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FOUR-VALUED TABLES AND MODAL LOGIC

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1. It would be extremely convenient to be able to use the four-valued tables devised by W. T. Parry[1] as a decision procedure in systems of propositional modal logic such as Lewis' S1-S5, in much the same way that truth tables are used in assertoric propositional logic. Furthermore, if one attempts to use the four-valued tables in this way it will be found that all theorems of such systems receive designated values for every value of the variables when one uses the appropriate table for the system; e.g. all S3 theorems receive designated values using the S3 tables. Furthermore, no known formula which is not intuitively valid receives designated values for every value of its variables, if one makes appropriate allowances for the difference in the systems S1-S5, and the four-valued aspect of the system, discussed below.[2]

As against these facts, which mean that in practice one runs into no difficulties in using the four-valued tables as quasi-truth tables, there is a result due to James Dugundji[3] which shows that there can be no matrix of a finite number of elements which satisfies those, and only those formulas which are provable in the systems S1-S5.

In what follows I wish to discuss Dugundji's paper, showing why the practical success of the tables is not surprising despite Dugundji's result, and how the tables can provide a decision method which is in practice reliable, as well as being infinitely more convenient than other proposed procedures.

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1. *Cf.* Lewis and Langford, *Symbolic Logic*, Second Edition, New York, Dover Publications (1957), Appendix II.
2. This statement is based on considerable experience: using techniques partly described in my paper "Doing Logic by Computer," *Notre Dame Journal of Formal Logic*, vol. 10 (1969), pp, 150-162, I have tested a large number of modal statements by a computer program which uses the four-valued tables as a decision procedure.
3. "Note on a Property of Matrices for Lewis and Langford's Calculi of Propositions," *The Journal of Symbolic Logic*, vol. 5 (1940), pp. 150-151.

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First let us take a quick look at Dugundji's result, especially in the context of four-valued tables. As we will see, there is a flaw in Dugundji's reasoning which will need to be corrected before his result is completely satisfactory.

The general strategy of Dugundji's paper is this: He first calls our attention to formulas, *Fn*, of the general pattern

$$(p_1 \equiv p_2) \vee (p_2 \equiv p_3) \vee \ldots . \vee (p_i \equiv p_k)$$

where $1 \leqslant i < k \leqslant n$. He then argues that for any matrix, M, of less than n elements which satisfies all the provable formulas of S1, S2, S3, S4 or S5, a formula *Fn* will receive a designated value for every value of the variables. This is correct, but, as we will see, not for the reason given by Dugundji.

Dugundji then proceeds to give a model which satisfies all of the provable formulas of any of the systems S1-S5, but which does not satisfy any formula of the type *Fn*. Thus formulas of type *Fn* are "table-tautologies"; that is, they receive a designated value for every value of their variables, but they are not provable in the system, since some model which is a model of all provable formulas in the system is not a model of formulas of type *Fn*.

Consider the following specification of Dugundji's result for four-valued tables. For convenience, I will speak only of an S3 system, but what I say applies equally to all the Systems S1-S5. The formula *F5* for this system will be

$$(p \equiv q) \vee (q \equiv r) \vee (r \equiv s) \vee (s \equiv t) \vee (p \equiv r) \vee (p \equiv s)$$
$$\vee (p \equiv t) \vee (q \equiv s) \vee (q \equiv t) \vee (r \equiv t)$$

As can easily be checked, this is a "table-tautology" using the usual four-valued tables for S3.

Now consider some higher valued table which satisfies the provable formulas of the system. For convenience, we may consider an eight-valued table, since eight-valued tables which satisfy the provable formulas of S3 have been worked out for other purposes. For these eight-valued tables *F5* is not a table-tautology. Thus, since some model of the provable formulas of S3 is not a model of *F5*, it follows that *F5* is not provable in the system. Therefore, some table-tautology for four-valued tables is not provable in S3, and thus the four-valued tables fail to satisfy *only* those formulas which are provable.

Let us now retrace our steps and ask just why *F5* satisfied the four-valued tables. The true reason is as follows: there are four values and five variables involved. This means of course that in any formula at least two variables have the same value and at least one disjunct of *F5* receives a designated value, which means that the whole disjunction receives a designated value. Thus *F5* must be a table-tautology for four-valued tables. Equally evident there is no such reason why *F5* must be a table-tautology for 5, 6, 7, 8 or higher-valued tables.[4] Thus it is clear just how the result arises.

Dugundji's own reasoning on this point is faulty. He states that formulas *Fn* are equivalent to formulas of the form

$$(p \equiv p) \vee B$$

where *B* is any formula and that this is the reason why they are in my terminology "table-tautologies." But this is false. For the four-valued case all the formulas

$$(p \equiv p) \vee B, (q \equiv q) \vee B, \text{ etc.}$$

are table-tautologies for the eight-valued tables as well as for the four-valued tables. They thus cannot be equivalent to *F5*. The same is true for Dugundji's more complex model.

Dugundji has evidently confused the fact that for every value of the variables one disjunct (i.e. some disjunct or other) is true with the mistaken idea that some single disjunct will be true for all values of the variables. This is simply false, as can be seen by inspecting tables. We have now stated Dugundji's result sufficiently for our purposes. Let us now turn to its significance.

2. In this section I will make a number of negative points about the significance of Dugundji's result. Positive points, even if closely related, will be deferred until section 3.

A. Nothing in Dugundji's result shows that there are unprovable table-tautologies with a number of variables equal to or smaller than the number of values in the tables used. Specifically, if one uses four-valued tables Dugundji has not shown that there are unprovable table-tautologies with one, two, three, or four variables. Since almost all putative theorems of S1-S5 discussed in the literature have four variables or less, Dugundji's method applied to a four-valued table will not refute any of these formulas.

B. Nothing in Dugundji's result shows that any provable formula of S1-S5 is not a table-tautology for the appropriate tables. Indeed, as we will discuss in more detail presently, his result assumes the opposite. Thus four-valued tables would be at the very least a disproof procedure, similar to those available for predicate logic.

C. Consider what the formula *F5* tells us about a system in which it is true. Such a system must be one with four or fewer values, given the definition of "≡" used in the S1-S5 systems. The parallel formula

$$(p \equiv q) \vee (q \equiv r) \vee (p \equiv r)$$

which is a table-tautology of assertoric propositional logic tells us that assertoric propositional logic is a two-valued system. Now nothing in the axioms of S1-S5 insist that these systems must be interpreted as four-

4. It receives a non-designated value for $p = 1, q = 2, r = 3, s = 4, t = 5$.

valued systems. Thus since the axioms fail to say that the system is four-valued, and *F5* asserts that the system is four-valued, it is not surprising that *F5* is not provable from the axioms. But nothing in Dugundji's result shows that any other sort of non-provable table-tautology can be discovered in the systems S1-S5. Indeed, we will presently see reasons for thinking that none such exist.

D. Suppose that we add to one or all of the systems S1-S5 an additional axiom which is simply *F5* or some variant of it. While inconvenient in some ways (e.g. we couldn't use eight-valued tables to prove the independence of some axiom from a set of axioms which included *F5*) there are some arguments in favor of this course. Nothing in Dugundji's result shows that the systems S1*-S5* resulting from the addition of *F5* as an axiom to the systems S1-S5 do not have finite characteristic matrices.

Thus, although Dujundji's result is extremely interesting from a systematic point of view, its practical results are small so far as undermining our confidence in four-valued tables as a practical decision procedure for S1-S5. (What would undermine this confidence is an intuitively invalid formula which was a table-tautology for some system S1-S5.)

3. In what follows in this section, I speak as a practically oriented "applied" logician. Specifically, I am interested in using modal logic to get philosophically interesting results. I realize that I run the risk of being totally uninteresting to systematically oriented mathematical logicians, but I refuse to agree that their concerns are the only legitimate ones in discussions of modal logic. I will present my remarks as a discussion of the strategy of handling interesting putative theorems in modal logic.

Rather obviously the *first* thing to do with any statement which may be a theorem in some system of modal logic is to test it by using four-valued tables. If it is not a table-tautology, one can eliminate it as a possible theorem. This follows from the fact, used by Dugundji in obtaining his result, that a proposed theorem is provable from a given set of axioms only if every model satisfied by the axioms is also satisfied by the proposed theorem. This same line of reasoning is basic to many independence proofs, including those which show that the systems S1, S2, S3, S4 and S5 are separate systems. The fact that four-valued tables are an effective disproof procedure in this sense, although shown by E. J. Lemmon in 1966,[5] is not mentioned in many treatments of modal logic.

Once a proposed theorem has been shown to be a table-tautology the *second* step would seem clearly to be as follows; the proposed theorem must be examined to see if the fact that it is a table-tautology depends in any way on the four-valued character of the tables. If it does so depend, it will not be provable in systems S1-S5, but will presumably be provable in systems S1*-S5* which have *F5* as an additional axiom.

5. "Algebraic semantics for modal logics," *The Journal of Symbolic Logic*, vol. 31 (1966), pp. 46-65, 191-218.

How does one discover that a statement is a table-tautology because of the four-valued character of the tables? In some cases it is possible to simply reflect on a "suspicious" statement and see that it receives a designated value because of some limitation of four-valued tables. An example of this can be found in A. N. Prior's book *Time and Modality*.[6] Prior, who is using four-valued tables to investigate a quasi-modal system of "temporal logic", mentions a counter-intuitive table-tautology and then shows how, in terms of his interpretation of the four-valued tables, the result is not surprising. We will return to Prior's discussion, which has certain points of interest, but we can also mention some other methods of checking suspect statements, which do not depend on using one's logical acumen.

The simplest such check is simply to use tables with a greater number of values. Thus if one suspects that a table-tautology receives a designated value only because of the four-valued character of the tables, one can check it with an eight-valued table. If one suspects that a statement receives a designated value from eight-valued tables because of their eight-valued character, one can use sixteen-valued tables, and so on. (There is a simple formula for constructing "doubling" tables which give a designated value to the axioms.) This is time-consuming if done by hand, but easy with a computer.

Interestingly enough, there is a shortcut method which has proved effective in practice. This shortcut method depends on several facts, which experience in working with four-valued tables bring to attention. First, we find that every known non-intuitive formula which is valid for S4 or S5 tables is also valid for the S3 tables. Second, although both 1 and 2 are designated values for S3 tables it develops that table-tautologies whose main connective is a "modal" one (i.e. "$\Box$", "$\Diamond$", "⥽", or "$\equiv$") receive only values of 2, while table-tautologies whose main connective is a "propositional" one (i.e. ".", "v", "$\supset$", or "$\equiv$") receive only values of 1. Every non-intuitive formula which is a table-tautology which is known to me is an exception to this rule. Dugundji's formulas, Prior's example, and all their transformations, are valid for S3 tables, their main connectives are "propositional" ones but they receive only values of 2. This interesting anomaly, which undoubtedly has some explanation in terms of the properties of the system, can be used as a quick check on suspect formulas.

On the other hand, one may become interested in the properties of the explicitly four-valued system which results from adding *F5* as an additional axiom to some acceptable set of axioms for S3, S4, or S5. The advantages of having *F5* as an additional axiom are that certain table-tautologies not previously provable will become provable. The disadvantages will arise from making the system explicitly four-valued. I myself do not find this disadvantage fatal. Interpreting the values as: 1 = *logically true*, 2 =

6. A. N. Prior, *Time and Modality*, Oxford University Press (1957), pp. 16-17.

factually true, 3 = *factually false*, and 4 = *logically false*, an extremely convincing interpretation of modal logic as a four-valued system can be built up. For instance, consider the S3 and S5 tables for '□' (The designated values are 1 and 2)

S3: p	$\Box p$	S5: p	$\Box p$
1	2	1	1
2	4	2	4
3	4	3	4
4	4	4	4

In both systems "$\Box p$" receives a designated value if and only if "p" has the value 1; which fits the interpretation given above. Furthermore, in an S3 system truths about modalities are contingent, and in an S5 system "modal status is always necessary" which again agrees with the interpretation.

The failures of this interpretation are many: all of them arise out of certain anomalous values for $p = 2$, $q = 3$ or $p = 3$, $q = 2$ in the tables for material and strict implication.

In fact, the basic reason that the four-valued tables are unsatisfactory can be found, although in a somewhat oblique and indirect form in the discussion by Prior cited above. Prior's point is that a four-valued logic is unsatisfactory for temporal logic because we need to consider an infinite number of different times for an adequate temporal logic. In terms of modal logic, the parallel difficulty is that an adequate modal logic must be free to consider an infinite range of possibilities (or "possible worlds" if you like). This is not possible with only four values. Specifically, we can think of a four-valued system as allowing only one alternate possibility, or "possible world". Let us call this the "Other World". We can then read the four values as follows:

1 = True both in this actual world and in the other world
2 = True in this world and false in the other world
3 = False in this world and true in the other world
4 = False in both worlds

On this interpretation, all the anomalous values can be seen to be precisely what is to be expected. However, it is equally obvious that this system is not sufficiently rich for many purposes. We are forced in the direction of an infinite-valued system for modal logic, but we can recognize that a great many truths of modal logic do not depend on the infinite-valued aspect of an adequate modal logic. For such truths four-valued tables are an adequate test. (In addition the four-valued system has a certain independent interest.)

Consider now the case of a table-tautology, not yet proved, and not depending on the four-valued character of the system. What confidence can we have that such a statement is provable in the system? I will argue that we can have considerable confidence. The character of a system of modal logic can be specified in two ways: by its axioms and by a tabular definition of its connectives and operators. Dugundji's result shows that these two means of specification are more independent than we might have supposed, but this result only holds because the axioms permit a more than four-valued interpretation. There would seem to be no other fact about the system which is not captured by the axioms, and even four-valuedness can be captured by adding *F5* as an axiom.

In the upshot, then, I argue that in the absence of any examples of intuitively false table-tautologies we can continue to use the four-valued tables as a practical decision procedure. So used they can give valuable indications, though not conclusive evidence, as to whether a proposed theorem is provable, and they can conclusively show that certain proposed theorems are not provable in a given system. These are facts which should be brought to the attention of those who wish to employ modal logic in the investigation of philosophically interesting problems, a use for which I believe it is particularly suited.[7]

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7. As some small evidence in this direction I cite my own papers "Moore's Modal Argument," *American Philosophical Quarterly*, Vol. 3, No. 3, July 1966. "Hartshorne's Modal Proof," *Journal of Philosophy*, Vol. LXIII, No. 14, July 1966. "Ontological Modalities," *Review of Metaphysics*, Vol. XXI, No. 2, December 1967.