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THE PRODUCT OF IMPLICATION AND COUNTER-IMPLICATION SYSTEMS

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1. *Introduction.** Rasiowa has obtained in [1] a finite axiomatization of the product system of "implication" and "equivalence". In this paper, we show that the logic system based on the single binary connective $\bigcirc$ with the logical matrix

$\bigcirc$	1	2	3	4
*1	1	1	3	3
2	2	1	4	3
3	1	1	1	1
4	2	1	2	1

that is the product connective of implication (C) and counter-implication ($\supset$) is finitely axiomatizable. The axiom and the rules of inference have been obtained by combining the axioms and the rules of inference of the complete axiomatizations of the implication system (C-system) and of the counter-implication system ($\supset$-system).

2. *Preliminary definitions.* In these definitions Δ and Δ_1 are arbitrary binary connectives.

2.1. *Δ-formulas.* Δ-formulas are defined recursively as follows:

i) a sentential variable, a small Roman letter, is a Δ-formula;

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ii) if a and b are Δ-formulas, then Δab is a Δ-formula;

iii) no formula is a Δ-formula unless its being so follows by a finite number of applications of i) and ii).

2.2. *Δ-formula schemata.* Δ-formula schemata are defined recursively as follows:

i) a small Greek letter is a Δ-formula schema;

ii) if α and β are Δ-formula schemata, then $\Delta\alpha\beta$ is a Δ-formula schema;

iii) no formula schema is a Δ-formula schema unless its being so follows by a finite number of applications of i) and ii) above.

2.3. *Corresponding Δ_1-formula of a Δ-formula.* A Δ_1-formula obtained by replacing every occurrence of Δ by the binary connective Δ_1 in a Δ-formula a is the corresponding Δ_1-formula of the Δ-formula a and is denoted $a\ \Delta_1$.

2.4. *Corresponding Δ_1-formula schema of a Δ-formula schema.* A Δ_1-formula schema obtained by replacing every occurrence of Δ by the binary connective Δ_1 in a Δ-formula schema Φ_Δ is the corresponding Δ_1-formula schema of the Δ-formula schema Φ_Δ and is denoted Φ_{Δ_1}.

2.5. *Δ-rule scheme.* A rule scheme that has a finite number of Δ-formula schemata for premisses and one Δ-formula schema for conclusion is a Δ-rule scheme. A Δ-rule scheme is also referred to as a Δ-rule of inference.

2.6. *The corresponding Δ_1-rule scheme of Δ-rule scheme.* The Δ_1-rule scheme, obtained by replacing every occurrence of Δ by the binary connective Δ_1 in a Δ-rule scheme (both in the premisses and in the conclusion) is called the corresponding Δ_1-rule scheme of a Δ-rule scheme.

3. *Derivation of a complete axiomatization of Ↄ-system.* Łukasiewicz's axiomatization of the C-system in [2] consists of the single axiom

Ł : *CCCpqrCCrpCsp*

and the two rules of inference—the rule of substitution and the rule of detachment for C, viz., $C\alpha\beta,\ \alpha \rightarrow \beta$. This is used to obtain a complete axiomatization of the Ↄ-system. As C and Ↄ are functionally complete with respect to each other, we translate the axiom Ł and the rules of inference and obtain the following complete axiomatization of the Ↄ-system consisting of the single axiom

Tr (Ł) : *ↃↃↃpsↃprↃrↃqp*

and the two rules of inference—the rule of substitution and the rule of detachment for Ↄ, viz., Ↄ$\alpha\beta,\ \beta \rightarrow \alpha$.

4. *Expressibility of the product connective $\mathsf{I}_1(p,q) \times \mathsf{I}_2(p,q)$ by an ○-formula.* The connectives $\mathsf{I}_1(p,q)$ and $\mathsf{I}_2(p,q)$ have the truth-tables:

$\mathsf{I}_1(p,q)$		q: 1	2
p:	1	1	1
	2	2	2

$\mathsf{I}_2(p,q)$		q: 1	2
p:	1	1	2
	2	1	2

Their product connective $\mathsf{I}_1(p,q) \times \mathsf{I}_2(p,q)$ has the truth-table:

$\mathsf{I}_1(p,q) \times \mathsf{I}_2(p,q)$	1	2	3	4
1	1	2	1	2
2	1	2	1	2
3	3	4	3	4
4	3	4	3	4

and is expressible by the ○-formula ○○*q*○*qq*○○*ppp*. The C-formula

$$CCqCqqCCppp$$

is the corresponding C-formula of this ○-formula and the Ↄ-formula

$$ↃↃqↃqqↃↃppp$$

is the corresponding Ↄ-formula of this ○-formula. They are respectively truth-functionally equivalent to the C-formula p and the Ↄ-formula q. The sentential letters p and q are ○-formulas as well. As p and q denote arbitrary ○-formulas, we can assert

Theorem 1. Let f be any two-valued truth-function expressible by a C-formula a. Let g be any two-valued truth-function expressible by a Ↄ-formula b. There is an ○-formula, d, such that the following two conditions hold:

i) *Its corresponding C-formula, d_C, is equivalent to a, i.e., f is expressible by d_C.*
ii) *Its corresponding Ↄ-formula, $d_Ↄ$, is equivalent to b, i.e., g is expressible by $d_Ↄ$.*

This theorem is crucial in the development of the axiomatization.

5. *A Complete axiomatization of the ○-system.* We now develop a complete axiomatization of the ○-system by using the complete axiomatizations of the C- and Ↄ-systems.

Łukasiewicz has shown in [2] that the single axiom

$$Ł : CCCp_1q_1r_1CCr_1p_1Cs_1p_1$$

and the two rules of inference—the rule of substitution and the rule of detachment for C, viz., $C\alpha\beta$, $\alpha \rightarrow \beta$ together constitute a complete axiomatation of the C-system. From section 2 it follows that the single axiom

$$K : ↃↃↃp_2s_2Ↄp_2r_2Ↄr_2Ↄq_2p_2$$

and the two rules of inference—the rule of substitution and the rule of detachment for $\supset$, viz., $\supset\alpha\beta$, $\beta \rightarrow \alpha$ together constitute a complete axiomatization of the $\supset$-system. By Theorem 1, we know that it is possible to obtain an $\bigcirc$-formula such that its corresponding C-formula is equivalent to the given C-formula and its corresponding $\supset$-formula is equivalent to the given $\supset$-formula. So the $\bigcirc$-formula

$$R : \bigcirc\bigcirc K_{\bigcirc}\bigcirc K_{\bigcirc}K_{\bigcirc}\bigcirc\bigcirc Ł_{\bigcirc}Ł_{\bigcirc}Ł_{\bigcirc}$$

where $K_{\bigcirc}$ and $Ł_{\bigcirc}$ are abbreviations for the $\bigcirc$-formulas $\bigcirc\bigcirc\bigcirc p_2 s_2 \bigcirc p_2 r_2 \bigcirc r_2 \bigcirc q_2 p_2$ and $\bigcirc\bigcirc\bigcirc p_1 q_1 r_1 \bigcirc\bigcirc r_1 p_1 \bigcirc s_1 p_1$ respectively, has for its corresponding C-formula, a C-formula that is equivalent to Ł, and has for its corresponding $\supset$-formula, a $\supset$-formula that is equivalent to K. It is clear that the $\bigcirc$-formula R is an $\bigcirc$-tautology. Let us accept R as an axiom. Let us consider the following four rules of inference:

i) the rule of substitution,
ii) $\bigcirc\bigcirc\alpha\bigcirc\alpha\alpha\bigcirc\bigcirc\alpha\alpha\alpha \rightarrow \alpha$,
iii) $\bigcirc\bigcirc\sigma\bigcirc\sigma\sigma\bigcirc\bigcirc\bigcirc\alpha\beta\bigcirc\alpha\beta\bigcirc\alpha\beta$, $\alpha \rightarrow \bigcirc\bigcirc\sigma\bigcirc\sigma\sigma\bigcirc\bigcirc\beta\beta\beta$,
iv) $\bigcirc\bigcirc\bigcirc\gamma\delta\bigcirc\bigcirc\gamma\delta\bigcirc\gamma\delta\bigcirc\bigcirc\sigma\sigma\sigma$, $\delta \rightarrow \bigcirc\bigcirc\gamma\bigcirc\gamma\gamma\bigcirc\bigcirc\sigma\sigma\sigma$.

We assert that the axiom R and these four rules of inference constitute a complete axiomatization of the $\bigcirc$-system.

Proof: Given any $\bigcirc$-tautology a, to prove that the above axiomatization of the $\bigcirc$-system is complete, we must show that a is provable in the $\bigcirc$-system. Given a, we find its corresponding C-tautology a_C, and its corresponding $\supset$-tautology $a_{\supset}$. Now we obtain a C-tautology $a_C^{(1)}$ from a_C by subscripting each of its variable occurrences by the numeral 1. (This is a substitution instance of a_C, containing variables subscripted with the numeral 1 instead of the variables in a_C). Similarly, we obtain a $\supset$-tautology $a_{\supset}^{(2)}$ from $a_{\supset}$ by subscripting each of its variable occurrences by the numeral 2. This makes the variables in $a_C^{(1)}$ and $a_{\supset}^{(2)}$ independent and we get the facility that $a_{\supset}^{(2)}$ will be left unaltered in any substitution we perform in $a_C^{(1)}$ and vice versa. In order to prove a, we first prove

$$\bigcirc\bigcirc a^{(2)}\bigcirc a^{(2)}a^{(2)}\bigcirc\bigcirc a^{(1)}a^{(1)}a^{(1)},$$

where $a^{(2)}$ is the corresponding $\bigcirc$-formula of the $\supset$-formula $a_{\supset}^{(2)}$, and $a^{(1)}$ is the corresponding $\bigcirc$-formula of the C-formula $a_C^{(1)}$. Because we have a complete axiomatization of the C-system, there is a proof of $a_C^{(1)}$ starting from Ł by one or more applications of the two rules of inference. If the proof of $a_C^{(1)}$ uses any of the variables in K, then we can modify the proof in such a way that none of the variables in K is used. Now we can be certain that we do not change any of the variables in $K_{\bigcirc}$ when we deduce starting from R. Corresponding to each step in the modified proof of $a_C^{(1)}$ starting from Ł, if starting from R we perform the corresponding step indicated below, it is clear that we obtain a proof of

$$\bigcirc\bigcirc K_{\bigcirc}\bigcirc K_{\bigcirc}K_{\bigcirc}\bigcirc\bigcirc a^{(1)}a^{(1)}a^{(1)}$$

If the rule of substitution is used in the modified proof of $a_C^{(1)}$, we use the rule of substitution and substitute for the variables corresponding ○-formulas; if the rule of detachment for C is used, we use the following rule of detachment for ○:

$$\bigcirc\bigcirc\sigma\bigcirc\sigma\sigma\bigcirc\bigcirc\bigcirc\alpha\beta\bigcirc\alpha\beta\bigcirc\alpha\beta,\ \alpha \rightarrow \bigcirc\bigcirc\sigma\bigcirc\sigma\sigma\bigcirc\bigcirc\beta\beta\beta$$

If the axiom Ł is used in the proof of $a_C^{(1)}$, we use the axiom R. Because we have a complete axiomatization of the Ɔ-system, there is a proof of $a^{(2)}$ starting from K by one or more applications of the two rules of inference. If the proof of $a^{(2)}$ uses any of the variables in $a_C^{(1)}$, then we can modify the proof in such a way that none of the variables in $a_C^{(1)}$ is used. Now we can be certain that we do not substitute for any of the variables in $a^{(1)}$. Corresponding to each step in the modified proof of $a_{Ɔ}^{(2)}$ starting from K, if starting from

$$\bigcirc\bigcirc K_{\bigcirc}\bigcirc K_{\bigcirc}K_{\bigcirc}\bigcirc\bigcirc a^{(1)}a^{(1)}a^{(1)}.$$

we perform the corresponding step indicated below, it is clear that we obtain a proof of

$$\bigcirc\bigcirc a^{(2)}\bigcirc a^{(2)}a^{(2)}\bigcirc\bigcirc a^{(1)}a^{(1)}a^{(1)}.$$

If the rule of substitution is used in the modified proof of $a_{Ɔ}^{(2)}$, we use the rule of substitution and substitute for the variables corresponding ○-formulas; if the rule of detachment for Ɔ is used, we use the following rule of detachment for ○:

$$\bigcirc\bigcirc\bigcirc\gamma\delta\bigcirc\bigcirc\gamma\delta\bigcirc\gamma\delta\bigcirc\bigcirc\sigma\sigma\sigma,\ \delta \rightarrow \bigcirc\bigcirc\gamma\bigcirc\gamma\gamma\bigcirc\bigcirc\sigma\sigma\sigma.$$

If the axiom K is used in the proof of $a_{Ɔ}^{(2)}$, we use the tautology $\bigcirc\bigcirc K_{\bigcirc}\bigcirc K_{\bigcirc}K_{\bigcirc}\bigcirc\bigcirc a^{(1)}a^{(1)}a^{(1)}$. We can by the rule of substitution assert

$$\bigcirc\bigcirc aa\bigcirc aa\bigcirc\bigcirc aaa$$

where a is a substitution instance of $a^{(2)}$ as well as $a^{(1)}$ (we substitute the original variables in a instead of the subscripted variables in $a^{(2)}$ and $a^{(1)}$). Finally, we can assert a by one application of the rule scheme:

$$\bigcirc\bigcirc\alpha\bigcirc\alpha\alpha\bigcirc\bigcirc\alpha\alpha\alpha \rightarrow \alpha.$$

We state without proof a theorem that permits us to verify easily the validity of these rule schemes.

Theorem 2: An ○-rule scheme is valid if and only if its corresponding C-rule scheme and its corresponding Ɔ-rule scheme are valid.

i) The ○-rule scheme:

$$\bigcirc\bigcirc\sigma\bigcirc\sigma\sigma\bigcirc\bigcirc\bigcirc\alpha\beta\bigcirc\alpha\beta\bigcirc\alpha\beta,\ \alpha \rightarrow \bigcirc\bigcirc\sigma\bigcirc\sigma\sigma\bigcirc\bigcirc\beta\beta\beta$$

is valid.

Proof: The corresponding C-rule scheme is:

$$CC\sigma_C C\sigma_C\sigma_C CCC\alpha_C\beta_C C\alpha_C\beta_C C\alpha_C\beta_C,\ \alpha_C \rightarrow CC\sigma_C C\sigma_C\sigma_C CC\beta_C\beta_C\beta_C$$

The first premiss is equivalent to $C\alpha_C\beta_C$, and the conclusion is equivalent to β_C. We know that $C\alpha_C\beta_C$, $\alpha_C \rightarrow \beta_C$ is valid. Therefore, the corresponding C-rule scheme of the $\bigcirc$-rule scheme is valid.

The corresponding Ɔ-rule scheme is

$$ƆƆ\sigma_ƆƆ\sigma_Ɔ\sigma_ƆƆƆƆ\alpha_Ɔ\beta_ƆƆ\alpha_Ɔ\beta_ƆƆ\alpha_Ɔ\beta_Ɔ,\ \alpha_Ɔ \rightarrow ƆƆ\sigma_ƆƆ\sigma_Ɔ\sigma_ƆƆƆ\beta_Ɔ\beta_Ɔ\beta_Ɔ$$

The first premiss is equivalent to $\sigma_Ɔ$ and so is the conclusion. Hence, the corresponding Ɔ-rule scheme is valid.

ii) The $\bigcirc$-rule scheme:

$$\bigcirc\bigcirc\bigcirc\gamma\delta\bigcirc\bigcirc\gamma\delta\bigcirc\gamma\delta\bigcirc\bigcirc\sigma\sigma\sigma,\ \delta \rightarrow \bigcirc\bigcirc\gamma\bigcirc\gamma\gamma\bigcirc\bigcirc\sigma\sigma\sigma$$

is valid.

Proof: The corresponding C-rule scheme is

$$CCC\gamma_C\delta_CCC\gamma_C\delta_CC\gamma_C\delta_CCC\sigma_C\sigma_C\sigma_C,\ \delta_C \rightarrow CC\gamma_CC\gamma_C\gamma_CCC\sigma_C\sigma_C\sigma_C.$$

The first premiss is equivalent to σ_C and so is the conclusion. Therefore, the corresponding C-rule scheme is valid.

The corresponding Ɔ-rule scheme is

$$ƆƆƆ\gamma_Ɔ\sigma_ƆƆƆ\gamma_Ɔ\delta_ƆƆ\gamma_Ɔ\delta_ƆƆƆ\sigma_Ɔ\sigma_Ɔ\sigma_Ɔ,\ \delta_Ɔ \rightarrow ƆƆ\gamma_ƆƆ\gamma_Ɔ\gamma_ƆƆƆ\sigma_Ɔ\sigma_Ɔ\sigma_Ɔ$$

The first premiss is equivalent to $Ɔ\gamma_Ɔ\delta_Ɔ$, and the conclusion is equivalent to $\gamma_Ɔ$. We know that $Ɔ\gamma_Ɔ\delta_Ɔ$, $\delta_Ɔ \rightarrow \gamma_Ɔ$ is valid; therefore, the corresponding Ɔ-rule scheme is valid.

iii) $\bigcirc\bigcirc\alpha\bigcirc\alpha\alpha\bigcirc\bigcirc\alpha\alpha\alpha \rightarrow \alpha$

The $\bigcirc$-formula schemata $\bigcirc\bigcirc\alpha\bigcirc\alpha\alpha\bigcirc\bigcirc\alpha\alpha\alpha$ and α are equivalent.

Hence, the $\bigcirc$-rule scheme above is valid.

6. *A complete axiomatization of the product system of "implication" and "equivalence"*. The method by which the axiomatization of the $\bigcirc$-system has been developed, is based on the following two facts: i) the logic systems based on the connectives C and Ɔ are completely axiomatizable, ii) the connective $\bigcirc$ has the property that it is possible to express the product connective $\mathsf{I}_1(p,q) \times \mathsf{I}_2(p,q)$ by an $\bigcirc$-formula. This makes us suspect that a similar method may be applicable for developing complete axiomatizations of logic systems based on other single product connectives.

We consider the product system of "implication" and "equivalence" and give an axiomatization that is different from the one given by Rasiowa.

Let us denote the product connective of "implication" and "equivalence" by W. The product connective $\mathsf{I}_1(p,q) \times \mathsf{I}_2(p,q)$ is expressible by the W-formula $WWpqp$. This axiomatization is based on Łukasiewicz's axiomatizations of the C-system in [2] and of the E-system in [3]. The former is the one used in section **4** and the latter consists of the single axiom

$$\mathrm{J}:\ EEp_2q_2EEr_2q_2Ep_2r_2$$

and the rules of inference—the rule of substitution and the rule of detachment for E, viz., $E\alpha\beta$, $\alpha \rightarrow \beta$. The axiomatization consists of the single axiom

$$WW\text{Ł}_W \mathrm{J}_W \text{Ł}_W,$$

where Ł_W and J_W are abbreviations for the W-formulas

$$WWWp_1q_1r_1WWr_1p_1Ws_1p_1$$

and

$$WWp_2q_2WWr_2q_2Wp_2r_2$$

respectively, and the following four rules of inference:

i) the rule of substitution,
ii) $WWW\alpha\beta\sigma W\alpha\beta$, $\alpha \rightarrow WW\beta\sigma\beta$,
iii) $WW\sigma W\gamma\delta\sigma$, $\gamma \rightarrow WW\sigma\delta\sigma$,
iv) $WW\alpha\alpha\alpha \rightarrow \alpha$

We can establish that this axiomatization is complete by a proof similar to the proof in section **4** that established the complete axiomatization of the $\bigcirc$-system.

Discussion: One might comment that the single axiom and the rules of inference that together constitute the axiomatization of the $\bigcirc$-system are very lengthy and cumbersome. But the method of development of the axiomatization of the $\bigcirc$-system makes use of the axiomatization of the implication system and counter-implication system in a straighforward manner. The crucial step in the axiomatization is the expressibility of the connective $\mathsf{I}_1(p,q) \times \mathsf{I}_2(p,q)$ by the product connective $\bigcirc$.

Rasiowa has raised in [1] a question as to whether every product of two axiomatizable systems of propositional calculus (with a single connective) is axiomatizable. The method of development of the axiomatization of the $\bigcirc$-system suggests that it is possible to axiomatize other products of two axiomatizable systems of propositional calculus (with a single connective) provided it is possible to express the product connective $\mathsf{I}_1(p,q) \times \mathsf{I}_2(p,q)$ by a formula in the product connective under consideration. This is a partial answer to the question raised by Rasiowa. In two-valued propositional calculus, if we consider two mutually dual connectives, it is clear that it is not possible to express the product connective $\mathsf{I}_1(p,q) \times \mathsf{I}_2(p,q)$ by a formula in the product connective of the two mutually dual connectives. But this product system is empty in the sense that there are no formulas in this product connective which are tautologies. So the problem as to whether it is possible to express the connective $\mathsf{I}_1(p,q) \times \mathsf{I}_2(p,q)$ by every product of two distinct connectives that are not mutually dual is interesting. As far as the product connectives of two distinct connectives other than the connective $\mathsf{Vr}(p,q)$ with the matrix

$\mathsf{Vr}(p,q)$	q:	1	2
p:	1	1	1
	2	1	1

the author has shown in [4] that the connective $\mathsf{I}_1(p,q) \times \mathsf{I}_2(p,q)$ may be expressed by all the product connectives that have non-empty sets of tautologies.

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