

## REFERENTIAL INVOLVEMENTS OF NUMBER WORDS

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*Introduction.*\* In recent formalistic analyses of the notion of number, philosophers tend to emphasize the falsity of the Frege-Russell doctrine that natural numbers are objects.<sup>1</sup> On this view number words do not have reference except to a set of numerals. Thus it is asserted that "on this view the sequence of number words is just that—a sequence of words or expressions with certain properties. There are not two kinds of things, numbers and number words, but just one, the words themselves."<sup>2</sup> It is also suggested that "what guarantees the existence of the number is the *existence* of an ordered set in which some object is the  $n^{\text{th}}$ . For any numeral, the numerals up to that one will be such a set. Then no ulterior fact beyond the generation of the numerals is needed to guarantee that they have reference."<sup>3</sup> Now while these conclusions admittedly could be meaningful and possibly true, some of the arguments which lead to these conclusions appear to be dubious.<sup>4</sup>

In this paper I wish to examine two of these arguments and show that they do not warrant the conclusions indicated above. Because of this, some alternative interpretation of numbers which perhaps could be classified as conceptualistic as well as constructivist will be suggested. However, I will not attempt here to adduce evidence for the presence of a referential semantics of number words in language. Nor will I attempt to give reasons for the construction of such a referential semantics in various discourses. In fact, it will be simply maintained that for an account of the meaning of number words, the question of their having reference can be answered in the affirmative, without implying that numbers are some special sort of things in the world.

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*Argument From Use of Number Words in Ordinary Language.* The first argument which has been advanced for showing that number words do not refer to objects or classes of objects is that our use of number words shows no evidence that they refer to objects or classes of objects. As Benacerraf has indicated, number words are not class predicates, and therefore cannot be names of classes or refer to classes. But the argument is not a conclusive one, for the contrast between class-predicates and number words need not be a genuine one. Let us compare indeed the statements:

- (1) There are three apples on the table.
- (2) There are red apples on the table.

From (2) *apparently* one can deduce that each of the apples on the table is red, but from (1) *apparently* one cannot deduce that each of the apples on the table is three—this conclusion is simply nonsensical. Granted that “red” is a predicate, the difference in the inferential potential between (1) and (2) seems to be a reason for rejecting “three” as a predicate. But we should note that this does not entail that on the level of deep structural semantical interpretation “three” does not predicate some intrinsic property of the three apples as a class of objects on the one hand, and predicate some other intrinsic property of each of the three apples in the class on the other. In fact, there is no grammatical inconsistency, nor intuitive forbiddenness in transforming (1) into:

- (3) The apples on the table are 3 in number.

in analogy to transforming (2) into:

- (4) The apples on the table are red in color.

Now “red in color” can be regarded as an intrinsic property of each of the apples on the table. Certainly nothing in the use of our language prevents us from similarly regarding “3 in number” as some intrinsic property of a class of three apples on the one hand, as well as some intrinsic property of each of the three apples in the class on the other. Furthermore, nothing in the use of our language prevents us from explaining and making explicit these intrinsic properties in terms of some suitable concepts.<sup>5</sup>

An intuitive understanding of (3) along these lines seems to suggest that the best explanation is that “are 3 in number” in (3) means “are three-membered” on the one hand, and means “belong to a 3-membered class” on the other. For to assert (1) or (3) is to assert that the objects mentioned in these statements can be correlated one-to-one with a number series—1, 2, 3, by counting. If counting a class of  $n$  objects presupposes that the class is  $n$ -membered, then to say that there are  $n$  objects such that  $Fx$  can be interpreted as saying that the class (extension) of the concept  $F$  is  $n$ -membered or is equivalent to an  $n$ -membered class. This should be precisely Frege’s analysis of the cardinal number of a class of 3 objects from which a definition of 3 as a class of all classes of three members can be easily reached.<sup>6</sup>

Since the notion of class and that of membership go together and form

a context independent of which each will lose its meaning in being used, one might suggest that, the meaning of (1) or (3) is to be expressed in terms of the class-membership context:

- (5) The class of apples on the table is three-membered (or has three members or is equivalent with a three-membered class.)
- (5') These apples on the table belong to a three-membered class.

Even though (5) and (5') do not appear to be sentences in non-technical English, the possibility of this semantico-syntactical interpretation, plus the absence of intuitive evidence to the contrary, clearly indicates that "three" in (1) or "three in number" in (3) could be construed as a predicate of a class of objects, and at the same time as a predicate of each of the objects in the class. This possibility is particularly significant, if we remember that for every statement of the form (1) or of the form (3), corresponding statements of the form (5) and the form (5') can be provided.<sup>7</sup> Now from (5') one may infer that each of these apples belongs to a three-membered class, (i.e. each is a unit or one.) Thus the dis-analogy between (1) and (2) in terms of their inferential potential breaks down.

In analogy to (5) and (5'), one might even *rewrite* (2) as:

- (6) The class of these apples on the table is red-membered (or has red members).
- (6') These apples on the table belong to a class of red objects.

Here the analogy between (5) and (5') on the one hand and (6) and (6') on the other consists in the fact that when one asks why these apples are red, the answer will be that each is red; and when one asks why these apples are three, the answer will be that each is one.

In order to strengthen the argument for the semantical similarity of number words and ordinary predicates, one might even proceed to point to a class of ordinary predicates (words in typical position "is (are) . . .") which are true only of a class of objects, but not of each of the objects in the class. These are exemplified by:

- The Indians are *vanishing*
- The sheep are *scattered*.
- The fog (the class of fog-particles) is *dense*.
- The series of numbers is *linearly ordered*, etc.

Even one can imagine that under certain circumstances color words could be only true of a class of things, but not each of the things in the classes.<sup>8</sup> A systematic inquiry into the use of collective or class predicates cannot be presented here.

What is shown in the above argument is that if class-membership can assume any intuitive sense, and therefore can be introduced in the analysis of (1) or (3), it cannot be denied that number words can be regarded as class predicates, and that certainly not on the basis of a deep level of a semantico-syntactical interpretation or in the analyses of the common use of ordinary language. For ordinary language is simply neutral and indeterminate on this point.

Now we wish to stress the fact that if number words ordinarily can be construed as predicates, they must be ordinarily construed not just as predicates of classes, but also as predicates of each of the objects in the classes. Frege has shown that number words apply to classes of things and further considered them as referring to the extension of the concept of such class—classes of classes.<sup>9</sup> Since it is not necessary for our purpose to show that under some suitable comprehension premises for classes, number words refer to classes of classes of objects, we may confine our attention to the thesis that number words refer to objects in a weaker sense, i.e. they are true of each of the objects in a class.<sup>10</sup>

As in the case of (5), where a number predicate is true of a class of objects, the number predicate has to be construed in such a way that both the notion of membership and that of class has to be presupposed, we notice in the case of (5') where number predicate is true of each of the objects in a class, the same presupposition has also to be made. Thus, if "n" is a number predicate, when it applies to a class, it would be construed as meaning "is n-membered (in symbolism equiv  $\{1, 2, \dots, n\}$ ); but when it applies to each of the objects in a class, it should be construed as meaning "belongs to a n-membered class (in symbolism  $\epsilon \{1, 2, 3, \dots, n\}$ ).

On the basis of the above construal, we may now define the referential function of number words as follows. A number word  $N$  applies to a class of objects if the class of objects is  $n$ -membered where  $n$  is the number represented by the number word  $N$ ; a number word  $N$  applies to each of the objects in a class if each of the objects belongs to a  $n$ -membered class where  $n$ , like in the above, is the number represented by the number word  $N$ . Therefore, to say (1) or (3) is to say that "are 3-membered" is true of the class of these apples on the table and that "belongs to a 3-membered class" is true of each of these apples on the table. In this sense number words do refer—they have a *divided* reference (each of the objects in the class) as well as an *undivided* reference (the class of objects as a whole). For the divided and undivided reference of number words, we don't have to assume that they must be abstract or logical objects. In other words, contrary to the proposal by Frege, we don't have to assume that natural numbers are such objects. In so far as ordinary language discourses are concerned, natural numbers are ways by which we refer to concrete finite things in the world, and statements in which they occur are statements referring to finite natural things and classes of finite natural things in the world.<sup>11</sup>

There are two consequences following from this account of the predicativeness and referentiality of number words. First, since number words can be considered as referring to classes of objects of finite sizes and simultaneously as referring to each of the objects in the classes, in so far as these objects can be correlated with the elements in a finite number series  $1, 2, 3, \dots$ , any physical exhibition of the objects in the class, or any explication of the notion of a given class of objects independently of the number, will suffice to identify the number. In this sense, a numerical statement about those objects can apply. It is therefore not surprising that the idiom:

There are  $n$  objects  $x$  such that  $Fx$  (where  $n$  is a *fixed* number).

can be explained in such a way that the number  $n$  need not be explicitly mentioned. The explanation consists in simply rewriting the statement in terms of identity theory and quantification theory. A simpler way of identifying a number say, 5, of course, is to produce a class of distinct and discrete objects (say marbles) which can be counted and correlated one-to-one with a series of five natural numbers.<sup>12</sup>

The second consequence from this predicative and referential account of number words is that number words can be regarded as predicates of some, though perhaps not of all, things in the world, when these things are considered as belonging to individual classes. That number words apply to distinct and discrete objects is a matter of experience in counting. In this sense, to study things in the world will certainly lead to their characterization (with respect to their belonging to classes) in terms of number words. On the other hand, the study of properties of numbers can be regarded as a study of certain relationships of things of which they are true. Among them, the most important is that of class-membership.

Now this account has two advantages: first, numerical statements such as  $2 + 3 = 5$  can be always regarded as applying to classes of natural things, because 2, 3, 5 can be defined as classes of natural objects. Second, discoveries of special properties of things and their relationships will lead to an introduction of new numbers. An example of this is the introduction of the real number  $\sqrt{2}$ , and the formulation of true statements about  $\sqrt{2}$  on the basis of studying the measurability of the hypotenuse of right triangles when each of their non-diagonal sides is one unit in length.<sup>13</sup>

*Argument from Incompatibilities of Set-Theoretical Descriptions.* The second argument against the possibility that number words have reference is also provided by Benacerraf. Benacerraf asks us to imagine that two children  $A$  and  $B$  have been taught different set-theoretical concepts of numbers. Suppose that they can communicate with each other and are able to apply their set-theoretical concepts of numbers to counting and measuring multiplicity. Benacerraf shows that they can not resolve their disagreements over questions of the relations between natural numbers. For each can prove that his assertions contradict the other's. Thus,  $A$  can prove that 3 belongs to 17 on the basis of a theorem in his von Neumann-type set-theoretical system:

For any two numbers  $x$  and  $y$ ,  $x$  is less than  $y$  if and only if  $x$  belongs to  $y$  and  $x$  is a proper subset of  $y$ .

Now this theorem is a result of defining 1, 2, 3, ... in terms of  $[\phi]$ ,  $[\phi, [\phi]]$ ,  $[\phi, [\phi, [\phi]]]$ , ... On the other hand,  $B$  can prove that 3 does not belong to 17 on the basis of a theorem in his Zermelo-type set-theoretic system:

For any two numbers  $x$  and  $y$ ,  $x$  belongs to  $y$  if and only if  $y$  is the immediate successor of  $x$ .

This theorem clearly is a result of defining 1, 2, 3, ... in terms of  $[\phi]$ ,  $[[\phi]]$ ,

$[[[\phi]]], \dots$  These definitions are apparently incompatible. The incompatibility in question derives from the fact that two definitions of natural numbers which could occur in the same set-theoretical framework assign different referents to the number words under analysis. Now there are three alternatives in regard to the correctness of the above two accounts of numbers:

- (a) Both accounts are correct, thus  $3 = [\phi, [\phi], [\phi, [\phi]]]$ , and  $3 = [[[\phi]]]$ .
- (b) One of the two accounts in question is correct.
- (c) Neither account is correct.

Benacerraf rejects (a) as absurd, and regards (b) as being impossible to establish for he argues that there does not seem to exist arguments for establishing which account is the really correct one, even though both share certain necessary conditions for a correct account of numbers.<sup>14</sup> Therefore, the only acceptable alternative for Benacerraf is (c) of which he has offered little explanation.

In fact, if we examine Benacerraf's argument concerning the incompatibility of the set-theoretical definitions of numbers and consequent incompatible theorems about numbers, we will see that Benacerraf is basically misled and perhaps mistaken. In the first place, one suspects that the conclusion that two set-theoretical definitions of, and theorems about, numbers, are incompatible is based on misguided premises. For one may point out that the question as to whether 3 belongs to 17, which invites divided answers, is an ambivalent one, because the term "belongs to" is ambivalent with respect to different set-theoretic accounts of numbers. Indeed it is a question which is not formulated in any particular set-theoretical account of numbers, but instead, *belongs to* our intuitive arithmetic; and its exact meaning, say, in terms of "less-than" or "membership" relation, need to be explicated.<sup>15</sup> Only when the question is translated into each individual set theory, the relation of "belonging to" will become determinate according to each individual set theory. Thus for set theory  $\theta_1$ , the question is whether or not 3 belongs to 17 relative to  $\theta_1$ . For set theory  $\theta_2$ , the question is whether or not 3 belongs to 17 relative to  $\theta_2$ , and similar translations of the question holds for  $\theta_3, \theta_4, \dots$ . In these reformulated questions, the relation of "belonging to" is suitably defined in each theory  $\theta_i$ .

When the above clarification is made, the apparent incompatibilities between different definitions and theorems in a set-theoretical framework disappear. The reason for this is simple. In  $A$ 's system 3 belongs to 17 in the sense that  $3 \in 17$ , and in  $B$ 's system 3 belongs to 17 in the sense that  $3 \in 4 \in 5 \in \dots \in 17$ , or  $3 \in^{14} 17$ ,<sup>16</sup> where  $\in$  is understood in an intuitive sense.<sup>17</sup> Thus the dispute between  $A$  and  $B$  is resolved in so far as they can still communicate with each other over laws of arithmetic and preserve their truth respectively in their systems. In so far as there is *sufficient* correspondence between their systems in regard to these laws, it is immaterial with which particular sets each of them identifies numbers in his own theory. Both identifications are correct, because each is correct for a set theory. The point is that it need not to be considered absurd to maintain

that the accounts by  $A$  and  $B$  are both correct, as Benacerraf does, in so far as they meet the minimal conditions that their sets form a progression in terms of which fundamental properties of numbers can be defined and deduced. In other words, in so far as they are intended to be unique representation of intuitive number theory and therefore are isomorphic to each other under a certain intended transformation.

In the light of the above, the possibility of identifying numbers with sets need not be denied, for it may be said that for any set-theoretical account of natural numbers, natural numbers are sets.<sup>18</sup> A decision as to what numbers could be in this (weak) sense, however, need not entail a decision as to what particular sets must be numbers. Indeed one may not *know* what particular sets numbers should be really identified with. One reason for this view is that we don't know what count as a sufficient set of conditions for being a natural number, i.e. we don't know a unique characterization of numbers in terms of sets. We only know that when numbers are *identified* with sets, the sets are intended as such unique characterization. Thus even though one might not be able to reach the decision as to what particular sets are numbers, this does not bar his using one of the infinitely many set-theoretical accounts of numbers for his purpose at hand.<sup>19</sup> And at the same time considering the given account as a correct one in an intended sense.

At this point one might simply say that the existing set theories need not be considered as rivals, but instead should be considered as devices motivated by different aims, and used to serve different purposes. Each has its own strengths, as each also has its own weaknesses. There is not a single criterion for deciding which set theory is better than others, for considerations of formal neatness, consistency, intuitive naturalness, and fruitfulness all have to be taken into account. For a certain purpose, numbers can be identified with certain *particular sets*. Indeed there is not incompatibility, nor puzzlement, in the fact that one knows something  $x$  has to be of  $y$  kind, but at the same time does not know whether  $x$  is  $z_1$  of  $y$ , or  $z_2$  of  $y$ , etc. It is possible that  $x$  is some  $z$  of  $y$  and one has to establish this by either analytical investigation or empirical inquiry. The same situation holds for the set specification of natural numbers. One's knowing that numbers are sets does not guarantee that one knows which particular sets numbers are. But this is not to say that numbers are not sets of some sort,<sup>20</sup> which may or may not have been specified.

To conclude, the argument from incompatible set-theoretical descriptions does not show that numbers are not sets, in so far as the question whether numbers are sets can be significantly raised. On the contrary, we have seen that numbers can be sets and that number words are both predicative and referential. In fact, constructively it can be shown that as the reference of any term or any concept in a theory is only defined in a larger theory, the reference of numbers must be so defined.

## NOTES

1. Paul Benacerraf: "What Numbers Could Not Be," *Philosophical Review*, LXXIV (1965), 47-73. Stephen F. Barker: *Philosophy of Mathematics*, Prentice-Hall Inc., New Jersey (1964), Chapter 5 "Transition to a Non-literalistic View of Numbers," pp. 82-104. Charles Parsons, "Frege's Theory of Numbers," *Philosophy in America*, edited by Max Black, Cornell University Press (1965), pp. 180-203.
2. Paul Benacerraf, *op. cit.*, p. 71.
3. Charles Parsons, *op. cit.*, p. 202.
4. The assumption that number words could be said to refer to themselves is also dubious. Perhaps the objection entertained by these philosophers to a referential account of number words is that number words should not be said to refer to intangible abstract objects, such as seemingly presupposed in statements like "natural numbers are either odd or even" and "number 3 is a prime number." The objection leaves open the question as to whether number words can also be said to refer to tangible concrete objects. Their arguments, however, do not *seem* to leave this question open. Thus a clarification, at least in order to remove misunderstanding, is warranted.
5. Here I intend to show that the surface-syntax of English permits us to exhibit certain similarity between the use of number words and that of color words. This similarity may appear to be illusory after a certain depth-syntax is discovered, and therefore may not possess any philosophical suggestiveness. But, on the other hand, it is not clear whether in a limited discourse the use of color words and that of number words may not share something in common, that is, the structures of sentences in which they occur may not share the same model. It should be pointed out here that the notion of a depth-syntax is compatible with a semantical interpretation of sentences involving two types of words which preserves the surface syntactic similarity in question. This is not to deny that there could not be other syntactical properties which are dissimilar. Perhaps, given any syntactic theory and any semantical interpretation for a given discourse, there will be always syntactical and semantical deviations when the given discourse is embedded in a larger one. What I am suggesting is that complete syntactic and semantic equivalence can not be defined over open-textured discourses the presence of which, of course, characterizes any natural language.
6. *Cf.*, G. Frege, *The Foundations of Arithmetic*, pp. 67-95. Also, *cf.*, Parsons' explication, *op. cit.*, p. 183. Geach points out that for Frege the identification of members with certain extensions is only a secondary and doubtful point. *Cf.*, *Three Philosophers*, Cornell University, Ithaca (1961), p. 158. *Cf.*, Frege, *The Grundlagen*, p. 107.
7. It might be suggested that this is not a convincing argument, since it might be agreed that "class  $\phi$  is three-numbered" means simply that "the number of elements of  $\phi$  is identical with the number 3." But here we do not assert an unique construal of the idiom: Class  $\phi$  is  $n$ -numbered. Our purpose is to bring out the predicative nature of number words as much as we can, on a semantical syntactical *basis*.
8. One could imagine that a group of objects look red only when put together, but look differently when placed alone.



9. See Frege, *op. cit.*, pp. 67-95. According to Frege, the extension of the concept "is (are)  $n$ -membered" is a class of classes of  $n$  members where  $n$  is a fixed number, not a variable. Here we might be interested in knowing what is the extension of the concept "belongs to a  $n$ -membered class." Clearly, the extension in question is a class of things in a class of classes of  $n$  members.
10. In fact, for an intuitive account of ordinary language number words it seems natural not to assume more than the availability of first order classes.
11. Körner suggests the distinction between the concept of natural numbers and that of Natural Numbers. The latter is inexact and empirical, because it is used in practical and ordinary language as referring to actual facts and objects in the world; the former, on the other hand, are natural numbers in mathematical systems or theories of numbers. Their properties are abstractly specified. Körner points out that one criterion for making this distinction is that natural numbers satisfy the axiom of infinity whereas Natural Numbers do not. One may also point out that another criterion, and perhaps a more important one, is that of mathematical induction. Whereas there are number series where mathematical induction holds true, it is not clear how mathematical induction applies to empirical generalizations. In fact empirical generalizations are never mathematically valid. For Körner, see his *Philosophy of Mathematics*, Harper, (1960), pp. 52-72.
12. Here a major question is what are the epistemological conditions for identifying 5 distinct and discrete objects, instead of 6 or 7 each. I will not go into this question here. It is to be granted that, as Frege makes clear, one has to recognize the units in which the objects are to be counted. One pair of shoes could be regarded as two shoes. But we could simply assume that what units are to be adopted depends upon the epistemological contexts and pragmatic purposes which orient the counting. One could examine situations and choices where one unit is relevant, another is not. We have to discuss this in another paper.
13. Here I do not intend to identify a specific reference of  $\sqrt{2}$  or any real number, though it is still possible to maintain, perhaps like intuitionists, that real numbers refer to things of certain sort in the world with regard to their class relationships, these things being ordinary things in the world.
14. To quote Benacerraf, "Is  $3 = [[[\phi]]]$ " *tout court* (and not elliptically for 'in Ernie's account?'), in the absence of any way of settling it, is to lose one's bearing completely. No, if such a question has an answer, there are arguments supporting it, and if there are no such arguments, then there is no 'correct' account that discriminates among all the accounts satisfying the conditions of which we reminded ourselves a couple of pages back." *Ibid.*, p. 58.
15. This is perhaps one of the reasons why set-theoretical accounts of numbers arise at all.
16. This representation of accumulation of memberships is taken from Quine, *Set Theory and Its Logic*, Harvard University, Cambridge, Massachusetts (1963), p. 36.
17. Here we may develop a notion of generalized membership and an associated generalized set theory. For given an intuitive understanding of  $\epsilon$ , we can give definition of a generalized  $\epsilon$  along the following line:  
 $A \in B$  if and only if  $A \in B$  or there is some  $C$  such that  $A \in C$ , and  $C \in B$  and so on.

A precise formulation of the definition need take account of the comprehension premises for sets.

18. Hereafter I use the term "set" interchangeably with the term "class."
19. Benacerraf seems to agree on this point. *Cf., ibid.*, p. 62. But my point is that the referentiality and predicativeness of number words do not consist in the existence of an answer to the question as to what particular sets could be identified with numbers. The question is meaningless.
20. If one takes our intuitive notion of numbers as a general criterion, perhaps many are inclined to identify numbers with the sets defined in the Frege-Russell system of set theory than with sets defined in other set theories.

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