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A PARADOX IN ILLATIVE COMBINATORY LOGIC

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Curry, in [1] and [2], has shown the inconsistency of a system of illative combinatory logic containing the axiom:

$$\vdash \mathsf{H}^k\mathfrak{X} \text{ for all obs } \mathfrak{X},$$

for $k = 2$ (and 1). ("$\mathsf{H}\mathfrak{X}$" stands for "$\mathfrak{X}$ is a proposition".) He also stated that the inconsistency held for $k > 2$, this more general result is proved below. Assume the following:

A1 $\mathsf{H}\mathfrak{X}, \mathsf{H}\mathfrak{Y}, \mathsf{H}\mathfrak{Z} \vdash \mathfrak{X} \supset . \mathfrak{Y} \supset \mathfrak{Z} : \supset : \mathfrak{X} \supset \mathfrak{Y} . \supset . \mathfrak{X} \supset \mathfrak{Z}.$

A2 $\mathsf{H}\mathfrak{X}, \mathsf{H}\mathfrak{Y} \vdash \mathfrak{X} \supset . \mathfrak{Y} \supset \mathfrak{X}.$

A3 $\mathfrak{X}, \mathsf{P}\mathfrak{X}\mathfrak{Y} \vdash \mathfrak{Y}.$

A4 $\mathfrak{X} \vdash \mathsf{H}\mathfrak{X}.$

A5 $\vdash \mathsf{H}^{k+1}\mathfrak{X}$ *for any* $\mathfrak{X}$ *and* $k \geqslant 0.$

A6 $\vdash \mathsf{H}\mathfrak{A}.$

A7 *If* $\vdash \mathsf{H}\mathfrak{X}$ *and* $\mathfrak{X} \vdash \mathsf{H}\mathfrak{Y}$ *then* $\vdash \mathsf{H}(\mathsf{P}\mathfrak{X}\mathfrak{Y}).$

From *A1*, *A2*, *A3* and *A7* it follows (as is proved in [4]) that if $\mathsf{T}(\mathfrak{X}_1, \ldots, \mathfrak{X}_n)$ is any theorem of pure implicational intuitionistic propositional calculus for indeterminates $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$, then

$$\mathsf{H}\mathfrak{X}_1, \mathsf{H}\mathfrak{X}_2, \ldots, \mathsf{H}\mathfrak{X}_n \vdash \mathsf{T}(\mathfrak{X}_1, \ldots, \mathfrak{X}_n).$$

This fact is used in several places below.

Let $$G_0 \equiv [x] . x \supset \mathfrak{A},$$

and for $n \geqslant 0$ let

$$G_{n+1} \equiv [x] . \mathsf{H}^{n+1}x \supset G_n x.$$

Now

$$\mathsf{H}^{n+1}x \vdash \mathsf{H}(G_n x) \tag{1}$$

is proved by induction, thus:

By *A6* and *A7*

$$\mathsf{H}x \vdash \mathsf{H}(G_0 x).$$

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Now assume

$$\mathsf{H}^{n+1}x \vdash \mathsf{H}(G_n x);$$

then by A7

$$\mathsf{H}(\mathsf{H}^{n+1}x) \vdash \mathsf{H}(\mathsf{H}^{n+1}x \supset G_n x),$$

so

$$\mathsf{H}^{n+2}x \vdash \mathsf{H}(G_{n+1}x).$$

This completes the inductive proof of (1). Now let

$$X \equiv \mathsf{Y}G_k;$$

(Y is the paradoxical combinator $\mathsf{WS(BWB)}$; see [3]), then

$$X \equiv G_k X.$$

But by (1)

$$\mathsf{H}^{k+1}X \vdash \mathsf{H}(G_k X),$$

so

$$\mathsf{H}^{k+1}X \vdash \mathsf{H}X,$$

so by *A5* and *A4* for $i \geqslant 1$,

$$\vdash \mathsf{H}^i X.$$

Thus also for $i \geqslant 1$, $\vdash \mathsf{H}(G_i X)$.

Now for $j \geqslant 1$,

$$X \supset G_j X \vdash X \supset . \mathsf{H}^j X \supset G_{j-1}X,$$

so by the propositional calculus, as above noted,

$$X \supset G_j X \vdash \mathsf{H}^j X \supset . X \supset G_{j-1}X.$$

Now for $j \geqslant 1$,

$$X \supset G_j X \vdash X \supset G_{j-1}X,$$

and so for $j \geqslant 1$,

$$X \supset G_j X \vdash X \supset G_1 X.$$

Now as $X = G_k X$ and

$$\vdash \mathsf{H}X, \vdash X \supset G_k X$$

$$\vdash \mathsf{H}X \supset : X \supset . X \supset \mathfrak{A} \tag{2}$$

as

$$G_0 X = X \supset \mathfrak{A}.$$

Now by the propositional calculus

$$\vdash X \supset . X \supset \mathfrak{A} : \supset . X \supset \mathfrak{A},$$

and thus using (2)

$$\vdash \mathsf{H}X \supset . X \supset \mathfrak{A}, \tag{3}$$

that is

$$\vdash G_1 X.$$

But also

$$\vdash \mathsf{H}^2 X;$$

so by *A2* and *A4*

$$\vdash \mathsf{H}^2 X \supset G_1 X,$$

that is

$$\vdash G_2 X.$$

Similarly

$$\vdash G_3 X, \ldots \qquad \vdash G_k X$$

that is

$$\vdash X;$$

and by (3)

$$\vdash \mathfrak{A}.$$

Now eliminating assumption *A6*, we have for any $\mathfrak{A}$,

$$\mathsf{H}\mathfrak{A} \vdash \mathfrak{A}.$$

therefore

$$\mathsf{H}(\mathsf{H}^k\mathfrak{A}) \vdash \mathsf{H}^k\mathfrak{A},$$

and by *A5*

$$\vdash \mathsf{H}^k\mathfrak{A}.$$

Similarly $\quad \vdash \mathsf{H}^{k-1}\mathfrak{A}, \ldots \quad \vdash \mathsf{H}\mathfrak{A}, \vdash \mathfrak{A},$

so $\vdash \mathfrak{A}$ has been proved for any $\mathfrak{A}$.

Of the assumptions used to derive this inconsistency, *A1*, *A2* and *A3* are ordinary propositional calculus results and *A4* merely says that if $\mathfrak{X}$ is true then it is a proposition. Thus it seems that we should reject either *A5* or *A7*. If

$$\mathsf{H}\mathfrak{X} . \mathsf{H}\mathfrak{Y} \vdash \mathsf{H}(\mathsf{P}\mathfrak{X}\mathfrak{Y}) \tag{4}$$

is taken instead of *A7*, the paradox does not go through. However in some systems *A7* is preferable to (4) and we have to reject *A5*.

REFERENCES

[1] Curry, H. B., "Some advances in the combinatory theory of quantification," *Proceedings of the National Academy of Sciences, USA,* vol. 28 (1942), pp. 564-569.

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