\mathbf{a}_n)(Q_2).$ [*L6*,13]

$[\exists P_2H].$

16) $\mathrm{BIS}(P_1P_2).$

17) $H^n(a_0, a_i, a_1, \ldots, \hat{a}_i, \hat{a}_j, \ldots, a_n, a_j P_1)(H)$.
18) $P_1 \cap P_2 = H$. [L39,10,11,15,12,14]
19) $\sim(p \in (P_2 \cap P_1))$. [18,17,6,4]
20) $\sim p \in P_2$: [19,7]
$[\exists P_2]$. $BIS(P_1 P_2)$. $\sim(p \in P_2)$ [16,20]

L42 $[P_1 p a_0, \ldots, a_n t_1, \ldots, t_n ij] : 1 \leqq i \leqq n . 1 \leqq j \leqq n . i \neq j$. $t_i + t_j > 1 . P^n(a_0, \ldots, a_n)(P_1) . p = a_0 + t_1 a_1 + \ldots + t_n a_n . p \in P_1 . \supset . [\exists P_2] . BIS(P_1 P_2) . \sim(p \in P_2)$

PF $[P_1 p a_0, \ldots, a_n t_1, \ldots, t_n ij]$ :: Hp(7) . ⊃ :·:
$[\exists b_0, \ldots, b_n]$:·:
8) $b_0 = a_0 + a_i + a_j$.
9) $b_1 = a_1, \ldots, \hat{b}_i, \hat{b}_j, \ldots, b_n = a_n$. [1,2,3,5]
10) $b_i = -a_i . b_j = -a_j$.
11) $P^n(b_0, \ldots, b_n)(P_1)$. [8,9,10,5,3]
12) $P^n(b_0, \ldots, \hat{b}_i, \ldots, b_n, b_i)(P_1)$.
13) $P^n(b_0, \ldots, \hat{b}_j, \ldots, b_n, b_j)(P_1)$. [L6,11]
14) $P^n(b_0, b_j, b_1, \ldots, \hat{b}_i, \hat{b}_j, \ldots, b_n, b_i)(P_1)$ ::
$[\exists Q_1 Q_2]$ ::
15) $P^n(b_0, \ldots, \hat{b}_i, \ldots, b_n a_i)(Q_1)$.
16) $P^n(b_0, \ldots, \hat{b}_j, \ldots, b_n a_j)(Q_2)$. [5,11,8,9,10]
17) $P^n(b_0, a_j, b_1, \ldots, \hat{b}_j, \ldots, b_n)(Q_2)$. [L6,16]
18) $ETG(P_1 Q_1)$. [L30,12,10,15]
19) $ETG(P_1 Q_2)$.·. [L30,13,10,16]
$[\exists P_2 H]$.·.
20) $BIS(P_1 P_2)$.
21) $H^n(b_0, b_j, b_1, \ldots, \hat{b}_i, \hat{b}_j, \ldots, b_n, b_i P_1)(H)$.
22) $P_1 \cap P_2 = H$. [L39,14,15,17,18,19]
23) $P_2 \cap P_1 = \{a_0 + r_1 a_1 + \ldots + \hat{r}_i \hat{a}_i + \ldots + \hat{r}_j \hat{a}_j + \ldots + r_n a_n + (1 - r_i) a_i + (1 - r_j) a_j \mid 0 \leqq r_i . 0 \leqq r_j . r_i + r_j \leqq 1 . 0 \leqq r_1 \leqq 1, \ldots, \hat{r}_i, \hat{r}_j, \ldots, 0 \leqq r_n \leqq 1\}$: [22,21,8,9,10]
24) $[r_i r_j] : 0 \leqq r_i . 0 \leqq r_j . r_i + r_j \leqq 1 . \supset . (1 - r_i) + (1 - r_j) \geqq 1$: [simple computation]
25) $\sim(p \in (P_2 \cap P_1))$. [23,6,24,4]
26) $\sim(p \in P_2)$:·: [25,7]
$[\exists P_2]$. $BIS(P_1 P_2)$. $\sim(p \in P_2)$ [20,26]

L43 $[P_1 p a_0, \ldots, a_n t_1, \ldots, t_n ij] : 1 \leqq i \leqq n . 1 \leqq j \leqq n . i \neq j$. $t_i < \frac{1}{2} . t_i + t_j = 1 . P^n(a_0, \ldots, a_n)(P_1) . p \in P_1$. $p = a_0 + t_1 a_1 + \ldots + t_n a_n . \supset . [\exists P_2] BIS(P_1 P_2)$. $\sim(p \in P_2)$.

PF $[P_1 \mathsf{p} \mathsf{a}_0, \ldots, \mathsf{a}_n t_1, \ldots, t_n ij] \therefore \mathrm{Hp}(8) . \supset :\cdot:$

$[\exists \mathsf{b}_0, \ldots, \mathsf{b}_n] :\cdot:$

9) $\mathsf{b}_0 = \mathsf{a}_0 + \mathsf{a}_i .$
10) $\mathsf{b}_1 = \mathsf{a}_1, \ldots, \hat{\mathsf{b}}_i, \ldots, \mathsf{b}_n = \mathsf{a}_n .$ [1,2,3,6]
11) $\mathsf{b}_i = -\mathsf{a}_i .$
12) $\mathsf{P}^n(\mathsf{b}_0, \ldots, \mathsf{b}_n)(P_1) .$ [6,9,10,11]
13) $\mathsf{P}^n(\mathsf{b}_0, \ldots, \hat{\mathsf{b}}_i, \ldots, \mathsf{b}_n \mathsf{b}_i)(P_1) .$
14) $\mathsf{P}^n(\mathsf{b}_0, \ldots, \hat{\mathsf{b}}_j, \ldots, \mathsf{b}_n \mathsf{b}_j)(P_1) .$ [*L6*,12]
15) $\mathsf{P}^n(\mathsf{b}_0, \mathsf{b}_j, \mathsf{b}_1, \ldots, \hat{\mathsf{b}}_i, \hat{\mathsf{b}}_j, \ldots, \mathsf{b}_n, \mathsf{b}_i)(P_1) ::$

$[\exists Q_1 Q_2] ::$

16) $\mathsf{P}^n(\mathsf{b}_0, \ldots, \hat{\mathsf{b}}_i, \ldots, \mathsf{b}_n \mathsf{a}_i)(Q_1) .$
17) $\mathsf{P}^n(\mathsf{b}_0, \ldots, \hat{\mathsf{b}}_j, \ldots, \mathsf{b}_n - \mathsf{a}_j)(Q_2)$ [6,12,9,10,11]
18) $\mathsf{P}^n(\mathsf{b}_0, -\mathsf{a}_j, \mathsf{b}_1, \ldots, \hat{\mathsf{b}}_j, \ldots, \mathsf{b}_n)(Q_2) .$ [*L6*,17]
19) $\mathsf{ETG}(P_1 Q_1) .$ [*L30*,13,11,16]
20) $\mathsf{ETG}(P_1 Q_2) \therefore$ [*L30*,14,17]

$[\exists P_2 H] \therefore$

21) $\mathsf{BIS}(P_1 P_2) .$
22) $\mathsf{H}^n(\mathsf{a}_0, \mathsf{a}_j, \mathsf{a}_1, \ldots, \hat{\mathsf{a}}_i, \hat{\mathsf{a}}_j, \ldots, \mathsf{a}_n, \mathsf{a}_i P_1)(H) .$ [*L39*,15,16,18,19,20]
23) $(P_1 \cap P_2 = H) .$
24) $P_1 \cap P_2 =$

$\{\mathsf{a}_0 + r_1 \mathsf{a}_1 +, \ldots, + r_{i-1} \mathsf{a}_{i-1} + (1 - r_i)\mathsf{a}_i + r_{i+1}\mathsf{a}_{i+1} + \ldots + r_n \mathsf{a}_n | 0 \leqq r_i . 0 \leqq r_j . r_i + r_j \leqq 1 . 0 \leqq r_1 \leqq 1, \ldots, 0 \leqq r_n \leqq 1\} :$ [23,9,10,11]

25) $[r_i r_j] : r_i + r_j \leqq 1 . (1 - r_i) < \frac{1}{2} . \supset . r_i > \frac{1}{2} . r_j < \frac{1}{2} :$ [simple computation]

26) $[r_i r_j] : r_i + r_j \leqq 1 . (1 - r_i) < \frac{1}{2} . \supset . (1 - r_i) + r_j < 1 :$ [25]
27) $\sim(\mathsf{p} \in (P_1 \cap P_2)) .$ [24,26,8,5]
28) $\sim(\mathsf{p} \in P_2) :\cdot:$ [27,7]

$[\exists P_2] . \mathsf{BIS}(P_1 P_2) . \sim(\mathsf{p} \in P_2)$ [21,28]

L44 $[P_1 \mathsf{p} \mathsf{a}_0, \ldots, \mathsf{a}_n t_1, \ldots, t_n ij] : 1 \leqq i \leqq n . 1 \leqq j \leqq n . i \neq j . t_i > \frac{1}{2} . t_i + t_j = 1 . \mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(P_1) . \mathsf{p} \in P_1 . \mathsf{p} = \mathsf{a}_0 + t_1 \mathsf{a}_1 + \ldots + t_n \mathsf{a}_n . \supset . [\exists P_2] . \mathsf{BIS}(P_1 P_2) . \sim(\mathsf{p} \in P_2)$

PF $[P_1 \mathsf{p} \mathsf{a}_0, \ldots, \mathsf{a}_n t_1, \ldots, t_n ij] \therefore \mathrm{Hp}(8) . \supset :\cdot:$

$[\exists \mathsf{b}_0, \ldots, \mathsf{b}_n] :\cdot:$

9) $\mathsf{b}_0 = \mathsf{a}_0 + \mathsf{a}_j .$
10) $\mathsf{b}_1 = \mathsf{a}_1, \ldots, \hat{\mathsf{b}}_j, \ldots, \mathsf{b}_n = \mathsf{a}_n .$ [1,2,3,6]
11) $\mathsf{b}_j = -\mathsf{a}_j .$
12) $\mathsf{P}^n(\mathsf{b}_0, \ldots, \mathsf{b}_n)(P_1) .$ [6,9,10,11]

13) $\mathsf{P}^n(\mathsf{b}_0, \ldots, \hat{\mathsf{b}}_j, \ldots, \mathsf{b}_n\mathsf{b}_j)(P_1)$.
14) $\mathsf{P}^n(\mathsf{b}_0, \ldots, \hat{\mathsf{b}}_i, \ldots, \mathsf{b}_n\mathsf{b}_i)(P_1)$. } [*L6*,12,1,2,3]
15) $\mathsf{P}^n(\mathsf{b}_0, \mathsf{b}_j, \mathsf{b}_1, \ldots, \hat{\mathsf{b}}_i, \hat{\mathsf{b}}_j, \ldots, \mathsf{b}_n\mathsf{b}_i)(P_1)$: : [*L19*,12,1,2,3]
$[\exists Q_1 Q_2]$: :
16) $\mathsf{P}^n(\mathsf{b}_0, \ldots, \hat{\mathsf{b}}_i, \ldots, \mathsf{b}_n\text{-}\mathsf{a}_i)(Q_1)$.
17) $\mathsf{P}^n(\mathsf{b}_0, \ldots, \hat{\mathsf{b}}_j, \ldots, \mathsf{b}_n, \mathsf{a}_j)(Q_2)$. } [6,12,9,10,11]
18) $\mathsf{P}^n(\mathsf{b}_0, \mathsf{a}_j, \mathsf{b}_1, \ldots, \hat{\mathsf{b}}_j, \ldots, \mathsf{b}_n)(Q_2)$. [*L6*,17]
19) ETG $(P_1 Q_1)$. [*L30*,13,16]
20) ETG $(P_1 Q_2)$.˙. [*L30*,14,17,11]
$[\exists P_2 H]$.˙.
21) BIS$(P_1 P_2)$.
22) $\mathsf{H}^n(\mathsf{a}_0, \mathsf{a}_j, \ldots, \hat{\mathsf{a}}_i, \hat{\mathsf{a}}_j, \ldots,$
$\mathsf{a}_n, \mathsf{a}_i P_1)(H)$.
23) $P_1 \cap P_2 = H$. } [*L39*,15,16,18,19,20]
24) $P_1 \cap P_2 =$
$\{\mathsf{a}_0 + r_1\mathsf{b}_1 + \ldots + r_{j-1}\mathsf{b}_{j-1} + (1 - r_j)\mathsf{b}_j + r_{j+1}\mathsf{b}_{j+1} +$
$\ldots + r_n\mathsf{b}_n | 0 \leqq r_i . 0 \leqq r_j . r_i + r_j \leqq 1 . 0 \leqq r_1 \leqq 1, \ldots,$
$0 \leqq r_n \leqq 1\}$: [23,9,10,11]
25) $[r_i r_j] : r_i > \frac{1}{2} . r_i + r_j \leqq 1 . \supset . r_j < \frac{1}{2}.$
$(1 - r_j) > \frac{1}{2}$: [simple computation]
26) $[r_i r_j] : r_i > \frac{1}{2} . r_i + r_j \leqq 1 . \supset .$
$r_i + (1 - r_j) > 1$: [25]
27) $\sim (\mathsf{p} \in (P_1 \cap P_2))$. [24,26,8,5]
28) $\sim (\mathsf{p} \in P_2)$:·: [27,7]
$[\exists P_2]$. BIS$(P_1 P_2)$. $\sim (\mathsf{p} \in P_2)$ [21,28]

L45 $[P_1 \mathsf{p}] : \bar{\mathsf{P}}^n(P_1) . \sim (\mathsf{p} \in P_1) . \supset . [\exists P_2]$. BIS$(P_1 P_2) . \sim (\mathsf{p} \in P_2)$

PF $[P_1 \mathsf{p}]$ ⁝·⁝ Hp(2) . ⊃⁝⁝
$[\exists \mathsf{q} \mathsf{a}_0, \ldots, \mathsf{a}_n R]$ ⁝⁝
3) $\mathsf{R}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n P_1)(R)$. [1]
4) $\mathsf{q} = \mathsf{a}_0 + \frac{1}{2}\mathsf{a}_1 + \ldots + \frac{1}{2}\mathsf{a}_n$:·: [3]
$[\exists \mathsf{p}_1 t]$:·:
5) $0 < t < 1$.
6) $\mathsf{p}_1 = \mathsf{p} + t(\mathsf{q} - \mathsf{p})$. } [*L12*,3,4,2]
7) $\mathsf{p}_1 \in R$: :
$[\exists ijt_1, \ldots, t_n]$: :

8) $1 \leqq i \leqq n.$
9) $1 \leqq j \leqq n.$
10) $i \neq j:$
11) $t_i + t_j > 1 . \vee . t_i + t_j < 1.$
$t_i + t_j = 1 . t_i < \frac{1}{2} . \vee . t_i + t_j = 1 . t_i > \frac{1}{2}:$
12) $\mathbf{p}_1 = \mathbf{a}_0 + t_1\mathbf{a}_1 + \ldots + t_n\mathbf{a}_n .\therefore$ [7,3]

$[\exists P_2] .\therefore$
13) $\mathrm{BIS}(P_1P_2) .$
14) $\sim(\mathbf{p}_1 \epsilon P_2) .$ [*L41*,*L42*,*L43*,*L44*,8,9,10,11,12,7,3]
15) $\mathbf{q} \ \epsilon\ P_2 :$ [*L40*,13,3,4]
16) $\mathbf{p}\ \epsilon\ P_2 . \supset . \mathbf{p}_1 \epsilon P_2 :$ [*L14*,3,15,5,6]
17) $\sim(\mathbf{p} \epsilon P_2) ::$ [16,14]

$[\exists P_2] . \mathrm{BIS}(P_1P_2) . \sim(\mathbf{p} \epsilon P_2)$ [13,17]

L46 $[P_1P_2\, \mathbf{a}_0\mathbf{b}_0, \ldots, \mathbf{a}_n\mathbf{b}_n] : \mathrm{CON}(P_1P_2) . \mathbf{P}^n(\mathbf{a}_0, \ldots, \mathbf{a}_n)(P_1) . \mathbf{P}^n(\mathbf{b}_0, \ldots, \mathbf{b}_n)(P_2) . \supset . \mathbf{a}_0 + \frac{1}{2}\mathbf{a}_1 + \ldots + \frac{1}{2}\mathbf{a}_n = \mathbf{b}_0 + \frac{1}{2}\mathbf{b}_1 + \ldots + \frac{1}{2}\mathbf{b}_n$

PF $[P_1P_2\, \mathbf{a}_0\mathbf{b}_0, \ldots, \mathbf{a}_n\mathbf{b}_n] :\cdot: \mathrm{Hp}(3) . \supset :\cdot:$
4) $[\mathbf{p}] : \sim(\mathbf{p} \epsilon P_1) . \supset . \sim(\mathbf{p} \epsilon \cap \{Q \mid \mathrm{BIS}(P_1Q)\}) ::$ [*L45*,2]
5) $[\mathbf{p}] :: \mathbf{p} \epsilon P_1 . \mathbf{p} \neq \mathbf{a}_0 + \frac{1}{2}\mathbf{a}_1 + \ldots + \frac{1}{2}\mathbf{a}_n . \supset .\therefore$
$[\exists t_1, \ldots, t_n ij] .\therefore$
$1 \leqq i \leqq n.$
$1 \leqq j \leqq n.$
$i \neq j:$
$t_i + t_j < 1 . \vee . t_i + t_j > 1 . \vee . t_i + t_j = 1.$
$t_i < \frac{1}{2} . \vee .$
$t_i + t_j = 1 . t_i > \frac{1}{2}:$
$\mathbf{p} = \mathbf{a}_0 + t_1\mathbf{a}_1 + \ldots + t_n\mathbf{a}_n ::$ [2]
6) $[\mathbf{p}] : \mathbf{p} \epsilon P_1 . \mathbf{p} \neq \mathbf{a}_0 + \frac{1}{2}\mathbf{a}_1 + \ldots + \frac{1}{2}\mathbf{a}_n . \supset . \sim(\mathbf{p} \epsilon \cap \{Q \mid \mathrm{BIS}(P_1Q)\}) :$ [*L41*,*L42*,*L43*,*L44*,5,2]
7) $\mathbf{a}_0 + \frac{1}{2}\mathbf{a}_1 + \ldots + \frac{1}{2}\mathbf{a}_n \ \epsilon \cap \{Q \mid \mathrm{BIS}(P_1Q)\}.$ [*L40*,2]
8) $\mathbf{a}_0 + \frac{1}{2}\mathbf{a}_1 + \ldots + \frac{1}{2}\mathbf{a}_n = \cap \{Q \mid \mathrm{BIS}(P_1Q)\}.$ [7,6,4]
9) $\mathbf{b}_0 + \frac{1}{2}\mathbf{b}_1 + \ldots + \frac{1}{2}\mathbf{b}_n \ \epsilon \cap \{Q \mid \mathrm{BIS}(P_1Q)\} ::$ [*L40*,1,3]

$\mathbf{a}_0 + \frac{1}{2}\mathbf{a}_1 + \ldots + \frac{1}{2}\mathbf{a}_n = \mathbf{b}_0 + \frac{1}{2}\mathbf{b}_1 + \ldots + \frac{1}{2}\mathbf{b}_n$ [9,8]

We now consider our definition of equivalence (*DV5*) and we will show that two parallelepipeds are equivalent iff they have the same "center". One way is easy and we state it in lemma *L47*.

L47 $[P_1P_2\mathsf{a}_0\mathsf{b}_0, \ldots, \mathsf{a}_n\mathsf{b}_n] : \mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(P_1)\,.$
$\mathsf{P}^n(\mathsf{b}_0, \ldots, \mathsf{b}_n)(P_2)\,.\sim(\mathsf{EQV}\,.(P_1P_2))\,.\supset.$
$\sim\left[\left(\mathsf{a}_0 + \frac{1}{2}\mathsf{a}_1 + \ldots + \frac{1}{2}\mathsf{a}_n\right) = \left(\mathsf{b}_0 + \frac{1}{2}\mathsf{b}_1 + \ldots + \frac{1}{2}\mathsf{b}_n\right)\right]$

PF $[P_1P_2\mathsf{a}_0\mathsf{b}_0, \ldots, \mathsf{a}_n\mathsf{b}_n] .\because. \mathrm{Hp}(3)\,.\supset:$

$[\exists Q_1Q_2]:$

4) $\mathsf{CON}(P_1Q_1)\,.$
5) $\mathsf{CON}(P_2Q_2)\,.$ [3]
6) $\mathsf{EXT}(Q_1Q_2)\,.$

$[\exists \mathsf{a}_0'\mathsf{b}_0', \ldots, \mathsf{a}_n'\mathsf{b}_n'R]\,.$

7) $\mathsf{P}^n(\mathsf{a}_0', \ldots, \mathsf{a}_n')(Q_1)\,.$
8) $\mathsf{R}^n(\mathsf{b}_0', \ldots, \mathsf{b}_n'Q_2)(R)\,.$ [4,5]

9) $\mathsf{a}_0' + \frac{1}{2}\mathsf{a}_1' + \ldots + \frac{1}{2}\mathsf{a}_n' = \mathsf{a}_0 + \frac{1}{2}\mathsf{a}_1 + \ldots + \frac{1}{2}\mathsf{a}_n.$ [*L46*,4,1,7]

10) $\mathsf{b}_0' + \frac{1}{2}\mathsf{b}_1' + \ldots + \frac{1}{2}\mathsf{b}_n' = \mathsf{b}_0 + \frac{1}{2}\mathsf{b}_1 + \ldots + \frac{1}{2}\mathsf{b}_n.$ [*L46*,5,1,8]

11) $\mathsf{a}_0' + \frac{1}{2}\mathsf{a}_1' + \ldots + \frac{1}{2}\mathsf{a}_n' \notin (Q_2 - R)\,.$ [*L13*,8,7,6,→←]

12) $\mathsf{b}_0' + \frac{1}{2}\mathsf{b}_1' + \ldots + \frac{1}{2}\mathsf{b}_n' \in (Q_2 - R)\,.$ [8]

13) $\mathsf{a}_0' + \frac{1}{2}\mathsf{a}_1' + \ldots + \frac{1}{2}\mathsf{a}_n' \neq \mathsf{b}_0' + \frac{1}{2}\mathsf{b}_1' + \ldots + \frac{1}{2}\mathsf{b}_n' :$ [11,12]

$\mathsf{a}_0 + \frac{1}{2}\mathsf{a}_1 + \ldots + \frac{1}{2}\mathsf{a}_n \neq \mathsf{b}_0 + \frac{1}{2}\mathsf{b}_1 + \ldots + \frac{1}{2}\mathsf{b}_n$ [13,9,10]

The other direction is not so easy and our method is as follows: Assume P_1 and Q_1 are parallelepipeds which do not have the same center. Using lemmas *L48-L51* we can show that there exist parallelepipeds P_1' and Q_1' such that P_1' has the same center as P_1, Q_1' has the same center as Q_1 and $P_1' \cap Q_1' = \phi$. This is not enough, however, to violate the definition of equivalence because P_1' and Q_1' are not necessarily concentric to P_1 and Q_1 respectively (see figure 18 below). So we prove further (Lemmas *L52-L54*) that we can construct parallelepipeds Q_2 and Q_2' contained in $P_1 \cap P_1'$ and $Q_1 \cap Q_1'$ respectively such that Q_2 is concentric to P_1 and Q_2' is concentric to Q_1. Now because $Q_2 \subset P_1'$ and $Q_2' \subset Q_1'$ we also have $Q_2 \cap Q_2' = \phi$ and therefore Q_2 and Q_2' are external. This gives us a contradiction to the definition of equivalence (*DV5*) and so we can conclude P_1 and Q_1 are not equivalent.

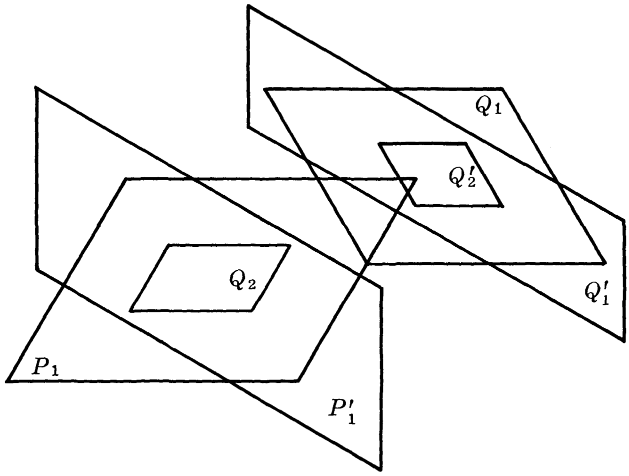

Fig. 18

Lemma *L48* is just a formal statement of the fact that given four distinct points $\mathsf{p}_1, \ldots, \mathsf{p}_4$ in $\mathfrak{A}^n$ there exists an affine transformation f (*cf.* Appendix B) such that $f(\mathsf{p}_1) = \mathsf{p}_3$ and $f(\mathsf{p}_2) = \mathsf{p}_4$.

L48 $[\mathsf{p}_1\mathsf{p}_2\mathsf{p}_3\mathsf{p}_4] : \mathsf{p}_1 \neq \mathsf{p}_2 \,.\, \mathsf{p}_3 \neq \mathsf{p}_4 \,.\supset.\, [\exists f] \,.\, f : \mathfrak{A}^n \to \mathfrak{A}^n.$
f is an affine transformation. $f(\mathsf{p}_1) = \mathsf{p}_3 \,.\, f(\mathsf{p}_2) = \mathsf{p}_4$

[Artzy [1],p. 88]

Our next two lemmas follow immediately from the definition of linear and the fact that independent vectors are mapped by linear maps into independent vectors. Therefore no proofs are given.

L49 $[P\mathsf{a}_0, \ldots, \mathsf{a}_n f] : f : \mathfrak{A}^n \to \mathfrak{A}^n.$ f is an affine transformation.
$\mathsf{P}^n(\mathsf{a}_0, \ldots, \mathsf{a}_n)(P) \,.\supset.\, [\exists \mathsf{b}_0, \ldots, \mathsf{b}_n Q].$
$\mathsf{P}^n(\mathsf{b}_0, \ldots, \mathsf{b}_n)(Q) \,.\, f(P) = Q \,.\, f\left(\mathsf{a}_0 + \frac{1}{2}\mathsf{a}_1 + \ldots + \frac{1}{2}\mathsf{a}_n\right) =$
$\mathsf{b}_0 + \frac{1}{2}\mathsf{b}_1 + \ldots + \frac{1}{2}\mathsf{b}_n$

L50 $[PQf] : \bar{\mathsf{P}}^n(P) \,.\, \bar{\mathsf{P}}^n(Q) \,.\, P \cap Q = \phi.$ f is an affine transformation.
$\supset. f(P) \cap f(Q) = \phi$

Lemma *L51* is the first step mentioned above, namely, if P_1 and Q_1 do not have the same centers then there exists P_1' and Q_1' such that P_1' and Q_1' have the same centers as P_1 and Q_1 and $P_1' \cap Q_1' = \phi$.

L51 $[PP'\mathfrak{a}_0, \ldots, \mathfrak{a}_n\mathfrak{b}_0, \ldots, \mathfrak{b}_n] : \mathsf{P}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n)(P)$.
$\mathsf{P}^n(\mathfrak{b}_0, \ldots, \mathfrak{b}_n)(P') . \mathfrak{a}_0 + \frac{1}{2}\mathfrak{a}_1 + \ldots + \frac{1}{2}\mathfrak{a}_n \neq \mathfrak{b}_0 + \frac{1}{2}\mathfrak{b}_1$
$+ \ldots + \frac{1}{2}\mathfrak{b}_n . \supset . [\exists QQ'\mathfrak{a}'_0, \ldots, \mathfrak{a}'_n\mathfrak{b}'_0, \ldots, \mathfrak{b}'_n]$.
$\mathsf{P}^n(\mathfrak{a}'_0, \ldots, \mathfrak{a}'_n)(Q) . \mathsf{P}^n(\mathfrak{b}'_0, \ldots, \mathfrak{b}'_n)(Q') . \mathfrak{a}'_0 + \frac{1}{2}\mathfrak{a}'_1 + \ldots +$
$\frac{1}{2}\mathfrak{a}'_n = \mathfrak{a}_0 + \frac{1}{2}\mathfrak{a}_1 + \ldots + \frac{1}{2}\mathfrak{a}_n . \mathfrak{b}'_0 + \frac{1}{2}\mathfrak{b}'_1 + \ldots + \frac{1}{2}\mathfrak{b}'_n =$
$\mathfrak{b}_0 + \frac{1}{2}\mathfrak{b}_1 + \ldots + \frac{1}{2}\mathfrak{b}_n . Q \cap Q' = \phi$

PF $[PP'\mathfrak{a}_0, \ldots, \mathfrak{a}_n\mathfrak{b}_0, \ldots, \mathfrak{b}_n] .\because . \mathrm{Hp}(3) . \supset :$
$[\exists \mathfrak{p}_0, \ldots, \mathfrak{p}_n\mathfrak{q}_0, \ldots, \mathfrak{q}_nP_1P_2] :$

4) $\mathfrak{p}_0 = 0$.
5) $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are n linearly independent vectors.
6) $\mathfrak{q}_0 = 2\mathfrak{p}_1$.
7) $\mathfrak{q}_1 = \mathfrak{q}_0 + \mathfrak{p}_1, \ldots, \mathfrak{q}_n = \mathfrak{q}_0 + \mathfrak{p}_n$.
8) $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are n linearly independent vectors.

[Definition of $\mathfrak{A}^n$]

9) $\mathsf{P}^n(\mathfrak{p}_0, \ldots, \mathfrak{p}_n)(P_1)$. [$DV\overline{P}$,4,5]
10) $\mathsf{P}^n(\mathfrak{q}_0, \ldots, \mathfrak{q}_n)(P_2)$. [$DV\overline{P}$,6,7,8]
11) $P_1 \cap P_2 = \phi$. [10,9,4,5,6,7]

$[\exists f]$.

12) $f: \mathfrak{A}^n \to \mathfrak{A}^n$.
13) f is an affine transformation.
14) $f\left(\mathfrak{p}_0 + \frac{1}{2}\mathfrak{p}_1 + \ldots + \frac{1}{2}\mathfrak{p}_n\right) = \mathfrak{a}_0 + \frac{1}{2}\mathfrak{a}_1 + \ldots \frac{1}{2}\mathfrak{a}_n$.
15) $f\left(\mathfrak{q}_0 + \frac{1}{2}\mathfrak{q}_1 + \ldots + \frac{1}{2}\mathfrak{q}_n\right) = \mathfrak{b}_0 + \frac{1}{2}\mathfrak{b}_1 + \ldots \frac{1}{2}\mathfrak{b}_n$:

[*L48*,11,9,10,3]

$[\exists QQ'\mathfrak{a}'_0, \ldots, \mathfrak{a}'_n\mathfrak{b}'_0, \ldots, \mathfrak{b}'_n] . \mathsf{P}^n(\mathfrak{a}'_0, \ldots, \mathfrak{a}'_n)(Q)$.
$\mathsf{P}^n(\mathfrak{b}'_0, \ldots, \mathfrak{b}'_n)(Q') . \mathfrak{a}'_0 + \frac{1}{2}\mathfrak{a}'_1 + \ldots + \frac{1}{2}\mathfrak{a}'_n =$
$\mathfrak{a}_0 + \frac{1}{2}\mathfrak{a}_1 + \ldots + \frac{1}{2}\mathfrak{a}_n . \mathfrak{b}'_0 + \frac{1}{2}\mathfrak{b}'_1 + \ldots + \frac{1}{2}\mathfrak{b}'_n =$
$\mathfrak{b}_0 + \frac{1}{2}\mathfrak{b}_1 + \ldots + \frac{1}{2}\mathfrak{b}_n . Q \cap Q' = \phi$

[*L49*, *L50*,9,10,11,12,13,14,15]

Lemma *L52* says that if P_1 is a parallelepiped and Q is a parallelepiped constructed in a special way (to be concentric to P_1 and arbitrarily small) then Q is, in fact, concentric to P_1, i.e., every bisector of P_1 is a bisector of Q (see figure 19).

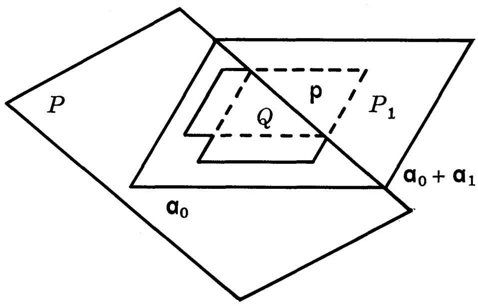

Fig. 19

L52 $[P_1PQ\mathsf{a}_0,\ldots,\mathsf{a}_n\mathsf{q}_0,\ldots,\mathsf{q}_n t\mathsf{p}]:\mathsf{P}^n(\mathsf{a}_0,\ldots,\mathsf{a}_n)(P_1).$
$\mathsf{P}^n(\mathsf{q}_0,\ldots,\mathsf{q}_n)(Q).\,0<t<1.p=\mathsf{a}_0+\frac{1}{2}\mathsf{a}_1+\ldots+$
$\frac{1}{2}\mathsf{a}_n.\,\mathsf{q}_0=\mathsf{p}+t(\mathsf{a}_0-\mathsf{p}).\,\mathsf{q}_1=t\mathsf{a}_1,\ldots,\mathsf{q}_n=t\mathsf{a}_n.$
$\mathsf{BIS}(P_1P).\supset.\mathsf{BIS}(QP)$

PF $[P_1PQ\mathsf{a}_0\mathsf{q}_0,\ldots,\mathsf{a}_n\mathsf{q}_n t\mathsf{p}]\vdots\cdot\vdots\,\mathrm{Hp}(7).\supset\vdots\ \vdots$

8) $[rs]:r+s\leqq 1.\equiv.\left(\frac{1}{2}(1-t)+rt\right)+\left(\frac{1}{2}(1-t)+st\right)\leqq 1:$ [3]

9) $[r]:0\leqq r.\supset.\,0\leqq\frac{1}{2}(1-t)+rt:$ [3]

10) $[r]:r\leqq 1.\supset.\frac{1}{2}(1-t)+rt\leqq 1\vdots\because$ [3]

$[\exists\mathsf{a}_0',\ldots,\mathsf{a}_n'\mathsf{b}_1\mathsf{b}_2Q_1Q_2H]\vdots\because$

11) $\mathsf{P}^n(\mathsf{a}_0',\ldots,\mathsf{a}_n')(P_1).$
12) $\mathsf{P}^n(\mathsf{a}_0',\ldots,\mathsf{a}_{n-1}',\mathsf{b}_1)(Q_1).$
13) $\mathsf{P}^n(\mathsf{a}_0',\mathsf{b}_2,\mathsf{a}_2',\ldots,\mathsf{a}_n')(Q_2).$
14) $\mathsf{ETG}(Q_1P_1).$
15) $\mathsf{ETG}(Q_2\mathrm{P}_1).$
16) $Q_1\subset P.$
17) $Q_2\subset P.$
18) $\mathsf{H}^n(\mathsf{a}_0',\ldots,\mathsf{a}_n'P_1)(H).$
19) $P_1\cap P=H\vdots\because$ [*L37*,7]

$[\exists s_1,\ldots,s_n]\vdots\because$

20) $\{s_1,\ldots,s_n\}\subset\{0,1\}$
21) $\mathsf{a}_0'=\mathsf{a}_0+s_1\mathsf{a}_1+\ldots+s_n\mathsf{a}_n.$
22) $\{\mathsf{a}_1',\ldots,\mathsf{a}_n'\}=\{(1-s_1)\mathsf{a}_1,\ldots,$
$(1-s_n)\mathsf{a}_n\}.$ [*L5*,1,11]

23) $\mathsf{p}=\mathsf{a}_0'+\frac{1}{2}\mathsf{a}_1'+\ldots+\frac{1}{2}\mathsf{a}_n'\because$ [4,20,21,22]

$[\exists u_1u_2]:\,:$

24) $u_1 < 0.$
25) $\mathbf{b}_1 = u_1 \mathbf{a}'_n .$
26) $u_2 < 0.$
27) $\mathbf{b}_2 = u_2 \mathbf{a}'_1 :\cdot:$ [L29,11,12,13]

$[\exists \mathbf{q}'_0, \ldots, \mathbf{q}'_n] :\cdot:$

28) $\mathbf{q}'_0 = \mathbf{q}_0 + s_1\mathbf{q}_1 + \ldots + s_n\mathbf{q}_n .$
29) $\mathbf{q}'_1 = (1 - 2s_1)\mathbf{q}_1, \ldots, \mathbf{q}'_n = (1 - 2s_n)\mathbf{q}_n .$ [20,11]
30) $\mathbf{P}^n(\mathbf{q}'_0, \ldots, \mathbf{q}'_n)(Q).$ [L6,28,29,11]
31) $\mathbf{q}'_0 = \mathbf{p} + t(\mathbf{a}'_0 - \mathbf{p}).$
32) $\mathbf{q}'_1 = t\mathbf{a}'_1, \ldots, \mathbf{q}'_n = t\mathbf{a}'_n :$ [28,29,4,5,6]
33) $[\mathbf{q} r_1, \ldots, r_n] : \mathbf{q} = \mathbf{q}'_0 + r_1\mathbf{q}'_1 + \ldots + r_n\mathbf{q}'_n . \supset .$

$\mathbf{q} = \mathbf{a}'_0 + \left[\frac{1}{2}(1-t) + r_1 t\right]\mathbf{a}'_1 + \ldots + \left[\frac{1}{2}(1-t) + r_n t\right]\mathbf{a}'_n ::$ [31,32,4,5,6]

$[\exists H_1 H_2] ::$

34) $\mathbf{H}^n(\mathbf{q}'_0, \ldots, \mathbf{q}'_n Q)(H_1) .$
35) $\mathbf{H}^n(\mathbf{a}'_0, \ldots, \mathbf{a}'_n P_1)(H_2) .$ [30,11]
36) $H_1 \subset H_2 :$ [33,8,9,10]
37) $[\mathbf{q}] : \mathbf{q} \in (Q - P) . \supset . \sim(\mathbf{q} \in H_1) :$ [36,19,→←]
38) $P \cap Q = H_1 .\cdot.$ [37,36]

$[Q'_1 Q'_2] .\cdot.$

39) $\mathbf{P}^n(\mathbf{q}'_0, \ldots, \mathbf{q}'_{n-1}, \mathbf{b}_1)(Q'_1) .$
40) $\mathbf{P}^n(\mathbf{q}'_0, \mathbf{b}_2, \mathbf{q}'_2, \ldots, \mathbf{q}'_n)(Q'_2) .$ [30,25,27,29]
41) $\mathrm{ETG}\,(Q'_1 Q) .$ [L30,30,24,25]
42) $\mathrm{ETG}\,(Q'_2 Q) :$ [L30,L6,30,24,27]
43) $[\mathbf{q}] : \mathbf{q} \in Q'_1 . \mathbf{q} = \mathbf{q}'_0 + r_1\mathbf{q}'_1 + \ldots + r_{n-1}\mathbf{q}'_{n-1} + r_n\mathbf{b}_1 = \mathbf{a}'_0 + \left[\frac{1}{2}(1-t) + r_1 t\right]\mathbf{a}'_1 + \ldots + \left[\frac{1}{2}(1-t) + r_{n-1}t\right]\mathbf{a}'_{n-1} + \left[\frac{1}{2}(1-t) + r_n u_1 t\right]\mathbf{a}'_n . \frac{1}{2}(1-t) + r_n u_1 t \geqq 0.$

$\supset . \mathbf{q} \in H :$ [18,19,39,3,24,9,10,8]

44) $[\mathbf{q}] : \mathbf{q} \in Q'_2 . \mathbf{q} = \mathbf{q}'_0 + r_1\mathbf{b}_2 + r_2\mathbf{q}'_2 + \ldots + r_n\mathbf{q}'_n = \mathbf{a}'_0 + \left[\frac{1}{2}(1-t) + r_1 u_2 t\right]\mathbf{a}'_1 + \left[\frac{1}{2}(1-t) + r_2 t\right]\mathbf{a}'_2 + \ldots + \left[\frac{1}{2}(1-t) + r_n t\right]\mathbf{a}'_n . \frac{1}{2}(1-t) + r_1 u_2 t \geqq 0.$

$\supset . \mathbf{q} \in H :$ [19,18,40,3,26,9,10,8]

45) $[\mathfrak{q}]:\mathfrak{q}\in Q_1'.\mathfrak{q}=\mathfrak{q}_0'+r_1\mathfrak{q}_1'+\ldots+$
$r_{n-1}\mathfrak{q}_{n-1}'+r_n\mathfrak{b}_1=\mathfrak{a}_0'+\left[\frac{1}{2}(1-t)+r_1t\right]\mathfrak{a}_1'$
$+\ldots+\left[\frac{1}{2}(1-t)+r_nu_1t\right]\mathfrak{a}_n'.\frac{1}{2}(1-t)+$
$r_nu_1t<0.\supset.\frac{1}{2}(1-t)+r_nu_1t>u_1.$
$\mathfrak{q}\in Q_1:$ [12,24,25,9,10]

46) $[\mathfrak{q}]:\mathfrak{q}\in Q_2'.\mathfrak{q}=\mathfrak{q}_0'+r_1\mathfrak{b}_2+r_2\mathfrak{q}_2'$
$+\ldots+r_n\mathfrak{q}_n'=\mathfrak{a}_0'+\left[\frac{1}{2}(1-t)+r_1u_2t\right]\mathfrak{a}_1'$
$+\ldots+\left[\frac{1}{2}(1-t)+r_nt\right]\mathfrak{a}_n'.\frac{1}{2}(1-t)+$
$r_nu_2t<0.\supset.\frac{1}{2}(1-t)+r_nu_2>u_2.$
$\mathfrak{q}\in Q_2:$ [13,27,26,9,10]

47) $Q_1'\subset P.$ [45,43,16]

48) $Q_2'\subset P\vdots$ [46,44,17]

BIS(QP) [*L38*,7,30,39,40,41,42,47,48,34,38]

In lemma *L53* we show the existence of Q_2 and Q_2' mentioned above. We shall make use of the fact that the cardinality of $\{(t_1,\ldots,t_n)\mid\{t_1,\ldots,t_n\}\in\{0,1\}\}$ is 2^n (see step 8 in proof of lemma *L53*).

L53 $[P_1P_2\mathfrak{a}_0\mathfrak{b}_0,\ldots,\mathfrak{a}_n\mathfrak{b}_n]:\mathsf{P}^n(\mathfrak{a}_0,\ldots,\mathfrak{a}_n)(P_1).\mathsf{P}^n(\mathfrak{b}_0,\ldots,\mathfrak{b}_n)(P_2).$
$\mathfrak{a}_0+\frac{1}{2}\mathfrak{a}_1+\ldots+\frac{1}{2}\mathfrak{a}_n=\mathfrak{b}_0+\frac{1}{2}\mathfrak{b}_1+\ldots+\frac{1}{2}\mathfrak{b}_n.$
$\supset.[\exists Q].\overline{\mathsf{P}}^n(Q).Q\subset P_2\cap P_1.\mathsf{CON}(P_1Q)$

PF $[P_1P_2\mathfrak{a}_0\mathfrak{b}_0,\ldots,\mathfrak{a}_n\mathfrak{b}_n]\vdots\mathrm{Hp}(3).\supset\vdots$
$[\exists R\mathfrak{p}\mathfrak{p}_0,\ldots,\mathfrak{p}_{2^n-1}]\vdots$

4) $\mathsf{R}^n(\mathfrak{b}_0,\ldots,\mathfrak{b}_nP_2)(R).$

5) $\mathfrak{p}=\mathfrak{a}_0+\frac{1}{2}\mathfrak{a}_1+\ldots+\frac{1}{2}\mathfrak{a}_n.$

6) $\mathfrak{p}_1=\mathfrak{a}_0+\mathfrak{a}_1,\ldots,\mathfrak{p}_n=\mathfrak{a}_0+\mathfrak{a}_n.$ [1,2]

7) $\mathfrak{p}_0=\mathfrak{a}_0.$

8) $\{\mathfrak{p}_j\mid 0\leqq j\leqq 2^n-1\}=\{\mathfrak{a}_0+t_1\mathfrak{a}_1+\ldots+t_n\mathfrak{a}_n\mid$
$\{t_1,\ldots,t_n\}\subset\{0,1\}\}:\cdot:$
$[\exists s_0,\ldots,s_{2^n-1}]:\cdot:$

9) $0<s_0,\ldots,s_{2^n-1}<1.$

10) $\mathfrak{p}+s_0(\mathfrak{p}_0-\mathfrak{p})\in P_2,\ldots,$ [*L14*,4,5,3,→←]
$\mathfrak{p}+s_{2^n-1}(\mathfrak{p}_{2^n-1}-\mathfrak{p})\in P_2::$
$[\exists t\mathfrak{p}_0',\ldots,\mathfrak{p}_{2^n-1}'\mathfrak{q}_0,\ldots,\mathfrak{q}_nQ]::$

11) $t=\min\{s_0,\ldots,s_{2^n-1}\}.$ [9]

12) $0<t<1.$

13) $\mathfrak{p}_0'=\mathfrak{p}+t(\mathfrak{a}_0-\mathfrak{p}),\ldots,$ [6,7,8,11]
$\mathfrak{p}_{2^n-1}'=\mathfrak{p}+t(\mathfrak{p}_{2^n-1}-\mathfrak{p}).$

14) $\{p'_0, \ldots, p'_{2^n-1}\} \subset P_1 \cap P_2$.˙. [*L14*,12,13,1,2,5,10]
$[\exists q_0, \ldots, q_n]$.˙.
15) $q_0 = p'_0$.
16) $q_1 = p'_1 - p'_0, \ldots$
$q_n = p'_n - p'_0$. [16,15,13,5,6,7]
17) $q_1 = t a_1, \ldots, q_n = t a_n$:
$[\exists Q]$:
18) $P^n(q_0, \ldots, q_n)(Q)$. [17,16,1,11]
19) $\{q_0 + t_1 q_1 + \ldots + t_n q_n \mid \{t_1, \ldots, t_n\} \subset \{0,1\}\} = \{p'_0, \ldots, p'_{2^n-1}\}$. [17,15,13,8]
20) $Q \subset P_1 \cap P_2$. [*L18*,1,2,18,19,14]
21) $[P]$: BIS$(P_1 P)$. ⊃ . BIS(QP): : [*L52*,1,18,12,5,17,15,13]

$[\exists Q]$. $\bar{P}^n(Q)$. $Q \subset P_1 \cap P_2$. CON$(P_1 Q)$ [18,20,21]

We now have enough information to show that if P_1 and Q_1 do not have the same center then they are not equivalent. We do this in our next lemma.

L54 $[P_1 Q_1 a_0 b_0, \ldots, a_n b_n]$: $P^n(a_0, \ldots, a_n)(P_1)$. $P^n(b_0, \ldots, b_n)(Q_1)$. $a_0 + \frac{1}{2}a_1 + \ldots + \frac{1}{2}a_n \neq b_0 + \frac{1}{2}b_1 + \ldots + \frac{1}{2}b_n$. ⊃ . ~(EQV$(P_1 Q_1)$)

PF $[P_1 Q_1 a_0 b_0, \ldots, a_n b_n]$.˙. Hp(3) . ⊃ :
$[\exists P'_1 Q'_1 a'_0, \ldots, a'_n b'_0, \ldots, b'_n]$:
4) $P^n(a'_0, \ldots, a'_n)(P'_1)$.
5) $P^n(b'_0, \ldots, b'_n)(Q'_1)$.
6) $a'_0 + \frac{1}{2}a'_1 + \ldots + \frac{1}{2}a'_n = a_0 + \frac{1}{2}a_1 + \ldots + \frac{1}{2}a_n$. [*L51*,1,2,3]
7) $b'_0 + \frac{1}{2}b'_1 + \ldots + \frac{1}{2}b'_n = b_0 + \frac{1}{2}b_1 + \ldots + \frac{1}{2}b_n$.
8) $P'_1 \cap Q'_1 = \phi$.
$[\exists Q_2 Q'_2]$.
9) $\bar{P}^n(Q_2)$.
10) $Q_2 \subset P'_1 \cap P_1$. [*L53*,1,4,6]
11) CON$(P_1 Q_2)$.
12) $\bar{P}^n(Q'_2)$
13) $Q'_2 \subset Q'_1 \cap Q_1$. [*L53*,2,5,7]
14) CON $(Q_1 Q'_2)$.
15) $Q_2 \cap Q'_2 = \phi$. [8,10,13]
16) EXT $(Q_2 Q'_2)$: [*L19*,15,9,12]
~(EQV$(P_1 Q_1)$) [10,13,15]

Summarizing *L47-L54* we have:

L55 $[P_1Q_1\mathfrak{a}_0\mathfrak{b}_0, \ldots, \mathfrak{a}_n\mathfrak{b}_n] .\because. \mathsf{P}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n)(P_1)$.
$\mathsf{P}^n(\mathfrak{b}_0, \ldots, \mathfrak{b}_n)(Q_1) . \supset : \mathfrak{a}_0 + \frac{1}{2}\mathfrak{a}_1 + \ldots + \frac{1}{2}\mathfrak{a}_n = \mathfrak{b}_0 +$
$\frac{1}{2}\mathfrak{b}_1, + \ldots + \frac{1}{2}\mathfrak{b}_n . \equiv . \mathsf{EQV}(P_1Q_1)$ [*L47,L54*]

We turn now to the construction of our function σ from $\{a \mid \mathsf{PNT}(a)\}$ to $\mathfrak{A}^n$. Essentially each a is mapped to the center of one of the parallelegrams P such that we have $a(P)$. The choice of P is arbitrary because of what we proved about "EQV".

L56 $[a] : \cdot : \mathsf{PNT}(a) . \supset : : [\exists \mathfrak{p}] .\because. [P\mathfrak{a}_0, \ldots, \mathfrak{a}_n] .\because. \mathsf{P}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n)(P)$.
$\supset : a(P) . \equiv . \mathfrak{p} = \mathfrak{a}_0 + \frac{1}{2}\mathfrak{a}_1 + \ldots + \frac{1}{2}\mathfrak{a}_n$

PF $[a] : : \mathrm{Hp}(1) . \supset .\therefore$
$[\exists Q] .\therefore$
2) $\bar{\mathsf{P}}^n(Q) :$
3) $[P] : a(P) . \equiv . \mathsf{EQV}(PQ) :$ [*DV6*,1]
$[\exists \mathfrak{q}_0, \ldots, \mathfrak{q}_n] :$
4) $\mathsf{P}^n(\mathfrak{q}_0, \ldots, \mathfrak{q}_n)(Q)$. [2]
$[\exists \mathfrak{p}]$.
5) $\mathfrak{p} = \mathfrak{q}_0 + \frac{1}{2}\mathfrak{q}_1 + \ldots + \frac{1}{2}\mathfrak{q}_n .\therefore$ [4]
$[P\mathfrak{a}_0, \ldots, \mathfrak{a}_n] : \mathsf{P}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n)(P) . \supset : a(P) . \equiv . \mathfrak{p} =$
$\mathfrak{a}_0 + \frac{1}{2}\mathfrak{a}_1 + \ldots + \frac{1}{2}\mathfrak{a}_n$ [*L55*,3,4,5]

L57 $[\exists\sigma] . \sigma : \{a \mid \mathsf{PNT}(a)\} \to \mathfrak{A}^n . \sigma$ is bijective

PF 1) $[a] : \cdot : \mathsf{PNT}(a) . \supset : : [\exists \mathfrak{p}] .\because. [P\mathfrak{a}_0, \ldots, \mathfrak{a}_n] .\because.$
$\mathsf{P}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n)(P) . \supset : a(P) . \equiv . \mathfrak{p} = \mathfrak{a}_0 + \frac{1}{2}\mathfrak{a}_1 + \ldots + \frac{1}{2}\mathfrak{a}_n : :$ [*L56*]
2) $[a] : \cdot : \mathsf{PNT}(a) . \supset : : [\exists Q] : : a(Q) .\therefore$ [*DV6*]
$[\exists\sigma] .\therefore$
3) $[aP\mathfrak{a}_0, \ldots, \mathfrak{a}_n] : \mathsf{PNT}(a) . \mathsf{P}^n(\mathfrak{a}_0, \ldots, \mathfrak{a}_n)(P)$.
$a(P) . \supset . \sigma(a) = \mathfrak{a}_0 + \frac{1}{2}\mathfrak{a}_1 + \ldots + \frac{1}{2}\mathfrak{p}_n . \sigma$
is a function [1,2]
$[ab] : \mathsf{PNT}(a) . \mathsf{PNT}(b) . a \neq b . \supset . [\exists PQ]$.
$\sim(\mathsf{EQV}(PQ)) . a(P) . a(Q) :$ [*DV6*]
5) $[ab] : \mathsf{PNT}(a) . \mathsf{PNT}(b) . a \neq b . \supset$.
$[\exists PQ\mathfrak{p}_0, \ldots, \mathfrak{p}_n\mathfrak{q}_0, \ldots, \mathfrak{q}_n] . \mathsf{P}^n(\mathfrak{p}_0, \ldots, \mathfrak{p}_n)(P)$.
$\mathsf{P}^n(\mathfrak{q}_0, \ldots, \mathfrak{q}_n)(Q) . a(P) . b(Q) . \mathfrak{q}_0 + \frac{1}{2}\mathfrak{q}_1$
$+ \ldots + \frac{1}{2}\mathfrak{q}_n \neq \mathfrak{p}_0 + \frac{1}{2}\mathfrak{p}_1 + \ldots + \frac{1}{2}\mathfrak{p}_n :$ [*L55*,4]
6) $[ab] : \mathsf{PNT}(a) . \mathsf{PNT}(b) . a \neq b . \supset . \sigma(a) \neq \sigma(b) :$ [5,3]
7) $[\mathfrak{p}] : \mathfrak{p} \in \mathfrak{A}^n . \supset . [\exists g 0 \mathfrak{a}_1, \ldots, \mathfrak{a}_n PQ] . \mathfrak{a}_1, \ldots,$
$\mathfrak{a}_n$ is a basis of $\mathfrak{A}^n . \mathsf{P}^n(0, \mathfrak{a}_1, \ldots, \mathfrak{a}_n)(P) . g$
is an affinity $. g\left(\frac{1}{2}\mathfrak{a}_1 + \ldots + \frac{1}{2}\mathfrak{a}_n\right) = \mathfrak{p} . g(P) = Q :$ [*L48,L49*]

8) $[\mathrm{p}]:\mathrm{p}\in\mathfrak{A}^n.\supset.[\exists a]\mathrm{PNT}(a).\sigma(a)=\mathrm{p}\,.\therefore$ [3, *L49*,7,*L6*]
$[\exists\sigma].\sigma:\{a\,|\,\mathrm{PNT}(a)\}\rightarrow\mathfrak{A}^n.\sigma$ is bijective [8,6,3]

§5 Having constructed our function σ we show in this section that it preserves betweenness in the following sense: $\mathrm{BTN}(abd)$ iff there exists $0<t<1$ and $\sigma(b)=\sigma(a)+t(\sigma(a)-\sigma(a))$. We shall show first that if a, b, and d are distinct point-classes and we have $\sim(\mathrm{BTN}(abd))$ then there does not exist a t such that $0<t<1$ and $\sigma(a)+t(\sigma(d)-\sigma(a))=\sigma(b)$. Then we shall show in lemmas *L59* to *L63* that if three distinct points p, q, and p' are such that there is no t for which we have $\mathrm{q}=\mathrm{p}+t(\mathrm{p}'-\mathrm{p})$ and $0<t<1$ then we have $\sim\mathrm{BTN}(\sigma^{-1}(\mathrm{p})\sigma^{-1}(\mathrm{q})\sigma^{-1}(\mathrm{p}'))$. We shall use the notation *L57* to refer to the construction in the proof of σ as well as to the statement of the lemma.

L58 $[abd]:\mathrm{PNT}(a).\mathrm{PNT}(b).\mathrm{PNT}(d).d\neq a.b\neq a.d\neq b.$
$\sim(\mathrm{BTN}(abd)).\supset.\sim([\exists t]0<t<1.\sigma(b)=\sigma(a)+t(\sigma(d)-\sigma(a)))$

PF $[abd]::\mathrm{Hp}(7).\supset:\cdot:$
$[\exists P_1P_2P_3Q]:\cdot:$
8) $a(P_1)$.
9) $b(P_3)$.
10) $d(P_2)$.
11) $\bar{\mathrm{P}}^n(Q)$.
12) $P_1\cup P_2\subset Q$.
13) $\mathrm{EXT}(P_3Q)::$ [*DV7*,1,2,3,4,5,6,7]
$[\exists\mathrm{a}_0\mathrm{b}_0\mathrm{d}_0\mathrm{q}_0,\ldots,\mathrm{a}_n\mathrm{b}_n\mathrm{d}_n\mathrm{q}_nR]::$
14) $\mathrm{P}^n(\mathrm{a}_0,\ldots,\mathrm{a}_n)(P_1)$.
15) $\mathrm{R}^n(\mathrm{b}_0,\ldots,\mathrm{b}_nP_3)(R)$.
16) $\mathrm{P}^n(\mathrm{d}_0,\ldots,\mathrm{d}_n)(P_2)$.
17) $\mathrm{P}^n(\mathrm{q}_0,\ldots,\mathrm{q}_n)(Q)\,.\therefore$ [1,2,3,8,9,10,11]
$[\exists\mathrm{a}'\mathrm{b}'\mathrm{d}']\,.\therefore$
18) $\mathrm{a}'=\mathrm{a}_0+\frac{1}{2}(\mathrm{a}_1+\ldots+\mathrm{a}_n)$.
19) $\mathrm{b}'=\mathrm{b}_0+\frac{1}{2}(\mathrm{b}_1+\ldots+\mathrm{b}_n)$.
20) $\mathrm{d}'=\mathrm{d}_0+\frac{1}{2}(\mathrm{d}_1+\ldots+\mathrm{d}_n)$. [14,15,16]
21) $\mathrm{a}'=\sigma(a)$. [*L57*,18]
22) $\mathrm{b}'=\sigma(b)$. [*L57*,19]
23) $\mathrm{d}'=\sigma(d)$: [*L57*,20]
24) $[t]:0<t<1.\mathrm{b}'=\mathrm{a}'+t(\mathrm{d}'-\mathrm{a}')$.
$\supset.\mathrm{b}'\in(Q\cap(P_3-R)):$ [12,19,15]
$[t]:0<t<1.\mathrm{b}'=\mathrm{a}'+t(\mathrm{d}'-\mathrm{a}')$.
$\supset.\sim(\mathrm{EXT}(P_3Q)):\cdot:$ [*L13*,15,24]
$\sim[\exists t].0<t<1.\sigma(b)=\sigma(a)+t(\sigma(d)-\sigma(a)))$ [25,13]

Our next lemma is illustrated by figure 20.

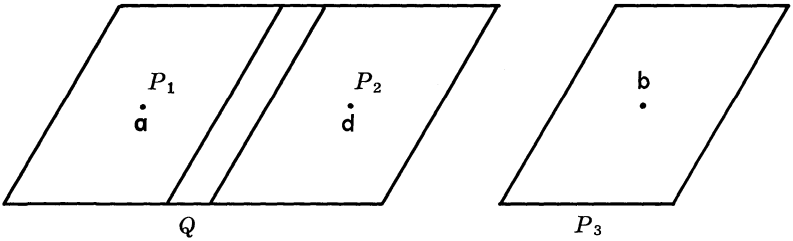

Fig. 20

L59 $[\mathsf{abd}t] : \mathsf{d} \neq \mathsf{a} \,.\, t > 1 \,.\, \mathsf{b} = \mathsf{a} + t(\mathsf{d} - \mathsf{a}) \,.\, \supset . [\exists P_1 P_2 P_3\, Q] .$
$(\sigma^{-1}(\mathsf{a}))(P_1) \,.\, (\sigma^{-1}(\mathsf{d}))(P_2) \,.\, (\sigma^{-1}(\mathsf{b}))(P_3) \,.\, \bar{\mathsf{P}}^n(Q) .$
$P_1 \cup P_2 \subset Q \,.\, \mathrm{EXT}(P_3 Q)$

PF $[\mathsf{abd}t] \vdots\because\vdots\, \mathrm{Hp}(3) \,.\supset . \vdots \cdot \vdots$

$[\exists \mathsf{b}_1, \ldots, \mathsf{b}_n] \vdots \cdot \vdots$

4) $\mathsf{b}_1, \ldots, \mathsf{b}_n$ forms a basis for $\mathfrak{A}^n$. [Definition of $\mathfrak{A}^n$]

$[\exists \mathsf{a}_1 \mathsf{a}_2] \vdots\, \vdots$

5) $\mathsf{a}_1 = \mathsf{b}_1 + \frac{1}{2}(\mathsf{b}_2 + \ldots + \mathsf{b}_n)$.

6) $\mathsf{a}_2 = 2\mathsf{b}_1 + \frac{1}{2}(\mathsf{b}_2 + \ldots + \mathsf{b}_n) \vdots \cdot \vdots$ [4]

$[\exists u] \vdots \cdot \vdots$

7) $u = \min\left\{\frac{1}{2}, \frac{1}{2}(t - 1)\right\}$. [2]

8) $u + 2 < \frac{1}{2}(t + u + 3) \vdots\,\vdots$ [7,2]

$[\exists P_1' P_2' Q'] \vdots\,\vdots$

9) $\mathsf{P}^n\left(\frac{1}{2}\mathsf{b}_1, \mathsf{b}_1, \ldots, \mathsf{b}_n\right)(P_1')$

10) $\mathsf{P}^n((2 - u)\mathsf{b}_1, 2u\mathsf{b}_1, \mathsf{b}_2, \ldots, \mathsf{b}_n)(P_2')$.

11) $\mathsf{P}^n\left(\frac{1}{2}\mathsf{b}_1, \left(\frac{3}{2} + u\right)\mathsf{b}_1, \mathsf{b}_2, \ldots, \mathsf{b}_n\right)(Q')$. [4,7]

12) $P_1' \cup P_2' \subset Q' : \cdot :$ [9,10,11,7]

$[\exists \mathsf{a}' f L] : \cdot :$

13) L is a linear transformation.

14) $[\mathsf{p}] \,.\, f(\mathsf{p}) = \mathsf{a}' + L(\mathsf{p})$.

15) $f(\mathsf{a}_1) = \mathsf{a}$.

16) $f(\mathsf{a}_2) = \mathsf{d} : :$ [*L48*,5,6,1]

$[\exists \mathsf{pq}] : :$

17) $\mathsf{p} = \frac{1}{2}(t + u + 3)\mathsf{b}_1$.

18) $\mathsf{q} = (t - u - 1)\mathsf{b}_1$. [2,4,7]

19) $\mathsf{p} + \frac{1}{2}\mathsf{q} = (1 + t)\mathsf{b}_1 \therefore$ [17,18]

$[\exists \mathsf{a}_3] \therefore$

20) $\mathfrak{a}_3 = p + \frac{1}{2}q + \frac{1}{2}(b_2 + \ldots + b_n).$ [17,18,4]

21) $\mathfrak{a}_3 = \mathfrak{a}_1 + t(\mathfrak{a}_2 - \mathfrak{a}_1)$. [20,9,10]

22) $f(\mathfrak{a}_3) = b$: [14,15,16,20]

$[\exists P_3']$:

23) $P^n(p,q,b_2,\ldots,b_n)(P_3')$. [17,18,4]

24) $Q' \cap P_3' = \phi$. [23,11,8]

$[\exists P_1 P_2 P_3 Q]$.

25) $P_1 = f(P_1')$.

26) $P_2 = f(P_2')$.

27) $P_3 = f(P_3')$.

28) $Q = f(Q')$. [*L49*,9,10,11,23]

29) $(\sigma^{-1}(\mathfrak{a}))(P_1)$. [*DV6*,*L57*,9,5,25,*L49*,15]

30) $(\sigma^{-1}(d))(P_2)$. [*DV6*,*L57*,10,6,26,*L49*,16]

31) $(\sigma^{-1}(b))(P_3)$. [*DV6*,*L57*,23,20,27,*L49*,22]

32) $P_1 \cup P_2 \subset Q$. [14,25,26,28,12]

33) $Q \cap P_3 = \phi$. [*L50*,13,14,11,23,24,27,28]

34) $\text{EXT}(P_3 Q)$ ⁝·⁝ [*L20*,33]

$[\exists P_1 P_2 P_3 Q] . (\sigma^{-1}(\mathfrak{a}))(P_1) . (\sigma^{-1}(d))(P_2)$.
$(\sigma^{-1}(b))(P_3) . \bar{P}^n(Q) . P_1 \cup P_2 \subset Q . \text{EXT}(P_3 Q)$ [29,30,31,32,34]

If we replace b_1 by $-b_1$, t by $|t|$, and negate step 7 we have a proof of the following lemma. (see figure 21).

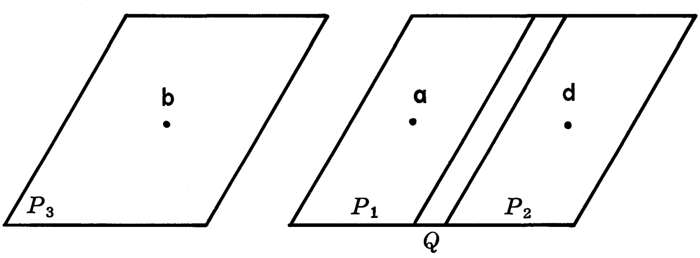

Fig. 21

L60 $[\mathfrak{a}bdt]: d \neq \mathfrak{a} . t < -1 . b = \mathfrak{a} + t(d - \mathfrak{a}) . \supset . [\exists P_1 P_2 P_3 Q]$.
$(\sigma^{-1}(\mathfrak{a}))(P_1) . (\sigma^{-1}(d))(P_2) . (\sigma^{-1}(b))(P_3) . \bar{P}^n(Q)$.
$P_1 \cup P_2 \subset Q . \text{EXT}'(P_3 Q)$ [*L59*]

In our next lemma we shall use the theorem given in Artzy [1] p. 88 that given any two triples of non-colinear points there is an affinity which maps the first triple to the second triple. We shall refer to this theorem in the proof below by using the notation (LT). The lemma is illustrated by figure 21′.

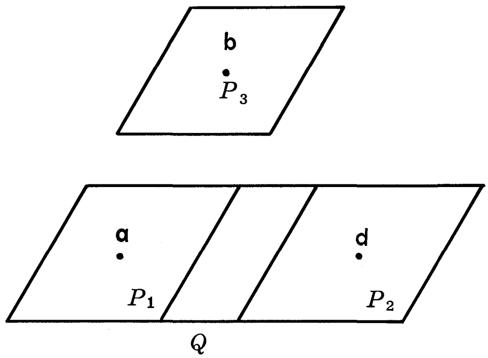

Fig. 21′

L61 $[\mathsf{abd}t] \therefore t \in F . \supset . \sim(\mathsf{b} = \mathsf{a} + t(\mathsf{d} - \mathsf{a})) : \supset . [\exists P_1 P_2 P_3 Q] .$
$\bar{\mathsf{P}}^n(Q) . (\sigma^{-1}(\mathsf{a}))(P_1) . (\sigma^{-1}(\mathsf{d}))(P_2) . (\sigma^{-1}(\mathsf{b}))(P_3) .$
$P_1 \cup P_2 \subset Q . \mathrm{EXT}(P_3 Q)$

PF $[\mathsf{abd}t] :\cdot: \mathrm{Hp}(1) . \supset ::$
$[\exists \mathsf{b}_1, \ldots, \mathsf{b}_n] ::$

2) $\mathsf{b}_1, \ldots, \mathsf{b}_n$ is a basis for $\mathfrak{A}^n . \therefore$ [Definition of $\mathfrak{A}^n$]
$[\exists P'_1 P'_2 P'_3 Q'] \therefore$
3) $\mathsf{P}^n(0, \mathsf{b}_1, \ldots, \mathsf{b}_n)(P'_1) .$
4) $\mathsf{P}^n(\mathsf{b}_1, \mathsf{b}_1, \ldots, \mathsf{b}_n)(P'_2) .$
5) $\mathsf{P}^n(2\mathsf{b}_n, \mathsf{b}_1, \ldots, \mathsf{b}_n)(P'_3) .$
6) $\mathsf{P}^n(0, 2\mathsf{b}_1, \mathsf{b}_2, \ldots, \mathsf{b}_n)(Q') .$ [2]
7) $P'_1 \cup P'_2 \subset Q'.$ [3,4,6]
8) $P'_3 \cap Q' = \phi :$ [5,6]
$[\exists \mathsf{a}_1 \mathsf{a}_2 \mathsf{a}_3]:$
9) $\mathsf{a}_1 = \frac{1}{2}(\mathsf{b}_1 + \ldots + \mathsf{b}_n) .$
10) $\mathsf{a}_2 = \mathsf{b}_1 + \frac{1}{2}(\mathsf{b}_1 + \ldots + \mathsf{b}_n) .$
11) $\mathsf{a}_3 = 2\mathsf{b}_n + \frac{1}{2}(\mathsf{b}_1 + \ldots + \mathsf{b}_n) .$ [2]
12) $[t] . \sim(\mathsf{a}_3 = \mathsf{a}_1 + t(\mathsf{a}_2 - \mathsf{a}_1)) .$ [9,10,11,2]
$[\exists \mathsf{a}' f L] .$
13) L is a linear transformation.
14) $[\mathsf{p}] . f(\mathsf{p}) = \mathsf{a}' + L(\mathsf{p}) .$
15) $f(\mathsf{a}_1) = \mathsf{a} .$
16) $f(\mathsf{a}_2) = \mathsf{d} .$
17) $f(\mathsf{a}_3) = \mathsf{b} .$
[LT,12,1]
18) $P_1 =$ $f(P'_1) .$
19) $P_2 = f(P'_2) .$
20) $P_3 = f(P'_3) .$
21) $Q = f(Q') .$ [*L49*,13,14,3,4,5,6]

22) $P_1 \cup P_2 \subset Q$. [14,18,19,21,7]
23) $P_3 \cap Q = \phi$. [14,20,21,8]
24) $\mathsf{EXT}(P_3 Q)$. [*L20*,23]
25) $(\sigma^{-1}(\mathsf{a}))(P_1)$. [*DV6*,*L57*,3,9,18,*L49*,15]
26) $(\sigma^{-1}(\mathsf{d}))(P_2)$. [*DV6*,*L57*,4,10,19,*L49*,16]
27) $(\sigma^{-1}(\mathsf{b}))(P_3)$: : [*DV6*,*L57*,5,11,20,*L49*,17]
$[_\exists P_1 P_2 P_3 Q] . \bar{\mathsf{P}}^n(Q) . (\sigma^{-1}(\mathsf{a}))(P_1) . (\sigma^{-1}(\mathsf{d}))(P_2)$.
$(\sigma^{-1}(\mathsf{b}))(P_3) . P_1 \cup P_2 \subset Q . \mathsf{EXT}(P_3 Q)$ [25,26,27,22,24]

L62 $[\mathsf{abdt}] : \mathsf{a} = \mathsf{b} . \mathsf{v} . \mathsf{d} = \mathsf{b} . \mathsf{v} . \mathsf{d} = \mathsf{a} . \supset$.
$\sim(\mathsf{BTN}(\sigma^{-1}(\mathsf{a}),\sigma^{-1}(\mathsf{b}),\sigma^{-1}(\mathsf{d}))$ [*L57*,*DV7*]

L63 $[\mathsf{abd}\,t] : \sim [_\exists t] . 0 < t < 1 . \mathsf{b} = \mathsf{a} + t(\mathsf{d} - \mathsf{a})) . \supset$.
$\sim(\mathsf{BTN})\sigma^{-1}(\mathsf{a}), \sigma^{-1}(\mathsf{b}), \sigma^{-1}(\mathsf{d}))$ [*L60*,*L61*,*L62*]

§6 In this final section of this paper we indicate how the construction of our function σ and the results of the previous section may be used to show that the interpretation of the axioms for the system $(\mathfrak{R}^n, \mathfrak{F})$ into the system $(\mathfrak{A}^n, \mathfrak{F})$ are provable in $\mathfrak{A}^n$. As examples we consider the axioms *A2′*, *A3′*, and *A4′* (and *DA1′*) where *A2′*, *A3′*, and *A4′* are obtained from *A2*, *A3*, and *A4* (and *DA1*) respectively by replacing pt by PNT and bet by BTN. We shall show that the interpretations of *A2′*, *A3′*, and *A4′* into $\mathfrak{A}^n$ are provable in lemmas *L64*, *L65*, and *L66* respectively.

L64 $[abd] : \mathsf{BTN}(abd) . \supset . \sim(\mathsf{BTN}(bda))$

PF $[abd] : : \mathrm{Hp}(1) . \supset .\therefore$
2) $a \neq b$.
3) $b \neq d$. [*DV7*,1]
4) $d \neq a$.
5) $\sigma(a) \neq \sigma(b)$.
6) $\sigma(b) \neq \sigma(d)$. [*L57*,*DV7*,1,2,3,4]
7) $\sigma(d) \neq \sigma(a) .\therefore$
$[_\exists t] .\therefore$
8) $0 < t < 1$.
9) $\sigma(b) = \sigma(a) + t(\sigma(d) - \sigma(a))$. [*L63*,*DV7*,1, →←]
10) $\sigma(d) = \sigma(b) + \left(1 - \frac{1}{t}\right)(\sigma(a) - \sigma(b))$: [9,8]
11) $[t'] : \sigma(d) - \sigma(b) + t'(\sigma(a) - \sigma(b)) . \supset$.
$t' = \left(1 - \frac{1}{t}\right)$: [5,6,7,8]
12) $1 - \frac{1}{t} < 0$. [8]
13) $\sim([_\exists t] . 0 < t < 1 . \sigma(d) = \sigma(b) + t(\sigma(a) - \sigma(b))) .\therefore$ [10,11,12]
$\sim(\mathsf{BTN}(abd))$ [*L63*,13, *L58*]

L65 $[ab] : \mathsf{PNT}(a) . \mathsf{PNT}(b) . a \neq b . \supset . [_\exists d] . \mathsf{BTN}(abd)$

PF $[ab] : \mathrm{Hp}(3) . \supset$.
$[_\exists d]$.
4) $\sigma(b) = \sigma(a) + \frac{1}{2}(\sigma(d) - \sigma(a))$. [*L57*,1,2]
$\mathsf{BTN}(abd)$ [*L58*,3,4, →←]

L66 $[abefl_1l_2]: \mathsf{S}^1(ab)(l_1) . \mathsf{S}^1(ef)(l_2) . e \in l_2 . f \in l_1 .$
$\supset . a \in l_2$
$[abefl_1l_2] :: \mathrm{Hp}(4) . \supset .\therefore$
5) $\mathsf{BTN}(eab) . \vee . \mathsf{BTN}(aeb) . \vee . \mathsf{BTN}(abe) . \vee . a = e . \vee . a = b :$ [*DA1'*,1,3]
6) $\mathsf{BTN}(fab) . \vee . \mathsf{BTN}(afb) . \vee . \mathsf{BTN}(abf) . \vee . f = e . \vee . f = b :$ [*DA1'*,1,4]
7) $f \neq e :$ [2]
$[\exists t_1 t_2] :$
8) $t_1 \neq t_2 .$
9) $\sigma(e) = \sigma(a) + t_1(\sigma(b) - \sigma(a)) .$
10) $\sigma(f) = \sigma(a) + t_2(\sigma(b) - \sigma(a)) .$ [*L63*,5,6,→←,7]
11) $\sigma(a) = \sigma(e) + (-t_1 | (t_2 - t_1))(\sigma(f) - \sigma(e)) :$ [9,10,8]
12) $\mathsf{BTN}(aef) . \vee . \mathsf{BTN}(eaf) . \vee . \mathsf{BTN}(efa) . \vee .$
$a = e . \vee . a = f .\therefore$ [*L58*,11,7,→←]
$a \in l_2$ [*DA1'*,2,12]

APPENDIX A

We shall present here a brief development of the deductive system of Mereology due to Leśniewski [5]. We note that much of what follows may be found in Clay [2].

Mereology may be considered as a formal system based on Ontology (see [4]) in which the relation "A is part of B" ($A \varepsilon \mathbf{pr}(B)$) is taken as primitive and we have the following axioms:

N1 $[AB]: A \varepsilon \mathbf{pr}(B) \cdot \supset \cdot \sim(B \varepsilon \mathbf{pr}(A))$
N2 $[ABD]: A \varepsilon \mathbf{pr}(B) \cdot B \varepsilon \mathbf{pr}(D) \cdot \supset \cdot A \varepsilon \mathbf{pr}(D)$
N3 $[AB]: A \varepsilon \mathbf{pr}(B) \cdot \supset \cdot B \varepsilon B$
DN1 $[AB] .\therefore A \varepsilon \mathbf{el}(B) \cdot \equiv : A \varepsilon A : A = B \cdot \vee \cdot A \varepsilon \mathbf{pr}(B)$
DN2 $[Aa] .\therefore A \varepsilon \mathbf{Kl}(a) \cdot \equiv : A \varepsilon A : [D] : D \varepsilon a \cdot \supset \cdot D \varepsilon \mathbf{el}(A) :$
$[D]: D \varepsilon \mathbf{el}(A) \cdot \supset \cdot [\exists EF] \cdot E \varepsilon a \cdot F \varepsilon \mathbf{el}(D) \cdot F \varepsilon \mathbf{el}(E)$
N4 $[ABa]: A \varepsilon \mathbf{Kl}(a) \cdot B \varepsilon \mathbf{Kl}(a) \cdot \supset \cdot A = B$
N5 $[Aa]: A \varepsilon a \cdot \supset \cdot [\exists B] \cdot B \varepsilon \mathbf{Kl}(a)$

It has been proved (see Tarski [8] p. 341 note 2) that Mereology is equivalent to a complete Boolean Algebra without 0. Also other primitive terms for Mereology have been given (see Sobociński [13]) such as $[ABD]: A \varepsilon B \times D . \equiv . A \varepsilon \mathbf{Kl}(\mathbf{el}(B) \cap \mathbf{el}(D))$. Finally another axiom system for Mereology is given in [9].

APPENDIX B

We give here the definitions of vector space, linear transformation, linearly independent, basis, the definition of an affinity and the definition of an ordered field.

DL1 An ordered tuple $(\mathbf{V}, \mathbf{F})$ is a vector space over the field $\mathbf{F}$ iff there exist operations + and $\cdot$ such that:

(1) $(\mathbf{V},+)$ is an abelian group and $\mathbf{0}$ denotes the identity element
(2) $r\cdot\mathbf{v} = \mathbf{v}\cdot r \ \varepsilon\ \mathbf{V}$ is uniquely defined for all $r\ \varepsilon\ \mathbf{F}$ and $\mathbf{v}\ \varepsilon\ \mathbf{V}$
(3) $r\cdot(\mathbf{v}+\mathbf{w}) = r\cdot\mathbf{v} + r\cdot\mathbf{w}$ for $r\ \varepsilon\ \mathbf{F}$ and $\mathbf{v}, \mathbf{w}\ \varepsilon\ \mathbf{V}$
(4) $(r+s)\cdot\mathbf{v} = r\cdot\mathbf{v} + s\cdot\mathbf{v}$ for $r, s\ \varepsilon\ \mathbf{F}$ and $\mathbf{v}\ \varepsilon\ \mathbf{V}$
(5) $(rs)\cdot\mathbf{v} = r\cdot(s\cdot\mathbf{v})$ for $r, s\ \varepsilon\ \mathbf{F}$ and $\mathbf{v}\ \varepsilon\ \mathbf{V}$
(6) $1\mathbf{v} = \mathbf{v}$ for 1 the identity of $\mathbf{F}$ and $\mathbf{v}\ \varepsilon\ \mathbf{V}$

DL2 L is defined to be a linear transformation if there exist two vector spaces (which may be equal) $(\mathbf{V}, \mathbf{F})$ and $(\mathbf{W}, \mathbf{F})$ such that:
(1) $L\colon\ \mathbf{V} \to \mathbf{W}$
(2) L is bijective
(3) $L(r\mathbf{v} + s\mathbf{w}) = r\cdot L(\mathbf{v}) + s\cdot L(\mathbf{w})$ for all $r, s\ \varepsilon\ \mathbf{F}$ and $\mathbf{v}, \mathbf{w}\ \varepsilon\ \mathbf{V}$

DL3 In a vector space $(\mathbf{V}, \mathbf{F})$ a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n\ \varepsilon\ \mathbf{V}$ is said to be linearly independent iff whenever $r_1\mathbf{v}_1 + \ldots + r_n\mathbf{v}_n = 0$ for $r_1, \ldots, r_n\ \varepsilon\ \mathbf{F}$ then $r_1 = \ldots = r_n = 0$

DL4 In a vector space $(\mathbf{V}, \mathbf{F})$ a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n\ \varepsilon\ \mathbf{V}$ form a basis for $(\mathbf{V}, \mathbf{F})$ iff $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent and for all $\mathbf{v}\ \varepsilon\ \mathbf{V}$ there exists $r_1, \ldots, r_n\ \varepsilon\ \mathbf{F}$ such that $\mathbf{v} = r_1\mathbf{v}_1 + \ldots + r_n\mathbf{v}_n$

DL5 A function $f\colon\ \mathbf{V} \to \mathbf{V}$ is called an affinity iff there exists a vector $\mathbf{a}\ \varepsilon\ \mathbf{V}$ and a linear transformation $L\colon\ \mathbf{V} \to \mathbf{V}$ such that $f(\mathbf{v}) = \mathbf{a} + L(\mathbf{v})$ for all $\mathbf{v}\ \varepsilon\ \mathbf{V}$

DL6 A field $\mathbf{F}$ is said to be ordered if there is a linear ordering $<$ on the elements of $\mathbf{F}$ such that $a < b$ implies $a + d < b + d$ for all $a, b, d\ \varepsilon\ \mathbf{F}$ and such that if 0 is the additive identity of $\mathbf{F}$ and we have $a\ \varepsilon\ \mathbf{F}, b\ \varepsilon\ \mathbf{F}, a > 0$, and $b > 0$ then we have $ab > 0$.

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