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REMARKS ON THE W. C. NEMITZ'S PAPER "SEMI-BOOLEAN LATTICES"

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In his paper [9] W. C. Nemitz considers the semi-Boolean lattices. We will show in this note that the classes of all semi-Boolean lattices and of all relative Stone lattices with a greatest element are equal. At the end we shall consider some equalities concerning the Brouwerian lattices.

We will start with some preliminaries. A Brouwerian (or implicative) lattice is a lattice L in which, for any given elements a and b, the set of all $x \epsilon L$ such that $a \cap x \leqslant b$ contains a greatest element $a * b$, the relative pseudo-complement of a in b. It is known (see [2]) that any Brouwerian lattice is distributive and contains a greatest element 1. A lattice L with a least element 0 is called *pseudo-complemented* if for every element $x \epsilon L$ there exists a relative pseudo-complement $x * 0$ which is denoted by x^{**}. A pseudo-complemented lattice need not be distributive but it always contains a greatest element 1. A *Stone lattice* is a distributive pseudo-complemented lattice which satisfies the equality

$$(1) \qquad x^* \cup x^{**} = 1 \textit{ for all } x \epsilon L.$$

A lattice is called *relative Stone* if all its (closed) intervals are Stone lattices.

In [1] or, more generally, in [5] the following statements were proved

Lemma 1. *A lattice L is a Brouwerian lattice if and only if the following conditions are satisfied.*

(i) *L is a distributive lattice with* 1;
(ii) *All (closed) intervals of L are pseudo-complemented.*

Lemma 2. *A lattice L with* 1 *is a relative Stone lattice if and only if it is a Brouwerian lattice which satisfies the following equality*

$$(2) \qquad (x * y) \cup (y * x) = 1 \textit{ for all } x, y \epsilon L.$$

A Brouwerian lattice L is called semi-Boolean (see [9]) if for all $x, y \epsilon L$

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$$(3) \qquad x \cup y = ((x * y) * y) \cap ((y * x) * x)$$

holds.

The following statement was proved in [7].

Lemma 3. *A universal algebra* $\langle L; \cup, \cap, *\rangle$, *where* $\cup, \cap$ *and* $*$ *are binary operations on* A, *is a Brouwerian lattice if and only if the following conditions are satisfied*:

(i) $\langle L; \cup, \cap\rangle$ *is a lattice*;
(ii) $x \cap [(x \cap y) * z] = x \cap (y * z)$ *for all* $x, y, z \in L$;
(iii) $x \cap [(y \cap z) * z] = x$ *for all* $x, y, z \in L$;
(iv) $x \cap (x * y) = x \cap y$ *for all* $x, y \in L$.[1]

The universal algebra $\langle L; \cup, \cap, *\rangle$ from Lemma 3 will be called a *Brouwerian algebra*. A *relative Stone* (*semi-Boolean*) algebra is a Brouwerian algebra which satisfies the equality (2) ((3)). It is clear that the classes of Brouwerian, relative Stone and semi-Boolean algebras are equationally definable.

First of all we will characterize the subdirectly irreducible Brouwerian and relative Stone algebras. To this we need some results about the congruence relations of Brouwerian algebras, which can be found in [8].

Let F_Θ denote the set $\{x \in L; x \equiv 1(\Theta)\}$ for a congruence relation Θ of a Brouwerian algebra L. It is clear that F_Θ forms a filter of L.

Lemma 4. *Let L be a Brouwerian algebra. If* Θ *is a congruence relation of L then*

$x \equiv y(\Theta)$ *if and only if* $x \cap d = y \cap d$ *for a suitable* $d \in F_\Theta$.

If F is a filter of L, then the binary relation $\Theta(F)$ *defined as follows*:

$x \equiv y(\Theta(F))$ *if and only if* $x \cap d = y \cap d$ *for a suitable* $d \in F$ *is a congruence relation of L.*

Lemma 5. *A non-trivial*[2] *Brouwerian algebra is subdirectly irreducible if and only if the set* $\{x \in L; x \neq 1\}$ *has a greatest element.*

Proof: Suppose $\langle L; \cup, \cap, *\rangle$ is a non-trivial subdirectly irreducible Brouwerian algebra. Then the system of all non-identical congruence relations of L contains a least element Θ_m. If $a \in L$, then $[a)$ denotes the set $\{x \in L; x \geqslant a\}$. $[a)$ is a filter of L generated by a. L contains the greatest element 1. For $x \neq 1$ it is clear that $\Theta([x)) \geqslant \Theta_m > 0 = \Theta([1))$. By Lemma 4 $\Theta_m = \Theta[F]$ for a suitable filter F of L. Hence $[x) \supseteq F$ for each $x \neq 1$. Further, $\Theta(F) \neq \Theta([1))$ implies $F \neq [1)$. There exists a element $c \neq 1$ of the filter F. Since $[c) \subseteq F$ therefore must be $\Theta([c)) \leqslant \Theta(F)$. However, $([c)) \geqslant \Theta(F)$, hence $\Theta([c)) = \Theta(F)$. Then $[c) = F$ and c is the greatest element of the set $\{x \in L; x \neq 1\}$. The proof of the converse is trivial.

1. Another system of postulates for Brouwerian lattices is due to A. A. Monteiro (see [2, II, §11]).
2. Containing more than one element.

Lemma 6. *A non-trivial relative Stone algebra L is subdirectly irreducible if and only if L forms a chain with a dual atom.*[3]

Proof. Let $\langle L; \cup, \cap, *\rangle$ be a non-trivial subdirectly irreducible relative Stone algebra. According to Lemma 5 $\{x \in L; x \neq 1\}$ contains a greatest element c. Moreover, L satisfies the identity

$$(x * y) \cup (y * x) = 1.$$

Hence $x * y \leq c$ or $y * x \leq c$ which is equivalent to $x * y = 1$ or $y * x = 1$. Thus $x \leq y$ or $y \leq x$ where "$\leq$" is the partial order determined by the lattice structure on L, which was to be proved. The proof of the converse is easy.

Lemma 7. *Any relative Stone algebra is a semi-Boolean algebra.*

Proof. At first we will prove that any subdirect irreducible relative Stone algebra is semi-Boolean. Suppose $\langle L; \cup, \cap, *\rangle$ is a subdirectly irreducible relative Stone algebra. By Lemma 6 L is a chain with a dual atom. Let $x, y \in L$. Then $x \leq y$ or $y \leq x$. If $x \leq y$ then $(x * y) * y = y$ and $(y * x) * x = 1$. Analogically, we can examine $y \leq x$. Thus L satisfies the equality (3) and hence L is a semi-Boolean algebra.

Since the class of all semi-Boolean algebras is equationally definable, therefore it contains with any system of algebras the subdirect product of these. Every relative Stone algebra is a subdirect product of suitable system of subdirect irreducible relative Stone algebras, because the class of all relative Stone algebras is also equationally definable. Therefore the class of all semi-Boolean algebras contains any relative Stone algebra. By Lemmas 2, 3 and 7 we can conclude

Lemma 8. *Any relative Stone lattice with* 1 *is semi-Boolean.*

For the next statement we will need some conceptions.

A element x of a pseudo-complemented lattice L is called closed if $x = x^{**}$. The set of all closed elements will be denoted by $B(L)$. It is known that if $\langle L; \cup, \cap, *\rangle$ is a pseudo-complemented lattice then $\langle B(L); \vee, \cap, *, 0, 1\rangle$ forms a Boolean algebra where for $a, b \in B(L)$ it holds:

$$a \vee b = (a^* \cap b^*)^*.$$

The following statement characterizes Stone lattices.

Lemma 9 (see [4]). *A distributive pseudo-complemented lattice L is a Stone lattice if and only if the Boolean algebra $B(L)$ of closed elements forms a sublattice of the lattice L.*

It is easy to prove

Lemma 10 (see [4]). *Any interval $[0, a]$ of a Stone lattice is itself a Stone lattice.*

Now we can prove

3. An element c in a lattice with 1 is called dual atom if $c < 1$ and $c < x \leq 1$ imply that $c = 1$.

Lemma 11. *Any semi-Boolean lattice is a relative Stone lattice with* 1.

Proof. Let L be a semi-Boolean lattice. Then L is a Brouwerian lattice by definition. By Lemma 1 and definition of a relative Stone lattice we will prove that the each interval $[a, b]$ $(a, b \in L)$ is a Stone lattice. According to Lemma 10 it is sufficient to choose $b = 1$ for the greatest element 1 of L. By Lemma 1 any interval $[a, 1]$ is pseudo-complemented and it is easy to prove that for $x \in [a, 1]$ $x * a$ is a pseudo-complement of x in this interval. Therefore we have $\mathrm{B}([a, 1]) = \{x * a; x \in [a, 1]\}$. We shall show that $\mathrm{B}([a, 1])$ is a sublattice of the interval $[a, 1]$. It is enough to prove that the operations "$\vee$" and "$\cup$" are equal on $\mathrm{B}([a, 1])$. For any $x, y \in \mathrm{B}([a, 1])$ we have

$$x \vee y \geqslant x \cup y.$$

If $x, y \in \mathrm{B}([a, 1])$ there exist such elements $x_1, y_1 \in [a, 1]$ that $x = x_1 * a$ and $y = y_1 * a$. Since x and y are closed elements in $[a, 1]$ therefore by [8] $(x * y) * y$ and $(y * x) * x$ are closed elements in $[a, 1]$, too. But $\mathrm{B}([a, 1])$ is a sublattice of the lattice $[a, 1]$ therefore the element

$$((x * y) * y) \cap ((y * x) * x)$$

belongs to $\mathrm{B}([a, 1])$. Hence by (4) we get $x \cup y \in \mathrm{B}([a, 1])$. Then it holds that

$$x \vee y = x \cup y.$$

$\mathrm{B}([a, 1])$ is a sublattice of the interval $[a, 1]$ and by Lemma 9 and the definition of a relative Stone lattice L is relative Stone.

By Lemmas 8 and 11 we can conclude

Theorem 1. *The classes of all semi-Boolean lattices and of all relative Stone lattices with* 1 *are equal.*

We can say now, by Theorem 1, that [9, Theorem 3] states the same as [3, Theorem 1] or [5, Theorem 5.16] and [6, Theorem 1] in a general formulation for $\cap$-semi-lattices.

We will mention now some remarks concerning the class of dual generalized Boolean lattices. It is easy to see that any Boolean lattice is a relative Stone lattice with a greatest element. Moreover, the same is true for any distributive relatively complemented lattice with 1. Such a lattice will be called a dual generalized Boolean lattice. Since any relative Stone lattice with 1 is a Brouwerian one, therefore the dual generalized Boolean lattice is a Brouwerian one as well. It is easy to see that a Brouwerian lattice L is a dual generalized Boolean one if and only if any interval $[a, 1]$ is a Boolean lattice. We obtain another characterization from

Lemma 12. *The Brouwerian lattice L is a dual generalized Boolean one if and only if the following equality is satisfied*:

$$x \cup (x * y) = 1 \quad \textit{for all } x, y \in L. \tag{4}$$

Proof. The necessity is trivial. Suppose now L is a Brouwerian lattice satisfying (4). Let $x \in [a, 1]$ for $a \in L$. Then $x \cap (x * a) = a$ and $x \cup (x * a) = 1$. Since L is distributive therefore $[a, 1]$ is a Boolean lattice, which was to be proved.

Now we shall consider the following identities:

(4′) $\qquad x \cup (x * y) = x * x \quad (=1);$

(5) $\qquad x \cup y = (x * y) * y;$

(6) $\qquad (x * y) * z = (x \cup z) \cap (y * z)$

and the classes of all Brouwerian algebras satisfying these identities. Let $K_i (1 \leq i \leq 3)$ be classes of the Brouwerian algebras defined by (or satisfying) (4′), (5) and (6) respectively. It is easy to see that (6) implies (5). Therefore $K_2 \supseteq K_3$. Further, by Theorem 1, K_2 is a subclass of the class of all relative Stone algebras.

Now assume that $\langle L; \cup, \cap, *\rangle$ is a non-trivial subdirectly irreducible algebra belonging to K_2. By Lemma 6 L is a chain with a dual atom. We will show that L is a two-element chain. Suppose on the contrary there exist elements $x, y \in L$ such that $1 > x > y$. Since L is a chain, therefore $x * y = y$ and hence $(x * y) * y = 1 = x = x \cup y$, which is a contradiction. Thus L is a two-element set. By an easy computation we see that a two-element Brouwerian algebra (= a subdirectly irreducible algebra satisfying (5)) belongs to the class K_3. Therefore $K_2 \subseteq K_3$ and hence $K_2 = K_3$.

Analogously we can obtain that any non-trivial subdirectly irreducible algebra L of the class K_1 is a two-element chain. Thus we conclude

Lemma 13. *The classes of algebras K_1, K_2 and K_3 are equal. The class K $(= K_1 = K_2 = K_3)$ is a proper subclass of the class of all relative Stone algebras.*

By Lemmas 12 and 13 we have

Theorem 2. *A Brouwerian lattice L is a dual generalized Boolean one if and only if L satisfies any of the identities* (4′), (5) *and* (6).

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