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CONCERNING SOME PROPOSALS FOR QUANTUM LOGIC

C. W. LEININGER

The suggestion made in 1936 by Birkhoff and von Neumann and discussed by Birkhoff in [1], pp. 156-163 that a propositional algebra appropriate to quantum theory should have the structure of an orthocomplemented lattice has been widely discussed. More recently Kochen and Specker [2], pp. 177-189 have presented the idea of a partial Boolean algebra. We will call a partial Boolean algebra B complete if each Boolean subalgebra of B is complete.

It is the purpose of this note to point out that a complete partial Boolean algebra has an extension to an orthocomplemented lattice L, and thus may be considered as such a lattice in which the join and meet of a pair x, y of elements of L is of logical significance if and only if each of x and y belongs to the same Boolean subalgebra of L, i.e., x, y is a commeasurable pair in the sense of [2]. Otherwise $x \smile y$ and $x \frown y$ are meaningless for quantum logic although existing in the lattice-theoretic sense. Perhaps this observation will help to clarify one of the problems frequently mentioned (e.g. in [3], p. 369) which is involved in the structure of a logic for quantum theory.

Instead of using the definition of a partial Boolean algebra, it will be more convenient for our purpose to have recourse to the properties of a model thereof, since [2] p. 184, every partial Boolean algebra is isomorphic to some case of the model. Hence the statement that B is a partial Boolean algebra will mean that B is a list $(M, \vee, \neg, 0, 1)$ and I is a set such that if $i \varepsilon I$, then B_i is a Boolean algebra $(M_i, \vee_i, \neg_i, 0, 1)$ and

(i) $M = \bigcup_{i \varepsilon I} M_i$;

(ii) if h, i, $j \varepsilon I$, a, $b \varepsilon M_h$, a, $c \varepsilon M_i$ and b, $c \varepsilon M_j$, then there is a $k \varepsilon I$ such that a, b, $c \varepsilon M_k$;

(iii) if i, $j \varepsilon I$, there is a $k \varepsilon I$ such that $M_i \cap M_j = M_k$;

(iv) if a, $b \varepsilon M$, then $a \vee b \varepsilon M$ if and only if there is an $i \varepsilon I$ such that a, $b \varepsilon M_i$, whence $a \vee b = a \vee_i b$;

(v) if $a \varepsilon M_i$, then $\neg a = \neg_i a$.

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Put less formally, B is a collection $\{B_i\}$ of Boolean algebras with a partial function $\vee$ having the domain $\bigcup_{i\varepsilon I} M_i{}^2$, where M_i denotes the set of elements of B_i, such that 0 and 1 are common to each B_i and the orthocomplement $\neg a$ of $a \varepsilon B$ is, if $a \varepsilon M_i$, the complement of $a \varepsilon B_i$. The partial function $\wedge$ and the relation $\leq$ are defined in the usual manner and thus are subject to restrictions analogous to those on $\vee$ and $\neg$. It will be convenient to resort to such informalities as "$x \varepsilon B_i$" rather than the more precise "$x \varepsilon M_i$ considered as the underlying set of B_i".

Theorem. *If* $(M, \vee, \neg, 0, 1)$ *is a complete partial Boolean algebra* B, *then there is an extension* $\smile$ *of the partial function* $\vee$ *such that* $(M, \smile, \neg, 0, 1)$ *is an orthocomplemented lattice* L.

Proof: If $x \varepsilon B_i$ and $y \varepsilon B_j$, let $S = \{u\colon u \varepsilon B_i \cap B_j,\ x \leq_i u\}$ and $T = \{v\colon v \varepsilon B_i \cap B_j, y \leq_j v\}$. Then $x \leq_i \bigcap S$ and $y \leq_j \bigcap T$, so that $\bigcap S \vee \bigcap T = \inf \{z\colon z \varepsilon B_i \cap B_j, x \leq z, y \leq z\} = x \smile y$. If $i = j$, then $x = \bigcap S$ and $y = \bigcap T$, so that $x \smile y = x \vee y$.

That the foregoing result depends upon the completeness of B is not inconsistent with the position of Birkhoff [1], p. 162 in suggesting as a candidate for a quantum logic a sublattice of the complete lattice of closed subspaces of a Hilbert space. On the other hand the failure in general of L to be modular would appear to be irrelevant, since L may be considered as being decomposed into classes—the Boolean subalgebras of B—such that the join and meet of elements in the same class are meaningful while those of elements of different classes are not.

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University of Dallas
Irving, Texas