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GENERALIZED REALS

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Real numbers fill the line, thanks to a postulational dictum—the axiom of completeness of the line. But since dimension depends on structure and not on cardinality, such an axiom is justified only by the simplicity it yields. In principle, there is no limit to the number of points that can be fitted on the line.

We shall assume that standard reals have already been introduced and shall refer to them as reals of the first kind, or simply reals; their set will be designated R. To the field R we now apply the operation of ultraproduct with R itself as an index set and with ultrafilter U containing all the co-countable subsets of R as well as all the upper halves of R (all sets of reals greater than, or greater than or equal to, a given r in R). The ultraproduct R_2 thus obtained is the quotient field of the set of all functions of R into R—denoted by A—modulo U. Members of R_2 will be called reals of the second kind, or r-reals. R_2 is a totally ordered non-Archimedean field containing infinites and infinitesimals. The monad $\mu(r)$ of a r-real r is the set of elements of R_2 infinitely close to r. For definitions see [3, p. 57].

Theorem 1. *R_2 is of cardinality $2^c = c_2$, which is also the cardinality of any monad.*

Proof: Because the cardinality of R_2 is the cardinality of any of its monads [2, p. 200], it suffices to show that R_2 has cardinality c_2. To do this, consider any one-to-one function f on R into R. The set P_f of f's permutations is of cardinality c_2 [4, p. 193]. We wish to show that there is a subset of P_f, also of cardinality c_2, composed exclusively of functions which are pairwise different modulo U. This will prove the theorem, since the cardinality of P_f/U is less than or equal to the cardinality of $A/U = R_2$. Let us consider, then, all chains (ordered by inclusion) of members of U in the Boolean algebra of all subsets of R. There must be c_2 of these chains because each is of cardinality c and the cardinality of U is c_2. There is no minimal element in any of these chains (otherwise the empty set would be in U). If for any permutations f_1 and f_2 of f, $f_1 = f_2$ (mod U), then there is a set u in U such that for all ξ in u, $f_1(\xi) = f_2(\xi)$. Also, for every η in every subset v of u in $U, f_1(\eta) = f_2(\eta)$. Therefore, as we move along a chain of

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members of U in the direction of the empty set ϕ we obtain smaller subsets v of R for all of whose elements $\eta, f_1(\eta) = f_2(\eta)$, all of these subsets having c elements and hence c_2 permutations. Further, no matter how far we move along these chains in the direction of ϕ we can always count c_2 of them, for given an arbitrary antichain of cardinality c_2 the finite intersection property of the ultrafilter can only yield antichains of the same cardinality. Regardless of how far we go in the direction of ϕ, then, there are always c_2 members of U which are pairwise incomparable with respect to inclusion. Let us now take an arbitrary antichain of cardinality c_2 and form the family F of subsets of f that we obtain by successively restricting f to each of the members of the antichain. The members of F are all different in a set-theoretic sense (not modulo U), and each has a set of c_2 permutations, providing altogether $c_2 \cdot c_2 = c_2$ set-theoretically different permutations. For each of these permutations $p(\xi)$ let us define $p(\nu) = f(\nu)$ for all ν outside the corresponding member of U. If the new family of permutations of f thus obtained does not contain c_2 of them which are different modulo U, then further along the c_2 chains in the direction of ϕ one must be able to find antichains of cardinality less than c_2. Since this is impossible, a set of c_2 different permutations of f modulo U exists and the theorem is proved.

Theorem 2. *The subset of multiples of an infinitesimal is not cofinal with the monad of all infinitesimals* $\mu(0)$.

Proof: It suffices to recall that the convergent sequences

$$\cdots > \frac{1}{n(\log n)^2} > \frac{1}{n^2} > \frac{1}{2^n} > \cdots$$

for every pair of finite natural numbers n and m satisfy

$$\frac{1}{n(\log n)^2} > \frac{m}{n^2}, \frac{1}{n^2} > \frac{m}{2^n}, \text{ etc.}$$

This implies that there are infinitesimals x and y such that $y > mx$ for every finite natural number m (define, for example, $x = 1/2^n$ and $y = 1/n^2$ for all ξ in the interval $n \leqslant \xi < n+1$). Further, it is well known that given a sequence that converges to zero there are always sequences that converge less rapidly, which implies that there are no "maximal" elements z in $\mu(0)$ in the sense that mz would be greater than any other w in $\mu(0)$ for finite natural numbers m greater than some m_0. Therefore, the theorem holds.

To R_2 we again apply the operation of ultraproduct with R_2 itself as an index set. The ultrafilter U_2 contains all the co-countable subsets of R_2 as well as the complements of every subset of R_2 of cardinality c (let us call them co-c subsets); in U_2 we also include all the upper halves of R_2. The ultraproduct R_3 thus obtained is a set of cardinality $2^{c_2} = c_3$, and its members will be called reals of the third kind, rr-reals. Rr-reals can be represented by families of r-reals $x = \{x_\xi\}_{\xi \in R}$, and in R_3 we find infinitesimals and infinites of a second kind in addition to those of the first kind that are carried over from R_2. Given a rr-real number x, we must distinguish between its monad of the first kind $\mu(x)$—empty for infinites of the second kind—and its monad of the second kind $\mu_2(x)$. The first one is the image of $\mu(x_0)$ in R_2 under the natural injection φ that maps any r-real r into the equivalence class modulo U_2 that contains the constant function $y = \{y_\xi | y_\xi = r$ for all $\xi\}_{\xi \in R}$, and where x_0 is that unique element of R_2 (if it exists) whose image under φ is infinitely close to x in R_3. On the other hand, $\mu_2(x)$ is the set of all z in R_3 such that $z - x$ is an infinitesimal of the second kind.

Iterating the operation of ultraproduct, we obtain reals of kind $k+1$ by forming the ultraproduct of R_k, with R_k as an index set and with ultrafilter U_k containing all the co-countable, co-c, co-$c_1, \ldots,$ co-c_{k-1} subsets of R_k (where $2^{c_n} = c_{n+1}$) as well as the upper halves of R_k. The cardinality of R_{k+1} is $2^{c_k} = c_{k+1}$; R_{k+1} contains infinitesimals and infinites of k kinds, and centered on each real x of the first kind there is a sequence of monads of k different kinds nested in the sense that every element of $\mu_{n+1}(x)$ is closer to x than every element of $\mu_n(x)$. Of course, infinites of kind k come after all the other infinites, and they only have monads of kind k. R_{k+1} is a totally ordered non-Archimedean field in which the set of reals of kind k is not dense. If x is a real of kind $n < k+1$ and y is a real of kind $k+1$, then $x+y$ is of kind $k+1$ (otherwise, $x + (y - x) = y$ with $y - x$ of a kind less than $k+1$). The measure of an interval determined by two reals of kind $k+1$ that belong to the same monad of kind k is an infinitesimal of kind k.

The presence of sequences of monads of various kinds centered around members of R_{k+1} is a necessary topological feature of sets of generalized reals and is a consequence of the fact that the Archimedean property imposes well-known restrictions on the cardinality of totally ordered fields. However, a pseudo-Archimedean property holds in R_{k+1}, that is, the property that if $x < y$ in R_{k+1} (x and y positive) there are infinite natural numbers of kind k such that $nx > y$ (as can be easily seen by extending some well-known properties of infinite sequences). Following the process described here, one can fill the line with points of real abscissae of various kinds, thus indefinitely increasing the cardinality of the line as a set of points, except if an axiom of completeness of the line is introduced at some stage.

Since a complete totally ordered field is Archimedean and therefore isomorphic to a subset of the reals of the first kind, the Dedekind completion of R_k (denoted R_k^*) is not a field. R_k^* is a double semigroup with identities and therefore is not algebraically interesting. However, both R_k and R_k^* have interesting, although intricate, set-theoretic structures, as the following considerations will make obvious.

The order type of R_2 is $\lambda_2 = \lambda_2(\theta^* + 1 + \theta)$ [2, p. 200] where θ is a dense order type without first and last element—the order type of the positive part of the quotient set R_2/R_2^0 where R_2^0 is the set of finite r-reals—θ^* is the inverse of θ and λ_2^0 is the order type of R_2^0. It should be pointed out that the equation that gives the order type of R_2 involves a *regressus ad infinitum*. Since

$$\mu = \lambda_2^0 \theta + 1 + \lambda_2^0 \theta^* \tag{1}$$

is the order type of any monad of R_2 and

$$\lambda_2^0 = \mu \lambda \tag{2}$$

λ being the order type of R, one finds inside any monad of R_2 c times the order type of this same monad. By iterating the substitutions of (2) in (1) and (1) in (2) successively ($\aleph_0$ times, since substitution is a denumerably recursive process), it becomes clear that the order type for a monad of R_2 is unusually complex and the order type for the various monads of R_k even more so.

Finally, one word on a possible use for generalized reals. Denjoy described the problem of a complete arithmetization of the class Z_2 of countable infinite ordinals as equivalent to the problem of "analytically formulating rules by which a determined, unique real number x_α would correspond to each ordinal of Z_2" [1, p. 210]. This statement can be extended so that the arithmetization of the classes Z_3, Z_4, etc., becomes equivalent to the one-to-one mapping of their elements into sets of r-reals, rr-reals, etc., respectively, a mapping whose characterization depends on solving the arithmetization of Z_2.

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